ERGODIC PROPERTIES OF COMPACT ACTIONS ON $C^*$-ALGEBRAS

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I. Introduction

Let $(A, G, \alpha)$ be a $C^*$-dynamical system. In [3] the classical notions of ergodic properties of topological dynamical systems such as topological transitivity, minimality, and uniquely ergodicity are extended and analyzed in the context of non-abelian $C^*$-dynamical systems. We showed in [2] that if $G$ is a compact group, then minimality, topological transitivity, uniquely ergodicity, and weakly ergodicity of the $C^*$-dynamical system $(A, G, \alpha)$ are equivalent. But we can give examples which show that the above statement is false if $G$ is not compact. Let $M_2^\infty$ be a Car-algebra, i.e. $M_2^\infty$ is an infinite tensor product $\otimes^\infty M_2$ of full $2 \times 2$ matrix algebra $M_2$. Let $\alpha$ be the shift of $M_2^\infty$ obtained by translating each tensor factor by one to the right. It is clear that $(M_2^\infty, \mathbb{Z}, \alpha)$ is ergodic. Let $p_\infty = \otimes_{-\infty}^\infty p$ where $p$ is the non-trivial projection of $M_2$. Then $p_\infty$ is the $\alpha$-invariant closed projection in the second dual $M_2^{\infty''}$ of $M_2^\infty$ because the sequence of projections $p_n = \otimes_{-n}^n p$ in $M_2^\infty$ is decreasing and converges $\sigma$-weakly to $p_\infty$. So $(I - p_\infty)M_2^{\infty''} (I - p_\infty) \cap M_2^\infty$ is the non-zero $\alpha$-invariant hereditary $C^*$-subalgebra of $M_2^\infty$ because $I - p_\infty$ is the open projection in $M_2^{\infty''}$. So $(M_2^\infty, \mathbb{Z}, \alpha)$ is not minimal. Let $p$ and $q$ be non-zero orthogonal projections in $M_2$ with $p + q = I$. Then with these projections we can make two $\alpha$-invariant hereditary $C^*$-subalgebras $B_1$ and $B_2$ of $M_2^\infty$ with $B_1B_2 = 0$. This means that $(M_2^\infty, \mathbb{Z}, \alpha)$ is not topologically transitive. Let $(A, G, \alpha)$ and $(A, G, \beta)$ be $C^*$-dynamical systems. It is said that two $C^*$-dynamical systems $(A, G, \alpha)$ and $(A, G, \beta)$ are exterior

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equivalent if there is a function \( t \to u_t \) from \( G \) to the unitary group of the multiplier algebra \( M(A) \) of \( A \) satisfying the conditions;

1. \( u_{st} = u_s \alpha_s(u_t) \),
2. \( \beta_t = Ad_{u_t} \alpha_t \),
3. \( t \to u_t x \) is norm continuous for each \( x \) in \( A \).

A function satisfying condition (1) is called a unitary cocyle. In \( W^* \)-dynamical systems the norm continuity of the condition (3) is replaced with the \( \sigma \)-weak continuity. In this paper we are going to discuss the exterior equivalence of dynamical systems. That is, if two \( C^* \)-dynamical systems (or \( W^* \)-dynamical systems) are exterior equivalent and one of them has some properties, then what happens to the other? In [4], it was known that if two \( C^* \)-dynamical systems (or \( W^* \)-dynamical systems) are exterior equivalent, they have the same Connes' spectrum.

2. Main result

Let \( (A, G, \alpha) \) be a \( C^* \)-dynamical system. The \( C^* \)-dynamical system \( (A, G, \alpha) \) is \textit{topologically transitive} if for any non-zero \( \alpha \)-invariant hereditary \( C^* \)-subalgebras \( B_1 \) and \( B_2 \) their product \( B_1 B_2 \) is not zero. If \( A \) is the only non-zero \( \alpha \)-invariant hereditary \( C^* \)-subalgebra of \( A \), then \( (A, G, \alpha) \) is called \textit{minimal}, [cf. 3]. A triple \( (\pi, u, H) \) is a covariant representation of \( (A, G, \alpha) \) if \( (\pi, H) \) is a representation of \( A \), \( (u, H) \) is a unitary representation of \( G \), and

\[
\pi(\alpha_t(x)) = u_t \pi(x) u_t^*
\]

for all \( x \) in \( A \) and \( t \) in \( G \). For \( W^* \)-dynamical systems the useful concepts are of course normal covariant representations.

**Lemma 2.1.** Let two \( C^* \)-dynamical systems \( (A, G, \alpha) \) and \( (A, G, \beta) \) be exterior equivalent with a unitary cocyle \( \{u_g | g \in G\} \). If \( (\pi, v, H) \) is a non-degenerate covariant representation of \( (A, G, \alpha) \), then \( (\pi, \pi(u)v, H) \) is a covariant representation of \( (A, G, \beta) \).

**Proof.** Since \( \pi \) is non-degenerate, there is a unique normal homomorphism \( \pi'' \) of the enveloping von Neumann algebra \( A'' \) of \( A \) onto the
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\( \sigma \)-weak closure \( \varpi(A)^{\sigma w} \) of \( \pi(A) \) which extends \( \pi \). Since \( \pi \) is nondegenerate, it is clear that \( g \to \pi(u_g)v_g \) is a unitary representation of \( G \). So we have for each \( \xi \in H \),

\[
\| \pi(u_g)v_g \xi - \xi \| \leq \| \pi(u_g) \| \| v_g \xi - \xi \| + \| \pi(u_g) \xi - \xi \|.
\]

Further more since \( \pi(\beta_g(x)) = \pi(u_g)v_g \pi(x)v_g^* \pi(u_g^*) \) for each \( g \in G \),
\( (\pi, \pi(u)v, H) \) is a covariant representation of \( (A, G, \beta) \).

Let \( (A, G, \alpha) \) be a \( C^* \)-dynamical system and \( \pi \) be a covariant representation of \( (A, G, \alpha) \). Then we can consider the \( W^* \)-dynamical system \( (\pi(A)'', G, \alpha'') \) on \( \pi(A)'' \) induced by \( (A, G, \alpha) \) where \( \alpha''(\pi(x)) = \pi(\alpha(x)) \) for all \( x \in A \).

**Proposition 2.2.** Let \( C^* \)-dynamical systems \( (A, G, \alpha) \) and \( (A, G, \beta) \) be exterior equivalent with a unitary cocyle \( \{ u_g \mid g \in G \} \). Let \( \pi \) be a covariant non-degenerate representation of \( (A, G, \alpha) \) on a Hilbert space \( H \). Then \( W^* \)-dynamical systems \( (\pi(A)'', G, \alpha'') \) and \( (\pi(A)'', G, \beta'') \) are also exterior equivalent with respect to the sense of \( W^* \)-dynamical systems.

**Proof.** We only have to show that the function \( g \to \pi(u_g)x \) from \( G \) into the unitary group of the multiplier algebra \( M(\pi(A)) \) of \( \pi(A) \) is \( \sigma \)-weakly continuous for each \( x \) in \( \pi(A)'' \). For each \( x \in \pi(A)'' \) we can choose a net \( \{ x_\alpha \}_{\alpha \in I} \) such that \( \{ \| x_\alpha \| \}_{\alpha \in I} \) is bounded and \( \{ x_\alpha \} \) converges \( \sigma \)-strongly to \( x \). Let \( \omega \) be a positive normal linear functional on \( \pi(A)'' \). Since \( \| \omega(y^*x) \|^2 \leq \omega(y^*y)\omega(x^*x) \) for all \( x, y \in \pi(A)'' \), we have

\[
|\omega(\pi(u_g)x - x)| \leq (\omega(1)\omega((x - x_\alpha)^*(x - x_\alpha)))^{\frac{1}{2}} \\
+ |\omega(\pi(u_g)x_\alpha - x_\alpha)| + |\omega(x_\alpha - x)|.
\]

Since \( x_\alpha - x \) converges \( \sigma \)-strongly to 0, \( (x_\alpha - x)^*(x_\alpha - x) \) converges \( \sigma \)-weakly to 0. Hence we see that \( \pi(u_g)x \) converges \( \sigma \)-weakly to \( x \) for each \( x \in \pi(A)'' \) as \( g \) goes to the identity of \( G \).
LEMMA 2.3. Let \((A, G, \alpha)\) be a \(C^*\)-dynamical system. Let \(p\) and \(q\) be equivalent projections in the fixed point algebra \(M(A)^\alpha\) of the multiplier algebra \(M(A)\). Assume that \((qAq, G, \alpha|_{qAq})\) is topologically transitive. Then \((pAp, G, \alpha|_{pAp})\) is also topologically transitive.

Proof. Suppose that \((pAp, G, \alpha|_{pAp})\) is not topologically transitive. Then there exist non-zero two elements \(x\) and \(y\) in \(pAp\) such that

\[ x\alpha_g(y) = 0, \quad g \in G. \]

Since \(p\) and \(q\) are equivalent in \(M(A)^\alpha\), there exists a partial isometry \(v\) in \(M(A)^\alpha\) such that

\[ v^*v = p, \quad vv^* = q. \]

Since \(x\) and \(y\) are contained in \(pAp\), we have \(x\alpha_g(y) = pxp\alpha_g(pyp)\). By \(\alpha\)-invariance of \(p\)

\[ x\alpha_g(y) = v^*vxv^*v\alpha_g(y)v^*v = 0 \]

for all \(g \in G\). Since \(v\) is fixed by \(\alpha_g\) for all \(g \in G\), we get for all \(g \in G\)

\[ 0 = v(v^*vxv^*v\alpha_g(y)v^*v)v^* \]

\[ = qvxv^*q\alpha_g(qvyv^*q). \]

Put \(x' = qvxv^*q\) and \(y' = qvyv^*q\). Then \(x'\) and \(y'\) are non-zero elements in \(qAq\). From the above calculation, \(x'\alpha_g(y') = 0\) for all \(g \in G\). Therefore \((qAq, G, \alpha|_{qAq})\) is not topologically transitive.

Let \((A, G, \alpha)\) be a \(C^*\)-dynamical system. When \(G\) is a compact group, \(A\) can be represented faithfully and covariantly.

LEMMA 2.4. Let \((A, G, \alpha)\) be a \(C^*\)-dynamical system and \(G\) be a compact group. Let \(A\) be represented faithfully and covariantly. If \((A, G, \alpha)\) is topologically transitive, then the \(W^*\)-dynamical system \((A'', G, \alpha'')\) induced by \((A, G, \alpha)\) is ergodic and the von Neumann algebra \(A''\) is finite.

Proof. Let \((A'', G, \alpha'')\) be the \(W^*\)-dynamical system induced by the \(C^*\)-dynamical system \((A, G, \alpha)\). Let \(P_0\) be the conditional expectation from \(A''\) to the fixed point algebra \(A''^{\alpha''}\) defined by

\[ P_0(x) = \int_G \alpha''_g(x) dg \]

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for all \( x \in A'' \). Since \( P_0 \) is \( \sigma \)-weak continuous and \( A \) is \( \sigma \)-weak dense in \( A'' \), \( P_0(A) = A^\alpha \) is \( \sigma \)-weak dense in \( A'''' \). By Corollary 2.3 in [2] \( A^\alpha \) is of one dimension, so \( A'''' \) is also trivial. Therefore \((A'', G, \alpha'')\) is ergodic and by Corollary 4.2 in [1] \( A'' \) is finite.

Let \( M \) be a finite von Neumann algebra and \( Z(M) \) be its center. The map \( \tau : M \rightarrow Z(M) \) is called a canonical central trace if it satisfies the following conditions:

1. \( \tau \) is linear and bounded,
2. \( \tau(xy) = \tau(yx) \) for any \( x, y \in M \),
3. \( \tau(z) = z \) for any \( z \in Z(M) \).

Let \( M \) be a finite von Neumann algebra and \( \tau \) be the canonical central trace on \( M \). Let \( p \) and \( q \) be projections in \( M \). \( \tau(p) = \tau(q) \) if and only if \( p \) and \( q \) are equivalent [cf 6].

**Lemma 2.5.** Let \((M, G, \alpha)\) be a \( W^* \)-dynamical system and \( G \) be a compact group. Let \( A \) be the set defined by

\[
A = \{ x \in M \mid x \rightarrow \alpha_g(x) \text{ is norm continuous} \}.
\]

If \((M, G, \alpha)\) is ergodic, the \( C^* \)-dynamical system \((A, G, \alpha|_A)\) is topologically transitive.

*Proof.* It is known that \( A \) is the \( \alpha \)-invariant \( C^* \)-algebra. Since \( G \) is compact, there exists a \( \sigma \)-weakly continuous expectation \( P_0 \) from \( M \) to the fixed point algebra \( M^\alpha \) as in the proof of Lemma 2.4. Since \( A \) is \( \sigma \)-weak dense in \( M \), \( P_0(A) \) is \( \sigma \)-weak dense in \( M^\alpha \). Therefore \( A^\alpha \) is trivial. By Corollary 2.3 in [2] \((A, G, \alpha|_A)\) is topologically transitive.

**Theorem 2.6.** Let \( W^* \)-dynamical systems \((M, G, \alpha)\) and \((M, G, \beta)\) be exterior equivalent and \( G \) be a compact group. Let \((M, G, \alpha)\) be ergodic. If \( Z(M^\beta) = Z(M)^\beta \), then the \( W^* \)-dynamical system \((M, G, \beta)\) is also ergodic.

*Proof.* Put \( p = I \otimes e_{11} \) and \( q = I \otimes e_{22} \) where \( \{ e_{ij} \mid i, j = 1, 2 \} \) is the matrix unit of \( M_2 \). Since \((M, G, \alpha)\) and \((M, G, \beta)\) are exterior equivalent, there exists the \( W^* \)-dynamical system \((M \otimes M_2, G, \gamma)\) such that

\[
\gamma_g \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} \alpha_g(x) & 0 \\ 0 & \beta_g(y) \end{pmatrix}
\]

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for $x, y \in M$ and $g \in G$. It is not difficult to show that

$$Z((M \otimes M_2)^{\gamma}) \subset \begin{pmatrix} Z(M^{\alpha}) & 0 \\ 0 & Z(M^{\beta}) \end{pmatrix}.$$ 

Since $Z(M^{\beta}) = Z(M)^{\beta}$, we have

$$Z(M \otimes M_2)^{\gamma} = Z((M \otimes M_2)^{\gamma}).$$

Since $M \otimes M_2$ is finite, we consider the canonical central trace $\tau : M \otimes M_2 \to Z(M \otimes M_2)$. We consider the restriction map

$$\tau|_{(M \otimes M_2)^{\gamma}} : (M \otimes M_2)^{\gamma} \to Z(M \otimes M_2)^{\gamma},$$

and denote it by $\tau^{\gamma}$. Since $Z(M \otimes M_2)^{\gamma} = Z((M \otimes M_2)^{\gamma})$, $\tau^{\gamma}$ becomes the canonical central trace on $(M \otimes M_2)^{\gamma}$. Since $p$ and $q$ are equivalent in $M \otimes M_2$, we have $\tau(p) = \tau(q)$. Since $p$ and $q$ are contained in $(M \otimes M_2)^{\gamma}$, we have $\tau^{\gamma}(p) = \tau^{\gamma}(q)$. Hence $p$ and $q$ are equivalent in $(M \otimes M_2)^{\gamma}$. So we can choose a partial isometry $v$ in $(M \otimes M_2)^{\gamma}$ such that $v^*v = p$ and $vv^* = q$. Let $A$ be the set defined as follows:

$$A = \{x \in M | x \to \alpha_g(x) \text{ is norm continuous}\}.$$ 

Then $A$ is the $\alpha$-invariant $C^*$-algebra and $\sigma$-weak dense in $M$ and $A \otimes M_2$ is $\sigma$-weak dense in $M \otimes M_2$. Since $(p(A \otimes M_2)p, G, \gamma|_{(A \otimes M_2)p})$ is isomorphic to $(A, G, \alpha)$, $(p(A \otimes M_2)p, G, \gamma|_{(A \otimes M_2)p})$ is topologically transitive by Lemma 2.5. Since $p$ and $q$ are equivalent with $\gamma$-invariant partial isometry $v$, $(q(M \otimes M_2)q, G, \gamma|_{q(M \otimes M_2)q})$ is also topologically transitive by Lemma 2.3. Hence by Lemma 2.4 $(M, G, \beta)$ is ergodic.

**Corollary 2.7.** Let a $W^*$-dynamical system $(M, G, \alpha)$ be ergodic and $G$ be a compact abelian group. Let $(M, G, \alpha)$ be exterior equivalent to a $W^*$-dynamical system $(M, G, \beta)$. If $Z(M^{\beta}) = Z(M)^{\beta}$, then $\text{Sp}(\alpha) = \text{Sp}(\beta)$.

**Proof.** Since $(M, G, \alpha)$ and $(M, G, \beta)$ are exterior equivalent, it was known in [4] that $\Gamma(\alpha) = \Gamma(\beta)$. By Theorem 2.6, we have $\text{Sp}(\alpha) = \text{Sp}(\beta)$.

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References


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