ON THE $M\bar{E}\!-\!C$ONNECTION IN $M\bar{E}X_n$

KI-JO YOO

I. Introduction

The usual Einstein's $n$-dimensional unified field theory is based on a generalized $n$-dimensional Riemannian manifold $X_n$, which is referred to a real coordinate system $x^\nu$ and obeys coordinate transformations $x^\nu \rightarrow \bar{x}^\nu$ (*) for which $Det \left( \frac{\partial \bar{x}}{\partial x} \right) \neq 0$.

The algebraic structure on $X_n$ is endowed with a general real non-symmetric tensor $g_{\lambda \mu}$ which may be split into its symmetric part $h_{\lambda \mu}$ and skew-symmetric part $k_{\lambda \mu}$:

$$g_{\lambda \mu} = h_{\lambda \mu} + k_{\lambda \mu}, \quad (1.1)$$

where

$$g = Det(g_{\lambda \mu}) \neq 0, \quad h = Det(h_{\lambda \mu}) \neq 0. \quad (1.2)$$

We may define a unique tensor $h^{\lambda \nu}$ by

$$h_{\lambda \mu} h^{\lambda \nu} = \delta^\nu_\mu. \quad (1.3)$$

The tensor $h_{\lambda \mu}$ and $h^{\lambda \nu}$ will serve for raising and/or lowering indices of tensors in $X_n$ in the usual manner.

The differential geometric structure on $X_n$ is imposed by the tensor $g_{\lambda \mu}$ by means of a real general connection $\Gamma^\nu_{\lambda \mu}$, which satisfies the transformation rule

$$\bar{\Gamma}^\nu_{\lambda \mu} = \frac{\partial \bar{x}^\nu}{\partial x^\alpha} \left( \frac{\partial x^\beta}{\partial \bar{x}^\lambda} \frac{\partial x^\gamma}{\partial \bar{x}^\mu} \Gamma^\alpha_{\beta \gamma} + \frac{\partial^2 x^\alpha}{\partial \bar{x}^\lambda \partial \bar{x}^\mu} \right) \quad (1.4)$$

Received June 2, 1993. Revised July 20, 1993.

(*) Throughout the present paper, Greek indices are used for the holonomic components of tensors in $X_n$. They take the values $1, 2, \ldots, n$ unless stated otherwise, and follow the summation convention.
and the system of Einstein's equations

\[ \partial_\omega g_{\lambda \mu} - \Gamma^\alpha_{\lambda \omega} g_{\sigma \mu} - \Gamma^\alpha_{\omega \mu} g_{\lambda \alpha} = 0, \]  

(1.5a)

or equivalently

\[ D_\omega g_{\lambda \mu} = 2S_{\omega \mu}^{\alpha} g_{\lambda \alpha}, \]  

(1.5b)

where \( D_\omega \) denotes the symbolic vector of the covariant derivative with respect to \( \Gamma^\nu_{\lambda \mu} \) and

\[ S_{\lambda \mu}^{\nu} = \Gamma^\nu_{[\lambda \mu]} \]  

(1.6)

is the torsion tensor of \( \Gamma^\nu_{\lambda \mu} \).

The Einstein's connection \( \Gamma^\nu_{\lambda \mu} \) which takes the form

\[ \Gamma^\nu_{\lambda \mu} = \left\{ \begin{array}{c} \nu \\ \lambda \mu \end{array} \right\} + 2\delta^\nu_{\lambda \mu} X_\mu - 2g_{\lambda \mu} X^\nu \]  

(1.7)

for a non-null vector \( X^\nu \) is called ME-connection, and a generalized Riemannian manifold \( X_n \) connected by this connection is called n-dimensional ME-manifold. We denote it by \( MEX_n \). In the representation of ME-connection the vector \( X^\nu \) will be called ME-vector.

The following quantities will be frequently used in our subsequent considerations:

\[ g = \frac{g}{h}, \quad k = \frac{k}{h}, \]  

(1.8)

\[ \sigma = \left\{ \begin{array}{c} 0 \quad \text{if } n \text{ is even}, \\ 1 \quad \text{if } n \text{ is odd}, \end{array} \right. \]  

(1.9)

\[ \kappa_\lambda^{\nu} = \delta_\lambda^{\nu}, \quad (p)\kappa_\lambda^{\nu} = (p-1)\kappa_\lambda^{\sigma_1 \sigma_2 \cdots \sigma_p} \quad (p = 1, 2, 3, \cdots), \]  

(1.10)

\[ K_0 = 1, \quad K_p = k^{\alpha_{1}}_{[\alpha_1} k^{\alpha_2} \cdots k^{\alpha_p]}_{\alpha_p}], \quad (p = 0, 1, 2, \cdots), \]  

(1.11)

\[ K_{\omega \mu}^\nu = \nabla_\omega k_{\nu \mu} + \nabla_\mu k_{\omega \nu} + \nabla_\nu k_{\omega \mu}, \]  

(1.12)

where \( \nabla_\omega \) is the symbolic vector of the covariant derivative with respect to the Christoffel symbols \( \left\{ \begin{array}{c} \nu \\ \lambda \mu \end{array} \right\} \) defined by \( h_{\lambda \mu} \).
On the $ME$-connection in $MEX_n$

It has been shown that the following relations hold in $X_n$ (see [5]):

$$K_p = 0 \quad \text{if } p \text{ is odd,} \quad K_n = k \quad \text{if } n \text{ is even},$$

$$\det(Mh_{\lambda\mu} + k_{\lambda\mu}) = \hbar \sum_{s=0}^{n-\sigma} K_s M^{n-s}, \quad (M \text{ is a real number}),$$

$$\sum_{n=0}^{n-\sigma} K_s^{(n+p-s)} k_{\lambda\nu} = 0, \quad (p = 0, 1, 2, \cdots),$$

(Generalized recurrence relation in $X_n$).

Here and in what follows, the induces $s$ and $t$ are assumed to take the values $0, 2, 4, 6, \cdots$ in the specified range.

If the system (1.5) admits a solution $\Gamma^{\nu}_{\lambda\mu}$, it must be of the form

$$\Gamma^{\nu}_{\lambda\mu} = \left\{ \begin{array}{c} \nu \\ \lambda \mu \end{array} \right\} + S^{\lambda\nu}_{\mu \lambda} + U^{\nu}_{\lambda\mu},$$

where

$$U^{\nu}_{\lambda\mu} = 2\hbar^{\nu\alpha} S^{\alpha(\lambda\beta)}_{\lambda\beta}(k_{\mu\beta}).$$

The skew-symmetric part of (1.5) gives the following relation satisfied by the torsion tensor $S_{\omega\mu\nu}$:

$$B_{\omega\mu\nu} = S^{101}_{\omega\mu\nu} + S^{011}_{\omega\mu\nu} + S^{112}_{\omega\mu\nu},$$

where

$$B_{\omega\mu\nu} = \frac{1}{2} (K_{\omega\mu\nu} + 3K_{[\alpha\beta\gamma]} k_{\omega}^{\alpha} k_{\mu}^{\beta} k_{\nu}^{\gamma}),$$

$$S^{(p)}_{\omega\mu\nu} = S^{(p)}_{\alpha\beta\gamma} (k_{\omega}^{\alpha} (q) k_{\mu}^{\beta} (r) k_{\nu}^{\gamma}, \quad (p, q, r = 0, 1, 2, \cdots).$$

Furthermore, the tensor field $k_{\lambda\mu}$ is said to belong to

1. the first class if $K_{n-\sigma} \neq 0$,
2. the second class with $j^{th}$ category if $K_{2j} \neq 0$,
   $$K_{2j+2} = K_{2j+4} = \cdots = K_{n-\sigma} = 0,$$
3. the third class if $K_2 = K_4 = \cdots = K_{n-\sigma} = 0.$
II. The \( ME \)-connection \( \Gamma^\nu_{\lambda\mu} \).

This section is denoted mainly to the proof of a necessary and sufficient condition for the existence of \( ME \)-connection and to that of an useful representation of the \( ME \)-connection.

**Agreement 2.1.** (a) All considerations in the present section are dealt for a general \( n \geq 2 \) and for all possible classes.

(b) In our further considerations, we use the word "present condition" to describe the situations that Einstein's connection, given by (1.16) and (1.17), takes the form (1.7).

**Theorem 2.2.** A necessary and sufficient condition that the present condition holds is that for a non-null vector \( X^\nu \)

(a) the torsion tensor \( S_{\lambda\mu}^\nu \) is given by

\[
S_{\lambda\mu}^\nu = 2\delta[\lambda^\nu X_\mu] - 2k_{\lambda\mu} X^\nu, \tag{2.1}
\]

(b) and the tensor field \( g_{\lambda\mu} \) satisfies

\[
\delta(\lambda^\nu X_\mu) - h_{\lambda\mu} X^\nu = k(\lambda^\nu X_\mu) + 2k(\lambda^\nu k_\mu)^\alpha X_\alpha, \tag{2.2a}
\]

or equivalently

\[
g_{\nu(\lambda X_\mu)} + 2k_{\nu(\lambda k_\mu)^\alpha} - h_{\lambda\mu} X_\nu = 0. \tag{2.2b}
\]

**Proof.** Suppose that the present condition holds. Then, in virtue of (1.16) and (1.7), we have

\[
S_{\lambda\mu}^\nu + U^\nu_{\lambda\mu} = \delta(\lambda^\nu X_\mu) - 2g_{\lambda\mu} X^\nu,
\]

or equivalently

\[
S_{\lambda\mu}^\nu = 2\delta[\lambda^\nu X_\mu] - 2k_{\lambda\mu} X^\nu, \tag{2.3a}
\]

\[
U^\nu_{\lambda\mu} = 2\delta(\mu^\nu X_\mu) - 2h_{\lambda\mu} X^\nu. \tag{2.3b}
\]

On the other hand, (1.17) and (2.3a) give

\[
U^\nu_{\lambda\mu} = 2k_{(\mu^\nu X_\lambda)} + 4k(\lambda^\nu k_\mu)^\alpha X_\alpha. \tag{2.4}
\]
On the $\mathcal{ME}$-connection in $\text{ME}X_n$

The condition (2.2a) immediately follows from (2.3b) and (2.4). Conversely, suppose that the statements (a) and (b) hold. Then

\[
U^\nu_{\lambda\mu} = 2\delta^\nu_{(\lambda^\nu X_\mu)} - 2h_{\lambda\mu}X^\nu
\]

\[
= 2k_{(\lambda^\nu X_\mu)} + 4k_{(\lambda^\nu k_\mu)^\alpha X_\alpha}
\]

\[
= 2h^{\nu\alpha} \left( h_{(\lambda|\alpha|X_\mu)} - \frac{1}{2}(k_{\mu\lambda} + k_{\lambda\mu})X_\alpha - 2k_{\alpha(\lambda k_\mu)^\beta X_\beta} \right)
\]

\[
= 2h^{\nu\alpha} \left( (k_{\mu[\alpha X_\lambda]} + k_{\lambda[\alpha X_\mu]} - 2k_{\alpha(\lambda k_\mu)^\beta X_\beta} \right)
\]

\[
= 2h^{\nu\alpha} \left( (\delta_{[\alpha}^\beta X_\lambda) - k_{\alpha\lambda X_\beta})k_{\mu\beta} + (\delta_{[\alpha}^\beta X_\mu) - k_{\alpha\mu X_\beta})k_{\lambda\beta} \right)
\]

\[
= h^{\nu\alpha} \left( S_{\alpha\lambda} k_{\mu\beta} + S_{\alpha\mu} k_{\beta\lambda} \right)
\]

\[
= 2h^{\nu\alpha} S_{\alpha(\lambda^\nu k_\mu)^\beta}
\]

Therefore, the Einstein's connection $\Gamma^\nu_{\lambda\mu}$ of the form (1.16) takes the form (1.7) in virtue of (2.3).

**Remark 2.3.** In virtue of Theorem 2.2, we note that under the present condition the $\mathcal{ME}$-connection $\Gamma^\nu_{\lambda\mu}$ is of the form

\[
\Gamma^\nu_{\lambda\mu} = \left\{ \begin{array}{c} \nu \\ \lambda\mu \end{array} \right\} + S_{\lambda\mu}^{\nu} + U^{\nu}_{\lambda\mu}, \quad (2.5)
\]

where

\[
S_{\lambda\mu}^{\nu} = 2\delta_{(\lambda^\nu X_\mu)} - 2k_{\lambda\mu}X^\nu, \quad (2.6a)
\]

\[
U^{\nu}_{\lambda\mu} = k_{(\lambda^\nu X_\mu)} + 2k_{(\lambda^\nu k_\mu)^\alpha X_\alpha} = \delta_{(\lambda^\nu X_\mu)} - h_{\lambda\mu}X^\nu. \quad (2.6b)
\]

In our further considerations, we need a symmetric tensor

\[
P_{\lambda\mu} = (2)k_{\lambda\mu} - h_{\lambda\mu} \quad (2.7a)
\]

and its unique inverse tensor $Q^{\lambda\nu}$ (see [5]) defined by

\[
P_{\lambda\mu}Q^{\lambda\nu} = \delta^{\lambda\nu}. \quad (2.7b)
\]

It has been shown that (see [6])

\[
Q^{\lambda\nu} = -\frac{1}{g} \sum_{s=0}^{n-1} \tilde{K}_s^{(n-2+\sigma-s)}k_{\lambda\nu}, \quad (2.8a)
\]

where

\[
\tilde{K}_s = 1 + K_2 + K_4 + \cdots + K_s. \quad (2.8b)
\]
THEOREM 2.4. Under the present condition, the vector $X^\nu$ satisfies
\[
(k_{\lambda\nu} - 2 (2) k_{\lambda\nu} + (n - 1) h_{\lambda\nu}) X^\nu = 0. \tag{2.9}
\]

Proof. The relation (2.8) may be obtained by putting $\nu = \mu$ in (2.2a).

THEOREM 2.5. Under the present condition, the following relation holds in $MEX_n$
\[
B_{\omega\mu\nu} = - 2P_{\nu[\omega} X_{\mu]} + 2k_\omega^\alpha P_{\alpha\mu} X_\nu. \tag{2.10}
\]

Proof. In view of the process of the proof of Theorem 2.2, it may be easily checked that the condition (2.2a) may be replaced by (1.18). Employing the abbreviations introduced in (1.20) and using the condition (2.1), it follows that
\[
S_{\omega\mu\nu} = (h_{\gamma\alpha} X_\beta - h_{\gamma\beta} X_\alpha - 2k_{\alpha\beta} X_\gamma)^{(p)} k_\omega^\alpha (q) k_\mu^\beta (r) k_{\nu\gamma} \tag{2.11}
\]
\[
= (-1)^r (p+r) k_{\omega\nu}^{(q)} k_\mu^\alpha X_\alpha - (-1)^q (q+r) k_{\nu\mu}^{(p)} k_\omega^\alpha X_\alpha
\]
\[
- 2(-1)^q (p+q+1) k_{\omega\mu}^{(r)} k_\nu^\alpha X_\alpha, \quad (p, q, r = 0, 1, 2, \cdots).
\]

Consequently, using (2.10) the relation (1.18) is reduced to (2.9) as in the following way:
\[
B_{\omega\mu\nu} = S_{\omega\mu\nu}^{101} + S_{\omega\mu\nu}^{011} + S_{\omega\mu\nu}^{110} + S_{\omega\mu\nu}^{110}
\]
\[
= 2h_{\nu[\omega} X_{\mu]} - 2k_{\omega\mu} X_\nu - (2) k_{\omega\nu} X_\mu - k_{\nu\mu} (k_\omega^\alpha X_\alpha)
\]
\[
- 2(2) k_{\omega\mu} (k_\nu^\alpha X_\alpha) - k_{\omega\nu} (k_\mu^\alpha X_\alpha) + (2) k_{\nu\mu} X_\omega
\]
\[
+ 2(2) k_{\omega\mu} (k_\nu^\alpha X_\alpha) + k_{\omega\nu} (k_\mu^\alpha X_\alpha) + k_{\nu\mu} (k_\omega^\alpha X_\alpha) + 2(3) k_{\omega\mu} X_\nu
\]
\[
= 2(h_{\nu[\omega} - (2) k_{\nu[\omega} X_{\mu]} + 2(3) k_{\omega\mu} - k_{\omega\mu}) X_\nu
\]
\[
= - 2P_{\nu[\omega} X_{\mu]} + 2k_\omega^\alpha P_{\alpha\mu} X_\nu.
\]
III. The $ME$-vector $X^\nu$.

This section is concerned mainly with an explicit representation of the present condition mentioned in Agreement 2.1 and a general representation of the $ME$-vector which holds for a general $n$ and all possible classes. Furthermore, we introduce a special kind of representation of $X_\lambda$ which holds for an even $n$ and for the first class.

In this section we use the following abbreviation for an arbitrary real vector $A_\lambda$:

$$^{(p)}A_\lambda = ^{(p)}k^{\lambda\alpha}A_\alpha, \quad (p = 0, 1, 2, \cdots) \quad (3.1a)$$

Then, since $^{(p)}k_{\lambda\mu} = (-1)^p \ ^{(p)}k^{\mu\lambda}$, we have

$$^{(p)}A^\nu = (-1)^p \ ^{(p)}k^{\alpha\nu}A^\alpha, \quad (p = 0, 1, 2, \cdots) \quad (3.1b)$$

**Theorem 3.1. (Recurrence relation)** In $MEX_n$ every vector $A_\omega$ satisfied the following recurrence relation:

$$\sum_{s=0}^{n-\sigma} K_s ^{(n-s)}A_\omega = 0. \quad (3.2)$$

**Proof.** Multiplying $A_\nu$ to both side of (1.15) and using the abbreviation (3.1a), we have (3.2).

We use the following quantities:

$$N = \frac{1-n}{2}, \quad (3.3a)$$

$$\hat{K}_s = \sum_{t=0}^{s} K_t N^{s-t} = N^s + K_2 N^{s-2} + \cdots + K_s, \quad (3.3b)$$

$$Y_\omega = \frac{1}{2} Q^{\nu\mu} B_{\omega\mu\nu}. \quad (3.3c)$$

The quantities introduced in (3.3) are involved in the relations stated in the following two Theorems:
THEOREM 3.2. We have

$$\hat{K}_s = K_s + \hat{K}_{s-2} N^2.$$  (3.4)

THEOREM 3.3. Under the present condition, the following relations hold in $MEX_n$:

$$(p) X_\omega = (p-1) Y_\omega + N (p-2) Y_\omega + N^2 (p-2) X_\omega, \quad (p = 1, 2, 3, \cdots),$$  (3.5)

$$(p) Y_\omega = -\frac{1}{4g} \sum_{s=0}^{n-1} K_s \left( p^{(s)}_{ij0} K_{\omega\alpha\beta} + p^{(s)}_{ij1} K_{\omega\alpha\beta} - p^{(s)}_{ij0} K_{\omega\alpha\beta} + p^{(s)}_{ij1} K_{\alpha\beta\omega} \right) h^{\alpha\beta},$$  (3.6)

where

$$p' = p + 1, \quad \tilde{s} = n + \sigma - 2 - s, \quad p = 0, 1, 2, \cdots.$$  (3.7)

Proof. The relation (3.4) is a direct consequence of (3.3a) and (3.3b). Multiply $Q^{\nu\mu}$ to both sides of (2.10) and use (2.7) to obtain

$$Q^{\nu\mu} B_{\omega\mu\nu} = (n-1) X_\omega + 2 k^{\omega}_{\alpha} X_\alpha = (n-1) X_\omega + 2 (1) X_\omega.$$  

Hence employing the notations introduced in (3.3), we have

$$(1) X_\omega = Y_\omega + N X_\omega.$$  (3.8)

Our assertion (3.5) immediately follows from (3.8) as in the following way:

$$(p) X_\omega = (p-1) k^{\omega}_{\alpha} (1) X_\alpha = (p-1) Y_\omega + N (p-1) X_\omega$$

$$= (p-1) Y_\omega + N (p-2) Y_\omega + N^2 (n-2) X_\omega.$$  

On the other hand, in virtue of (1.19), (1.20), (2.8a), (3.3c), and (3.7), the representation (3.6) may be proved as in the following way:

$$(p) Y_\omega = \frac{1}{2} Q^{\nu\mu} B_{\gamma\mu} (p) k^{\omega}_{\gamma}$$

$$= -\frac{1}{4g} \sum_{s=0}^{n-1} \tilde{K}_s^{(\tilde{s})} k^{\nu\mu}_{(p)} k^{\omega}_{\gamma} \left( K_{\gamma\mu\nu} + K^{110}_{\gamma\mu\nu} + K^{011}_{\nu\gamma\mu} + K^{101}_{\mu\nu\gamma} \right)$$

$$= \text{The right-hand side of (3.6)}.$$

Now, we are ready to prove a general representation of the $MEX$-vector in the following theorem.
THEOREM 3.4. Under the present condition, the ME-vector \(X_\omega\) in \(MEX_n\) may be given by

\[
(\sigma - 1 - \sigma N)\hat{K}_{n-\sigma}X_\omega
= \sum_{s=0}^{n-\sigma-2} \hat{K}_s \left( (n-s-1)Y_\omega + N^{(n-s-2)}Y_\omega \right) + \sigma \hat{K}_{n-\sigma}Y_\omega.
\] (3.9)

Proof. Substituting (3.5) into (3.2b) with \(A_\omega\) replaced by \(X_\omega\) and using (3.3b) and (3.4), we have

\[
\hat{K}_0 \left( (n-1)Y_\omega + N^{(n-2)}Y_\omega \right) + (K_2 + N^2)^{(n-2)}X_\omega + K_4^{(n-4)}X_\omega
+ \cdots + K_{n-\sigma-2}^{(\sigma+2)}X_\omega + K_{n-\sigma}^{(\sigma)}X_\omega = 0.
\] (3.10a)

Substituting again for \((n-2)X_\omega\) into (3.10a) from (3.5), we have

\[
\hat{K}_0 \left( (n-1)Y_\omega + N^{(n-2)}Y_\omega \right)
+ \hat{K}_2 \left( (n-3)Y_\omega + N^{(n-4)}Y_\omega \right)
+ (K_4 + N^2)^{(n-4)}X_\omega + K_6^{(n-6)}X_\omega + \cdots + K_{n-\sigma-2}^{(\sigma+2)}X_\omega
+ K_{n-\sigma}^{(\sigma)}X_\omega = 0.
\] (3.10b)

After \(\frac{n-\sigma}{2}\) steps of successive repeated substitutions for \((p)X_\omega\), we have

\[
\hat{K}_0 \left( (n-1)Y_\omega + N^{(n-2)}Y_\omega \right)
+ \hat{K}_2 \left( (n-3)Y_\omega + N^{(n-4)}Y_\omega \right)
+ \hat{K}_4 \left( (n-5)Y_\omega + N^{(n-6)}Y_\omega \right) + \cdots + \hat{K}_2 \left( (\sigma+1)Y_\omega + N^{(\sigma)}Y_\omega \right)
+ \hat{K}_{n-\sigma}^{(\sigma)}X_\omega = 0.
\] (3.10c)

On the other hand, it follows from (3.1a) and (3.8) that

\[
^{(\sigma)}X_\omega = \sigma Y_\omega + (\sigma N - \sigma + 1)X_\omega.
\] (3.11)

Substituting (3.11) into (3.10c), we finally have the relation (3.9).

As a consequence of Theorem 3.4, we have
THEOREM 3.5. There exists a unique $ME$-vector (and hence, a unique $ME$-connection) if, and only if the following condition holds for $g_{\lambda\mu}$:

$$\hat{K}_{n-\sigma} \neq 0.$$  

(3.12)

Proof. In virtue of (3.9), there exists a unique $X_\omega$ if $(\sigma-1-\sigma N)\hat{K}_{n-\sigma} \neq 0$. Hence the relation (3.12) immediately follows since $(\sigma-1-\sigma N) \neq 0$.

REMARK 3.6. (a) In virtue of Theorems 3.4 and 3.5, it is obvious that the present condition holds if, and only if the condition (3.12) holds for the tensor field $g_{\lambda\mu}$.

(b) The representation (3.9) of the $ME$-vector $X_\omega$ is the most general one which holds for a general $n \geq 2$ and for all possible classes.

Now, we present a quite different type of representation of the $ME$-vector from the general one, which holds in an even-dimensional $ME$-manifold with a certain special condition imposed on $g_{\lambda\mu}$. We need a tensor $F_{\lambda\mu}$ defined by

$$F_{\lambda\mu} = k_{\lambda\mu} - 2(2) k_{\lambda\mu}.$$  

(3.13)

THEOREM 3.7. The tensor $F_{\lambda\mu}$ is of rank $n$ if, and only if the tensor field $g_{\lambda\mu}$ satisfied the following condition:

$$\ell \sum_{s=0}^{n-\sigma} 2^s K_s \neq 0.$$  

(3.14)

Proof. The tensor $F_{\lambda\mu}$ may be written as

$$F_{\lambda\mu} = 2k_{\lambda\alpha} \left( \frac{1}{2} h_{\mu\beta} + k_{\mu\beta} \right) h^{\alpha\beta}.$$  

(3.15)

Our assertion follows from the following relation which may be obtained from (3.15) and (1.14):

$$\text{Det}(F_{\lambda\mu}) = 2^n \ell \left( \ell \sum_{s=0}^{n-\sigma} K_s \left( \frac{1}{2} \right)^{n-s} \right) \frac{1}{\ell}$$

$$= \ell \sum_{s=0}^{n-\sigma} 2^s K_s.$$
Agreement 3.8. In our further considerations, in this section we restrict ourselves to an even-dimensional $ME$-manifold $MEX_n$ and use the word "special condition" to describe the situations that the tensor field $g_{\lambda\mu}$ of an even-dimensional $MEX_n$ satisfies the condition

$$\sum_{s=0}^{n-\sigma} 2^s K_s \neq 0.$$  \hspace{1cm} (3.16)

Therefore, under the special condition the tensor $F_{\lambda\mu}$ is of rank $n$, so that there exists a unique inverse tensor $G^{\nu\lambda}$ defined by

$$G^{\lambda\nu} F_{\lambda\mu} = G^{\nu\lambda} F_{\mu\lambda} = \delta_{\mu}^{\nu}.$$ \hspace{1cm} (3.17)

Theorem 3.9. Under the special condition in an even-dimensional $MEX_n$, the $ME$-vector $X_\nu$ may be given by the following special type of representation:

$$X_\nu = -\frac{1}{2} G^{\nu\alpha} \partial_\alpha \log g.$$ \hspace{1cm} (3.18)

Proof. Multiplying $*g^{\lambda\mu}$, defined by

$$*g^{\lambda\nu} g_{\lambda\mu} = *g^{\nu\lambda} g_{\mu\lambda} = \delta_{\mu}^{\nu},$$ \hspace{1cm} (3.19)
to both sides of (1.5b), we have

$$\partial_\omega \log g - 2\Gamma^\alpha_{\alpha\omega} = *g^{\lambda\mu} D_\omega g_{\lambda\mu} = 2S_{\omega\alpha}^\alpha.$$ \hspace{1cm} (3.20)

On the other hand, multiply $h^{\lambda\mu}$ to both sides of the symmetric part of (1.5b) and make use of (2.1) to obtain

$$\partial_\omega \log h - 2\Gamma^\alpha_{\alpha\omega} = h^{\lambda\mu} D_\omega h_{\lambda\mu} = 2h^{\lambda\mu} S_{\omega(\mu}^\alpha g_{\lambda)\alpha}$$ \hspace{1cm} (3.21)

$$= 2S_{\omega\alpha}^\alpha - 2S_{\omega\beta}^\alpha k_\beta^\alpha$$

$$= 2S_{\omega\alpha}^\alpha - 4(\delta_{[\omega} X_{\beta]} - k_{\omega\beta} X^\alpha) k_\beta^\alpha$$

$$= 2S_{\omega\alpha}^\alpha - 2(k_{\omega}^\alpha + 2(2)k_{\omega}^\alpha) X_\alpha.$$  \hspace{1cm} (3.21)

Subtraction of (3.21) from (3.20) gives in virtue of (1.8)

$$\partial_\omega \log g = 2(k_{\omega}^\alpha + 2(2)k_{\omega}^\alpha) X_\alpha = -2F_{\nu\omega} X^\nu.$$ \hspace{1cm} (3.22)

The relation (3.18) immediately follows by multiplying $G^{\lambda\omega}$ to both sides of (3.22) and making use of (3.17).
REMARK 3.10. Incidentally, we note that the present condition (3.12) and the special condition (3.16) are coincident when \( n = 2 \). That is

\[
1 + 4K \neq 0. \tag{3.23}
\]

REMARK 3.11. In virtue of Theorem 3.9, our investigation of the ME-vector under the special condition is reduced to the study of the tensor \( G^{\lambda \mu} \). In order to know the ME-vector it is necessary and sufficient to know an explicit representation of \( G^{\lambda \mu} \) in terms of \( g_{\lambda \mu} \).

In our further considerations, we need the abbreviation \( (p)X^{\lambda \mu} \) for an arbitrary tensor \( X^{\lambda \mu} \) and notations \( K_s \) defined by

\[
(0)X^{\lambda \mu} = X^{\lambda \mu}, \quad (p)X^{\lambda \mu} = (p)k^\lambda_{\alpha}X^{\alpha \nu}, \quad (p = 1, 2, 3, \ldots) \tag{3.24}
\]

\[
\frac{1}{4} \sum_{t=0}^{s} \frac{1}{2^t} K_{s-t} \tag{3.25}
\]

The followings are immediate consequences of (3.22) and (3.23).

\[
(q)k^\lambda_{\alpha}(p)X^{\alpha \nu} = (p+q)X^{\lambda \nu}, \tag{3.26}
\]

\[
\frac{t}{4}, \quad \frac{1}{4}(K_2 + \frac{1}{4}), \quad \frac{1}{4}(K_4 + \frac{1}{4}K_2 + \frac{1}{16}), \quad \ldots, \tag{3.27a}
\]

\[
\frac{t}{4}(K_s + \frac{t}{4}s) \tag{3.27b}
\]

THEOREM 3.12. (Recurrence relation) In an even-dimensional \( MEX_n \), the tensor \( (p)G^{\lambda \nu} \) satisfies the following recurrence relation:

\[
\sum_{s=0}^{n} K_s (n-s)G^{\lambda \nu} = 0, \tag{3.28a}
\]

or equivalently

\[
(n)G^{\lambda \nu} + K_2 (n-2)G^{\lambda \nu} + \cdots + K_{n-2} (2)G^{\lambda \nu} + K_n G^{\lambda \nu} = 0. \tag{3.28b}
\]

Proof. Multiplying \( G^{\lambda \mu} \) to both sides of (1.15) and making use of (3.24), we have (3.28). Note that \( n - s \) is even, so that \( (n-s)k^\lambda_{\mu} \) is symmetric.
On the $MEX_n$ connection in $MEX_n$

**Theorem 3.13.** Under the special condition, the following relations hold in $MEX_n$:

$$(p+2)G^{\lambda\nu} + \frac{1}{2}(p+1)G^{\lambda\nu} + \frac{1}{2}(p)k^{\lambda\nu} = 0, \quad (p = 0, 1, 2, \cdots), \quad (3.29)$$

$$(q)G^{\lambda\nu} = \frac{1}{4}(q-2)G^{\lambda\nu} - \frac{1}{2}(q-2)k^{\lambda\nu} + \frac{1}{4}(q-3)k^{\lambda\nu}, \quad (q = 3, 4, 5, \cdots). \quad (3.30)$$

**Proof.** Substitution of (3.13) into (3.17) gives

$$2^{(2)}G^{\lambda\mu} + {^{(1)}}G^{\lambda\mu} + h^{\lambda\mu} = 0. \quad (3.31)$$

The relation (3.29) may be obtained by multiplying $\frac{1}{2}(p)k^{\nu\lambda}$ to both sides of (3.31). Using (3.29) twice, we have (3.30).

Now, we are ready to prove the following Theorem, which present a representation of the $G^{\lambda\mu}$ under the special condition.

**Theorem 3.14.** Under the special condition in an even-dimensional $MEX_n$, the tensor $G^{\lambda\nu}$ may be given by

$$2k\hat{K}_nG^{\lambda\nu} = \hat{K}_{n-2}(k^{\lambda\nu} + 2\hat{K}_{n-2}k^{\lambda\nu}) - 2\hat{K}_n\Lambda^{\lambda\nu} - \hat{K}_{n-2}{^{(1)}}\Lambda_{\lambda\nu}, \quad (3.32)$$

where

$$\Lambda^{\lambda\nu} = \sum_{s=0}^{n-4} \hat{K}_s \left(-2^{(n-s-2)}k^{\lambda\nu} + (n-3-s)k^{\lambda\nu}\right). \quad (3.33)$$

**Proof.** Substituting (3.30) into (3.28b) for $^{(n)}G^{\lambda\nu}$ and making use of (3.27), we have

$$\hat{K}_0(-2^{(n-2)}k^{\lambda\nu} + (n-3)k^{\lambda\nu}) + 4\hat{K}_4^{(n-4)}G^{\lambda\nu}$$

$$+ \cdots + \hat{K}_{n-2}{^{(2)}}G^{\lambda\nu} + \hat{K}_nG^{\lambda\nu} = 0. \quad (3.34a)$$

265
Substituting again for \((n-2)G^\lambda_\nu\) into (3.34a) from (3.30) gives

\[
\frac{1}{n-2}(n-2)K^{k\lambda_\nu} + (n-3)K^{k\lambda_\nu} + \frac{1}{n-2}(n-4)K^{k\lambda_\nu} + (n-5)K^{k\lambda_\nu} + 4K^{(n-4)}_4 G^\lambda_\nu + \ldots + K^{(2)}_{n-2} G^\lambda_\nu + K_n G^\lambda_\nu = 0.
\] (3.34b)

After \(\frac{n-2}{2}\) steps of successive repeated substitution for \((q)G^\lambda_\nu\), we have in virtue of (3.33)

\[
\Lambda^\lambda_\nu + 4K^{(2)}_{n-4} G^\lambda_\nu + K_n G^\lambda_\nu = 0.
\] (3.34c)

Comparison of (3.34) with \(A^{(2)} G^\lambda_\nu + B G^\lambda_\nu + \Lambda^\lambda_\nu = 0\) gives

\[
A = 4K^{(2)}_{n-2}, \quad B = K_n = k.
\] (3.35)

Consequently, the relation (3.32) follows by substituting (3.35) into \(B(A + 4B)G^\lambda_\nu = 2ABh^\lambda_\nu + A^2 k^\lambda_\nu - (A + 4B)\Lambda^\lambda_\nu - 2A^{(1)} \Lambda^\lambda_\nu\) and making use of (3.37b).

Now that we have obtained a representation of \(G^\lambda_\nu\) in Theorem 3.14, under the special condition it is possible for us to represent the \(ME\)-vector \(X^\nu\) in terms of \(g_{\lambda\mu}\) by only substituting (3.32) into (3.18).

**Theorem 3.15.** Under the special condition in an even-dimensional \(MEX_n\), the \(ME\)-vector \(X^\nu\) may be given by

\[
4kK_n X^\nu
\] (3.36)

\[= -\left(\frac{1}{n-2}(kh_{\nu\alpha} + 2K^{(2)}_{n-2} k^{\nu\alpha}) - 2K^\lambda_\nu \Lambda^\nu_\alpha - K^{(1)}_{n-2} \Lambda^\nu_\alpha\right) \partial_\alpha(\log g).
\]

**References**

On the $ME$-connection in $MEX_n$

4. K. T. Chung and T. S. Han, *n*-dimensional representations of the unified field tensor $g^{\lambda \nu}$, International J. of Theoretical Physics 20 (1981), 739-747.


Department of Mathematics, Mokpo National University, MuAn 534-729, Korea