ON NEW CLASSES OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS. II

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1. Introduction

Let $A$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$  \hspace{1cm} (1.1)

which are analytic in the unit disc $U = \{z : |z| < 1\}$. The Hadamard product of two functions $f(z) \in A$ and $g(z) \in A$ will be denoted by $f \ast g(z)$, that is, if $f(z)$ is given by (1.1) and $g(z)$ is given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k,$$  \hspace{1cm} (1.2)

then

$$f \ast g(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k.$$  \hspace{1cm} (1.3)

Ruscheweyh [12] introduced the classes $K_n$ of functions $f(z) \in A$ satisfying

$$\text{Re} \left\{ \frac{(z^n f(z))^{(n+1)}}{(z^{n-1} f(z))^{(n)}} \right\} > \frac{n + 1}{2}$$  \hspace{1cm} (1.4)

for $n \in N_0 = NU\{0\}$ and $z \in U$, where $N = \{1, 2, \cdots \}$. Ruscheweyh [12] showed the basic property

$$k_{n+1} \subset k_n, \quad n \in N_0.$$  \hspace{1cm} (1.5)

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Note that $K_0$ is the class $S^*\left(\frac{1}{2}\right)$ of starlike functions of order $\frac{1}{2}$.

Let

$$D^n f(z) = \frac{z^n f(z)(n)}{n!}, \quad n \in \mathbb{N}_0.$$  \hspace{1cm} (1.6)

This symbol $D^n f(z)$ was named the $n$-th order Ruscheweyh derivative of $f(z)$ by Al-Amiri [1]. We note that $D^0 f(z) = f(z)$ and $D^1 f(z) = zf'(z)$. Using Hadamard product, Ruscheweyh [12] observed that if

$$D^\alpha f(z) = \frac{z}{(1-z)^{\alpha+1}} \ast f(z) \quad (\alpha \geq -1)$$  \hspace{1cm} (1.7)

then (1.6) is equivalent to (1.7) when $\alpha = n \in \mathbb{N}_0$. Thus it follows from (1.4) that the necessary and sufficient condition for $f(z) \in A$ to belong to $K_n$ is

$$\text{Re} \left\{ \frac{D^{n+1} f(z)}{D^n f(z)} \right\} > \frac{1}{2}, \quad (z \in U).$$  \hspace{1cm} (1.8)

Note that $K_{-1}$ is the class of functions $f(z) \in A$ satisfying

$$\text{Re} \left\{ \frac{f(z)}{z} \right\} > \frac{1}{2}, \quad (z \in U).$$  \hspace{1cm} (1.9)

It is easy to see that

$$D^n f(z) = z + \sum_{k=2}^{\infty} \delta(n,k) a_k z^k,$$  \hspace{1cm} (1.10)

where

$$\delta(n,k) = \binom{n+k-1}{n}.$$  \hspace{1cm} (1.11)

Let $T$ denote the subclass of $A$ consisting of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \quad (a_k \geq 0).$$  \hspace{1cm} (1.12)

In [10] Owa studied the classes $K^*_n (n \in \mathbb{N}_0)$ by using the $n$-th order Ruscheweyh derivative of $f(z)$, defined as follows:
DEFINITION 1. We say that the function \( f(z) \) defined by (1.12) is in the class \( K_n^* \) if \( f(z) \) satisfies the condition (1.8) for \( n \in N_0 \).

In order to show our results, we need the following lemmas given by Owa [10].

LEMMA 1. Let the function \( f(z) \) defined by (1.12). Then \( f(z) \in K_n^* \) if and only if
\[
\sum_{k=2}^{\infty} \left( \frac{2k + n - 1}{n + 1} \right) \delta(n, k)a_k \leq 1
\]  
(1.13)
for \( n \in N_0 \). The result is sharp for the function
\[
f(z) = z - \frac{(n + 1)}{(2k + n - 1)\delta(n, k)} z^k \quad (k \geq 2).
\]  
(1.14)

LEMMA 2. The extreme points of the class \( K_n^* \) are \( f_1(z) = z \) and \( f_k(z) = z - \frac{(n+1)}{(2k+n-1)\delta(n,k)} z^k \quad (k \geq 2) \).

2. Some properties of the class \( K_n^* \)

THEOREM 1. \( K_{n+1}^* \subset K_n^* \) for each \( n \in N_0 \).

Proof. Let the function \( f(z) \) defined by (1.12) be in the class \( K_{n+1}^* \); then
\[
\sum_{k=2}^{\infty} \left( \frac{2k + n}{n + 2} \right) \delta(n + 1, k)a_k \leq 1
\]  
(2.1)
and since
\[
\delta(n, k) \leq \delta(n + 1, k) \quad \text{for} \quad k \geq 2,
\]  
(2.2)
we have
\[
\sum_{k=2}^{\infty} \left( \frac{2k + n - 1}{n + 1} \right) \delta(n, k)a_k \leq \sum_{k=2}^{\infty} \left( \frac{2k + n}{n + 2} \right) \delta(n + 1, k)a_k \leq 1
\]  
(2.3)

The result follows from Lemma 1.
Theorem 2. The class $K_n^*$ is closed under convex linear combinations.

Proof. Let the functions

$$f_i(z) = z - \sum_{k=2}^{\infty} a_{k,i} z^k \quad (a_{k,i} \geq 0, \ i = 1, 2) \quad (2.4)$$

be in the class $K_n^*$. It is sufficient to show that the function $h(z)$ defined by

$$h(z) = \lambda f_1(z) + (1 - \lambda) f_2(z) \quad (0 \leq \lambda \leq 1) \quad (2.5)$$

is in the class $K_n^*$. Since

$$h(z) = z - \sum_{k=2}^{\infty} \{\lambda a_{k,1} + (1 - \lambda) a_{k,2}\} z^k, \quad (2.6)$$

with the aid of Lemma 1, we have

$$\sum_{k=2}^{\infty} \left(\frac{2k + n - 1}{n + 1}\right) \delta(n,k) \{\lambda a_{k,1} + (1 - \lambda) a_{k,2}\}$$

$$= \lambda \sum_{k=2}^{\infty} \left(\frac{2k + n - 1}{n + 1}\right) \delta(n,k) a_{k,1}$$

$$+ (1 - \lambda) \sum_{k=2}^{\infty} \left(\frac{2k + n - 1}{n + 1}\right) \delta(n,k) a_{k,2} \leq 1$$

which implies that $h(z) \in K_n^*$.

3. Integral operators

Theorem 3. Let the function $f(z)$ defined by (1.12) be in the class $K_n^*$, and let $c$ be a real number such that $c > -1$. Then the function $F(z)$ defined by

$$F(z) = \frac{c + 1}{z^c} \int_0^z t^{c-1} f(t) dt \quad (3.1)$$

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also belongs to the class $K_n^*$.

**Proof.** From the representation of $F(z)$, it follows that

$$F(z) = z - \sum_{k=2}^{\infty} b_k z^k,$$  \hspace{1cm} (3.2)

where

$$b_k = \left( \frac{c+1}{c+k} \right) a_k.$$ \hspace{1cm} (3.3)

Therefore,

$$\sum_{k=2}^{\infty} \left( \frac{2k+n-1}{n+1} \right) \delta(n,k) b_k = \sum_{k=2}^{\infty} \left( \frac{2k+n-1}{n+1} \right) \delta(n,k) \left( \frac{c+1}{c+k} \right) a_k$$

$$\leq \sum_{k=2}^{\infty} \left( \frac{2k+n-1}{n+1} \right) \delta(n,k) a_k \leq 1,$$ \hspace{1cm} (3.4)

since $f(z) \in K_n^*$. Hence, by Lemma 1, $F(z) \in K_n^*$.

**Theorem 4.** Let the function $F(z)$ defined by (1.12) be in the class $K_n^*$, and let $c$ be a real number such that $c > -1$. Then the function $f(z)$ defined by (3.1) is univalent in $|z| < r^*$, where

$$r^* = \inf_k \left[ \frac{(c+1)(2k+n-1)\delta(n,k)}{k(c+k)(n+1)} \right]^{\frac{1}{k-1}} \hspace{1cm} (k \geq 2).$$ \hspace{1cm} (3.5)

The result is sharp.

**Proof.** From (3.1), we have

$$f(z) = z^{1-c} \frac{z^c F'(z)}{(c+1)} \hspace{1cm} (c > -1)$$ \hspace{1cm} (3.6)

$$= z - \sum_{k=2}^{\infty} \left( \frac{c+k}{c+1} \right) a_k z^k.$$ \hspace{1cm} (3.7)
In order to obtain the required result it suffices to show that

\[ |f'(z) - 1| < 1 \text{ in } |z| < r^*. \]

Now

\[ |f'(z) - 1| \leq \sum_{k=2}^{\infty} \frac{k(c+k)}{(c+1)} a_k |z|^{k-1}. \]

Thus \( |f'(z) - 1| < 1 \) if

\[ \sum_{k=2}^{\infty} \frac{k(c+k)}{(c+1)} a_k |z|^{k-1} < 1. \quad (3.8) \]

But Lemma 1 confirms that

\[ \sum_{k=2}^{\infty} \left( \frac{2k+n-1}{n+1} \right) \delta(n,k) a_k \leq 1. \quad (3.9) \]

Hence (3.8) will be satisfied if

\[ \frac{k(c+k)|z|^{k-1}}{(c+1)} < \frac{(2k+n-1)\delta(n,k)}{(n+1)} \]

or if

\[ |z| < \left[ \frac{(c+1)(2k+n-1)\delta(n,k)}{k(c+k)(n+1)} \right]^{\frac{1}{k-1}} (k \geq 2). \quad (3.10) \]

Therefore \( f(z) \) is univalent in \( |z| < r^* \). Sharpness follows if we take

\[ f(z) = z - \frac{(n+1)(c+k)}{(2k+n-1)\delta(n,k)(c+1)} z^k \quad (k \geq 2). \quad (3.11) \]
4. Radii of close-to-convexity, starlikeness and convexity

**Theorem 5.** Let the function \( f(z) \) defined by (1.12) be in the class \( K^*_n \), then \( f(z) \) is close-to-convex of order \( \rho (0 \leq \rho < 1) \) in \( |z| < r_1(n, \rho) \), where

\[
r_1(n, \rho) = \inf_k \left[ \frac{(1-\rho)(2k+n-1)\delta(n,k)}{k(n+1)} \right]^{\frac{1}{k-1}} (k \geq 2). \tag{4.1}
\]

The result is sharp, with the extremal function \( f(z) \) given by (1.14).

**Proof.** We must show that \( |f'(z) - 1| \leq 1 - \rho \) for \( |z| < r_1(n, \rho) \). We have

\[
|f'(z) - 1| \leq \sum_{k=2}^{\infty} ka_k |z|^{k-1}.
\]

Thus \( |f'(z) - 1| \leq 1 - \rho \) if

\[
\sum_{k=2}^{\infty} \left( \frac{k}{1-\rho} \right) a_k |z|^{k-1} \leq 1. \tag{4.2}
\]

Hence, by (3.9), (4.2) will be true if

\[
\frac{k |z|^{k-1}}{(1-\rho)} \leq \frac{(2k+n-1)\delta(n,k)}{(n+1)}
\]

or if

\[
|z| \leq \left[ \frac{(1-\rho)(2k+n-1)\delta(n,k)}{k(n+1)} \right]^{\frac{1}{k-1}} (k \geq 2). \tag{4.3}
\]

The theorem follows easily from (4.3).

**Theorem 6.** Let the function \( f(z) \) defined by (1.12) be in the class \( K^*_n \), then \( f(z) \) is starlike of order \( \rho (0 \leq \rho < 1) \) in \( |z| < r_2(n, \rho) \), where

\[
r_2(n, \rho) = \inf_k \left[ \frac{(1-\rho)(2k+n-1)\delta(n,k)}{(k-\rho)(n+1)} \right]^{\frac{1}{k-1}} (k \geq 2). \tag{4.4}
\]
The result is sharp, with the extremal function \( f(z) \) given by (1.14).

**Proof.** It is sufficient to show that \( \left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \rho \) for \( |z| < r_2(n, \rho) \). We have

\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{\sum_{k=2}^{\infty} (k - 1) a_k |z|^{k-1}}{1 - \sum_{k=2}^{\infty} a_k |z|^{k-1}}.
\]

Thus \( \left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \rho \) if

\[
\sum_{k=2}^{\infty} \frac{(k - \rho)a_k |z|^{k-1}}{(1 - \rho)} \leq 1. \tag{4.5}
\]

Hence, by (3.9), (4.5) will be true if

\[
\frac{(k - \rho)|z|^{k-1}}{(1 - \rho)} \leq \frac{(2k + n - 1)\delta(n, k)}{(n + 1)}
\]

or if

\[
|z| \leq \left[ \frac{(1 - \rho)(2k + n - 1)\delta(n, k)}{(k - \rho)(n + 1)} \right]^{\frac{1}{k-1}} \quad (k \geq 2). \tag{4.6}
\]

The theorem follows easily from (4.6).

**Corollary 1.** Let the function \( f(z) \) defined by (1.12) be in the class \( K^*_n \), then \( f(z) \) is convex of order \( \rho (0 \leq \rho < 1) \) in \( |z| < r_3(n, \rho) \), where

\[
r_3(n, \rho) = \inf_k \left[ \frac{(1 - \rho)(2k + n - 1)\delta(n, k)}{k(k - \rho)(n + 1)} \right]^{\frac{1}{k-1}} \quad (k \geq 2). \tag{4.7}
\]

The result is sharp, with the extremal function \( f(z) \) given by (1.14).
5. Fractional calculus

Many essentially equivalent definitions of fractional calculus (that is, fractional derivatives and fractional integrals) have been given in the literature (c.f., e.g., [2], [5, Chapter 13], [6], [7], [8], [11], [13], [14], [15, p. 28 et seq.], [17], [18], [19] and [21]). We find it to be convenient to recall here the following definitions which were used recently by Owa [9] (and by Srivastava and Owa [16]).

**Definition 2.** The fractional integral of order \( \lambda \) is defined, for a function \( f(z) \), by

\[
D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\xi)}{(z-\xi)^{1-\lambda}} d\xi \quad (\lambda > 0),
\]  

(5.1)

where \( f(z) \) is an analytic function in a simply connected region of the \( z \)-plane containing the origin, and the multiplicity of \((z-\xi)^{\lambda-1}\) is removed by requiring \( \log(z-\xi) \) to be real when \( z-\xi > 0 \).

**Definition 3.** The fractional derivative of order \( \lambda \) is defined, for a function \( f(z) \), by

\[
D_z^\lambda f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^\lambda} d\xi \quad (0 \leq \lambda < 1),
\]

(5.2)

where \( f(z) \) is constrained, and the multiplicity of \((z-\xi)^{-\lambda}\) is removed, as in Definition 2.

**Definition 4.** Under the hypotheses of Definition 3, the fractional derivative of order \( n + \lambda \) is defined by

\[
D_z^{n+\lambda} f(z) = \frac{d^n}{dz^n} D_z^\lambda f(z) \quad (0 \leq \lambda < 1; n \in \mathbb{N}_0).
\]

(5.3)

**Theorem 7.** Let the function \( f(z) \) defined by (1.12) be in the class \( K_n^* \). Then we have

\[
|D_z^{-\lambda} f(z)| \geq \frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)} \left\{ 1 - \frac{2}{(n+3)(2+\lambda)} |z| \right\}
\]

(5.4)
and
\[ |D_z^{-\lambda} f(z)| \leq \frac{|z|^{1+\lambda}}{\Gamma(2 + \lambda)} \left\{ 1 + \frac{2}{(n + 3)(2 + \lambda)} |z| \right\} \]  \hspace{1cm} (5.5)

for \( \lambda > 0, \ n \in \mathbb{N}_0, \) and \( z \in \mathcal{U}. \) The result is sharp.

Proof. Let
\[ F(z) = \Gamma(2 + \lambda)z^{-\lambda}D_z^{-\lambda} f(z) \]  \hspace{1cm} (5.6)
\[ = z - \sum_{k=2}^{\infty} \frac{\Gamma(k + 1)\Gamma(2 + \lambda)}{\Gamma(k + 1 + \lambda)} a_k z^k = z - \sum_{k=2}^{\infty} \Psi(k)a_k z^k, \]

where
\[ \Psi(k) = \frac{\Gamma(k + 1)\Gamma(2 + \lambda)}{\Gamma(k + 1 + \lambda)} \quad (k \geq 2). \]  \hspace{1cm} (5.7)

Since
\[ 0 < \Psi(k) \leq \Psi(2) = \frac{2}{2 + \lambda}, \]  \hspace{1cm} (5.8)
by using Lemma 1, we have
\[ \sum_{k=2}^{\infty} a_k \leq \frac{1}{n + 3}. \]  \hspace{1cm} (5.9)

Therefore, by using (5.8) and (5.9), we can see that
\[ |F(z)| \geq |z| - \psi(2)|z|^2 \sum_{k=2}^{\infty} a_k \geq |z| - \frac{2}{(n + 3)(2 + \lambda)} |z|^2 \]  \hspace{1cm} (5.10)
and
\[ |F(z)| \leq |z| + \psi(2)|z|^2 \sum_{k=2}^{\infty} a_k \leq |z| + \frac{2}{(n + 3)(2 + \lambda)}|z|^2 \]  \hspace{1cm} (5.11)
which prove the inequalities of theorem 7. Further, equalities are attained for the function \( f(z) \) defined by
\[ D_z^{-\lambda} f(z) = \frac{z^{1+\lambda}}{\Gamma(2 + \lambda)} \left\{ 1 - \frac{2}{(n + 3)(2 + \lambda)} z \right\} \]  \hspace{1cm} (5.12)
or
\[ f(z) = z - \frac{1}{n + 3} z^2. \]  \hspace{1cm} (5.13)
COROLLARY 2. Under the hypotheses of Theorem 7, $D_z^{-\lambda} f(z)$ is included in the disc with center at the origin and radius $R_1$ given by

$$R_1 = \frac{1}{\Gamma(2 + \lambda)} \left\{ 1 + \frac{2}{(n + 3)(2 + \lambda)} \right\}.$$  

(5.14)

THEOREM 8. Let the function $f(z)$ defined by (1.12) be in the class $K_n^*$. Then we have

$$|D_z^\lambda f(z)| \geq \frac{|z|^{1-\lambda}}{\Gamma(2 - \lambda)} \left\{ 1 - \frac{2}{(n + 3)(2 - \lambda)} |z| \right\}$$  

(5.15)

and

$$|D_z^\lambda f(z)| \leq \frac{|z|^{1-\lambda}}{\Gamma(2 - \lambda)} \left\{ 1 + \frac{2}{(n + 3)(2 - \lambda)} |z| \right\}$$  

(5.16)

for $0 \leq \lambda < 1$, $n \in N_0$, and $z \in U$. The result is sharp.

Proof. Let

$$G(z) = \Gamma(2 - \lambda) z^\lambda D_z^\lambda f(z)$$  

(5.17)

$$= z - \sum_{k=2}^{\infty} \frac{\Gamma(k + 1) \Gamma(2 - \lambda)}{\Gamma(k + 1 - \lambda)} a_k z^k = z - \sum_{k=2}^{\infty} \phi(k) k a_k z^k,$$

where

$$\phi(k) = \frac{\Gamma(k) \Gamma(2 - \lambda)}{\Gamma(k + 1 - \lambda)} \quad (k \geq 2).$$  

(5.18)

Since

$$0 < \phi(k) \leq \phi(2) = \frac{1}{2 - \lambda},$$  

(5.19)

by using Lemma 1, we have

$$\sum_{k=2}^{\infty} k a_k \leq \frac{2}{n + 3}.$$  

(5.20)

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Therefore, by using (5.19) and (5.20), we can see that

$$|G(z)| \geq |z| - \phi(2)|z|^2 \sum_{k=2}^{\infty} k \ a_k \geq |z| - \frac{2}{(n+3)(2 - \lambda)}|z|^2$$  \hspace{1cm} (5.21)

and

$$|G(z)| \leq |z| + \phi(2)|z|^2 \sum_{k=2}^{\infty} k \ a_k \leq |z| + \frac{2}{(n+3)(2 - \lambda)}|z|^2$$  \hspace{1cm} (5.22)

which give the inequalities of Theorem 8. Since equalities are attained for the function \( f(z) \) defined by

$$D_z^\lambda f(z) = \frac{z^{1-\lambda}}{\Gamma(2 - \lambda)} \left\{ 1 - \frac{2}{(n+3)(2 - \lambda)}z \right\}$$  \hspace{1cm} (5.23)

that is, by (5.13), we complete the assertion of Theorem 8.

**Corollary 3.** Under the conditions of Theorem 8, \( D_z^\lambda f(z) \) is included in the disc with center at the origin and radius \( R_2 \) given by

$$R_2 = \frac{1}{\Gamma(2 - \lambda)} \left\{ 1 + \frac{2}{(n+3)(2 - \lambda)} \right\}.$$  \hspace{1cm} (5.24)

6. Fractional integral operator

We need the following definition of fractional integral operator given by Srivastava, Saigo and Owa [20].

**Definition 5.** For real numbers \( \beta > 0 \), \( \gamma \) and \( \eta \), the fractional integral operator \( I_{0,z}^{\beta,\gamma,\eta} \) is defined by

$$I_{0,z}^{\beta,\gamma,\eta} f(z) = \frac{z^{-\beta-\gamma}}{\Gamma(\beta)} \int_0^z (z-t)^{\beta-1} F(\beta+\gamma,-\eta;\beta;1-\frac{t}{z}) f(t) \, dt$$  \hspace{1cm} (6.1)

where \( f(z) \) is an analytic function in a simply connected region of the \( z \)-plane containing the origin with the order

$$f(z) = O(|z|^\varepsilon), \ z \to 0,$$
where

\[ \varepsilon > \text{Max} \left( 0, \gamma - \eta \right) - 1, \]

\[ F(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k(1)_k} z^k, \quad (6.2) \]

where \((\nu)_k\) is the Pochhammer symbol defined by

\[ (\nu)_k = \frac{\Gamma(\nu + k)}{\Gamma(\nu)} = \begin{cases} 1 & (k = 0) \\ \nu(\nu + 1) \cdots (\nu + k - 1) & (k \in \mathbb{N}), \end{cases} \quad (6.3) \]

and the multiplicity of \((z - t)^{\beta - 1}\) is removed by requiring \(\log(z - t)\) to be real when \(z - t > 0\).

**Remark.** For \(\gamma = -\beta\), we note that

\[ I_{0, z}^{\beta, -\beta, \eta} f(z) = D_z^{-\beta} f(z). \]

In order to prove our result for the fractional integral operator, we have to recall here the following lemma due to Srivastava, Saigo and Owa [20].

**Lemma 3.** If \(\beta > 0\) and \(k > \gamma - \eta - 1\), then

\[ I_{0, z}^{\beta, \gamma, \eta} z^k = \frac{\Gamma(k + 1)\Gamma(k - \gamma + \eta + 1)}{\Gamma(k - \gamma + 1)\Gamma(k + \beta + \eta + 1)} z^{k - \gamma}. \quad (6.4) \]

With the aid of Lemma 3, we prove

**Theorem 9.** Let \(\beta > 0, \gamma < 2, \beta + \eta > -2, \gamma - \eta < 2, \gamma(\beta + \eta) \leq 3\beta\). If the function \(f(z)\) defined by (1.12) is in the class \(K_n^*\), then

\[ \left| J_{0, z}^{\beta, \gamma, \eta} f(z) \right| \geq \frac{\Gamma(2 - \gamma + \eta)|z|^{1-\gamma}}{\Gamma(2 - \gamma)\Gamma(2 + \beta + \eta)} \left\{ 1 - \frac{2(2 - \gamma + \eta)}{(n + 3)(2 - \gamma)(2 + \beta + \eta)}|z| \right\}. \quad (6.5) \]

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and
\[
\left| I_{0, z}^{\beta, \gamma, \eta} f(z) \right| \leq \frac{\Gamma(2 - \gamma + \eta)|z|^{1-\gamma}}{\Gamma(2-\gamma)\Gamma(2+\beta+\eta)} \left\{ 1 + \frac{2(2 - \gamma + \eta)}{(n+3)(2-\gamma)(2+\beta+\eta)}|z| \right\}
\]

for \( z \in U_0 \), where
\[
U_0 = \begin{cases} 
U & (\gamma \leq 1) \\
U - \{0\} & (\gamma > 1).
\end{cases}
\]

The equalities in (6.5) and (6.6) are attained by the function \( f(z) \) given by (5.13).

**Proof.** By using Lemma 3, we have
\[
I_{0, z}^{\beta, \gamma, \eta} f(z) = \frac{\Gamma(2 - \gamma + \eta)}{\Gamma(2-\gamma)\Gamma(2+\beta+\eta)} z^{1-\gamma} - \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(k-\gamma+\eta+1)}{\Gamma(k-\gamma+1)\Gamma(k+\beta+\eta+1)} a_k z^{k-\gamma}.
\]

Letting
\[
H(z) = \frac{\Gamma(2 - \gamma)\Gamma(2 + \beta + \eta)}{\Gamma(2-\gamma)} z^{\gamma} I_{0, z}^{\beta, \gamma, \eta} f(z)
\]
\[
= z - \sum_{k=2}^{\infty} h(k) a_k z^k,
\]

where
\[
h(k) = \frac{(2 - \gamma + \eta)_{k-1}(1)_k}{(2-\gamma)_{k-1}(2+\beta+\eta)_{k-1}} \quad (k \geq 2),
\]

we can see that \( h(k) \) is non-increasing for integers \( k \geq 2 \), and we have
\[
0 < h(k) \leq h(2) = \frac{2(2 - \gamma + \eta)}{(2-\gamma)(2+\beta+\eta)}.
\]
Therefore, by using (5.9) and (6.10), we have

\[ |H(z)| \geq |z| - h(2)|z|^2 \sum_{k=2}^{\infty} a_k \]

\[ \geq |z| - \frac{2(2 - \gamma + \eta)}{(n + 3)(2 - \gamma)(2 + \beta + \eta)} |z|^2 \]  

(6.11)

and

\[ |H(z)| \leq |z| + h(2)|z|^2 \sum_{k=2}^{\infty} a_k \]

\[ \leq |z| + \frac{2(2 - \gamma + \eta)}{(n + 3)(2 - \gamma)(2 + \beta + \eta)} |z|^2. \]

(6.12)

This completes the proof of Theorem 9.

**Remark.** Taking \( \beta = -\gamma = \lambda \) in Theorem 9, we get the result of Theorem 7.

7. **Modified Hadamard Product**

Let the functions \( f_i(z) \) \( (i = 1, 2) \) be defined by (2.4). The modified Hadamard product of \( f_1(z) \) and \( f_2(z) \) is defined by

\[ f_1 \ast f_2(z) = z - \sum_{k=2}^{\infty} a_{k,1} a_{k,2} z^k. \]  

(7.1)

**Theorem 10.** Let \( f_1(z) \in K_{n_1}^* \) and \( f_2(z) \in K_{n_2}^* \). Then the modified Hadamard product \( f_1 \ast f_2(z) \in K_{n_1}^* \cap K_{n_2}^* \).

**Proof.** Since \( f_2(z) \in K_{n_2}^* \), we have from (5.9) that

\[ a_{k,2} \leq \frac{1}{n_2 + 3}. \]  

(7.2)

From Lemma 1, since \( f_1(z) \in K_{n_1}^* \), we have

\[ \sum_{k=2}^{\infty} \left( \frac{2k + n_1 - 1}{n_1 + 1} \right) \delta(n_1, k) a_{k,1} \leq 1. \]  

(7.3)
Now, from (7.2) and (7.3),
\[ \sum_{k=2}^{\infty} \left( \frac{2k + n_1 - 1}{n_1 + 1} \right) \delta(n_1, k) a_{k,1} a_{k,2} \leq \frac{1}{n_2 + 3} \sum_{k=2}^{\infty} \left( \frac{2k + n_1 - 1}{n_1 + 1} \right) \delta(n_1, k) a_{k,1} \leq \frac{1}{n_2 + 3} \leq 1. \]

Hence \( f_1 \ast f_2(z) \in K_{n_1}^* \). Interchanging \( n_1 \) and \( n_2 \) by each other in the above, we get \( f_1 \ast f_2(z) \in K_{n_2}^* \). Hence the theorem.

8. Linear combination of functions in the class \( K_n^* \)

**Theorem 11.** Let each of the functions \( f_1(z), f_2(z), \ldots, f_m(z) \) defined by
\[ f_i(z) = z - \sum_{k=2}^{\infty} a_{k,i} z^k (a_{k,i} \geq 0; \ i = 1, 2, \ldots, m) \quad (8.1) \]
be in the same class \( K_n^* \). Then the function \( h(z) \) given by
\[ h(z) = \frac{1}{m} \sum_{i=1}^{m} f_i(z) \quad (8.2) \]
is also in the class \( K_n^* \).

**Proof.** By the definition (8.2) of \( h(z) \), we have the expansion
\[ h(z) = z - \sum_{k=2}^{\infty} \left[ \frac{1}{m} \sum_{i=1}^{m} a_{k,i} \right] z^k. \quad (8.3) \]
Since \( f_i(z) \in K_n^* \) for every \( i = 1, 2, \ldots, m \), by using Lemma 1, we obtain
\[ \sum_{k=2}^{\infty} \left( \frac{2k + n - 1}{n + 1} \right) \delta(n, k) \left[ \frac{1}{m} \sum_{i=1}^{m} a_{k,i} \right] \leq 1, \quad (8.4) \]
which, in view of Lemma 1, yields Theorem 11.
9. Support points

A function \( f(z) \) in \( K_n^* \) is said to be a support point of \( K_n^* \) if there exists a continuous linear functional \( J \) on \( T \) such that \( \text{Re} \ (J(f)) \geq \text{Re} \ (J(g)) \) for all \( g(z) \in K_n^* \), and \( \text{Re} \ (J) \) is non-constant on \( K_n^* \). We denote by \( \text{Supp} \ K_n^* \) the set of support points of \( K_n^* \), and by \( \text{Ext} \ K_n^* \) the set of extreme points of \( K_n^* \).

Let \( F \) be a subfamily of univalent functions in the unit disc \( U \) whose set of extreme points is countable, suppose \( f_0(z) \) is a support point of \( F \), and let \( J \) be a corresponding continuous linear functional. Let

\[
G_J = \{ f \in F : \text{Re} \ (J(f)) = \text{Re} \ (J(f_0)) \}.
\] (9.1)

Then Deeb [4] showed the following lemma.

**Lemma 4.** Let \( G_J \) be defined by (9.1). Then \( G_J \) is convex, \( \text{Ext} \ G_J \subset \text{Ext} \ F \), and

\[
G_J = \left\{ f \in F : f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z), \ \lambda_k \geq 0, \sum_{k=1}^{\infty} \lambda_k = 1, f_k(z) \in \text{Ext} \ G_J \right\}.
\] (9.2)

Let \( A_1 \) be the class of functions of the form

\[ f(z) = \sum_{k=0}^{\infty} a_k z^k \]

which are analytic in the unit disc \( U \). Then, Brickman, MacGregor and Wilken [3] have proved the following result.

**Lemma 5.** Let \( \{b_k\} \) be a sequence of complex numbers such that

\[
\lim_{k \to \infty} \sup |b_k|^{\frac{1}{k}} < 1,
\]

and set \( J(f) = \sum_{k=0}^{\infty} a_k b_k \) for \( f(z) \in A_1 \). Then \( J \) is a continuous linear functional on \( A_1 \).

Conversely, any continuous linear functional on \( A_1 \) is given by such a sequence \( \{b_k\} \).

Now, we prove
Theorem 12. The set \( \text{Supp} \, K_n^* \) of support points of \( K_n^* \) is given by

\[
\text{Supp} \, K_n^* = \left\{ f \in K_n^* : f(z) = z - \sum_{k=2}^{\infty} \frac{(n+1)\lambda_k}{(2k+n-1)\delta(n,k)} z^k, \quad \lambda_k \geq 0, \sum_{k=2}^{\infty} \lambda_k \leq 1, \lambda_j = 0 \text{ for some } j \right\}.
\]

Proof. Let

\[
f_0(z) = z - \sum_{k=2}^{\infty} \frac{(n+1)\lambda_k}{(2k+n-1)\delta(n,k)} z^k \tag{9.3}
\]

be in the class \( K_n^* \), where \( \lambda_k \geq 0, \sum_{k=2}^{\infty} \lambda_k \leq 1 \), and \( \lambda_j = 0 \) for some \( j \geq 2 \). If \( b_k = 0 \) (\( k \geq 2, k \neq j \)) and \( b_1 = b_j = 1 \), then

\[
\lim_{k \to \infty} \sup \left| \frac{b_k}{k} \right| < 1.
\]

Therefore, by using Lemma 5, we define the continuous linear functional \( J \) given by \( \{b_k\} \). It follows that \( J(f_0) = 1 \) and \( J(f) = 1 - a_j \leq 1 \) for \( f(z) \in K_n^* \). This implies that \( \text{Re} \, (J(f_0)) \geq \text{Re} \, (J(f)) \) for all \( f(z) \in K_n^* \). Hence, \( f_0(z) \) is a support point of the class \( K_n^* \).

Conversely, suppose that \( f_0(z) \) is a support point of \( K_n^* \), and that its continuous linear functional \( J \) is given by \( \{b_k\} \). Then \( \text{Re} \, (J) \) is also continuous and linear on \( K_n^* \). Therefore, by the Krein - Milman theorem, there exists an extreme point \( f_k(z) \) of \( K_n^* \) such that

\[
\text{Re} \, (J(f_0)) = \text{Max} \left\{ \text{Re} \, (J(f)) : f \in K_n^* \right\} = \text{Re} \, (J(f_k)). \tag{9.4}
\]

Let

\[
G_J = \{ f_k : \text{Re} \, (J(f_0)) = \text{Re} \, (J(f_k)), f_k \in \text{Ext} \, K_n^* \}.
\]

Then, we note that \( \text{Ext} \, K_n^* \) is countable by means of Lemma 2. If \( G_J = \text{Ext} \, K_n^* \), then \( \text{Re} \, (J) \) must be constant on \( K_n^* \). This contradicts that \( f_0(z) \) is a support point of \( K_n^* \). Consequently, there exists a \( j \) such that \( \text{Re} \, (J(f_0)) > \text{Re} \, (J(f_j)) \). It follows from this fact that

\[
\text{Ext} \, G_J \subset \{ f_k : f_k \in \text{Ext} \, K_n^*, k \geq 2, k \neq j \}. \tag{9.5}
\]

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Consequently, with the help of Lemma 4, we have

\[ f_0(z) = \sum_{k=2}^{\infty} \lambda_k f_k(z), \quad (9.6) \]

where \( \lambda_k \geq 0, \sum_{k=2}^{\infty} \lambda_k = 1, \) and \( f_k(z) \in \text{Ext} \ G_J, \ k \geq 2, k \neq j. \)

Further, by using Lemma 2, we obtain

\[ f_0(z) = z - \sum_{k=2}^{\infty} \frac{(n+1)\lambda_k}{(2k+n-1)\delta(n,k)} z^k \quad (9.7) \]

which completes the proof of Theorem 12.

References


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