CAUCHY PROBLEM FOR THE EULER EQUATIONS OF A NONHOMOGENEOUS IDEAL INCOMPRESSIBLE FLUID

Shigeharu Itoh

1. Introduction

Let us consider the Cauchy problem

\[
\begin{aligned}
&\rho_t + v \cdot \nabla \rho = 0 \\
&\rho [v_t + (v \cdot \nabla)v] + \nabla p = \rho f \\
&\text{div } v = 0 \\
&\rho|_{t=0} = \rho_0(x) \\
&v|_{t=0} = v_0(x)
\end{aligned}
\tag{1.1}
\]

in \( Q_T = \mathbb{R}^3 \times [0, T] \), where \( f(x, t), \rho_0(x) \) and \( v_0(x) \) are given, while the density \( \rho(x, t) \), the velocity vector \( v(x, t) = (v^1(x, t), v^2(x, t), v^3(x, t)) \) and the pressure \( p(x, t) \) are unknowns. The equations (1.1) describe the motion of a nonhomogeneous ideal incompressible fluid.

The aim of the present paper is to establish the unique solvability, local in time, of the problem (1.1). For the problem in a bounded domain with the boundary condition \( v \cdot n = 0 \), where \( n \) is the unit out normal to the boundary, that fact has been proved under appropriate assumptions by many authors, for instance, [1], [2], [3], [7], [8]. On the other hand, as far as we know, there is no investigation in an unbounded domain.

Our theorem is the following.

**Theorem.** Assume that

\[
\rho_0(x) - \bar{\rho} \in H^3(\mathbb{R}^3) \text{ for some positive constant } \bar{\rho}, \tag{1.2}
\]

\[
\inf \rho_0(x) \equiv m > 0 \text{ and } \sup \rho_0(x) \equiv M < \infty, \tag{1.3}
\]

\[
v_0(x) \in H^3(\mathbb{R}^3) \text{ and } \text{div } v_0 = 0 \tag{1.4}
\]
and
\[ f(x,t) \in L^1(0,T : H^3(\mathbb{R}^3)) \quad \text{and} \quad \text{div } f \in L^1(Q_T). \] (1.5)

Then there exists \( T^* \in (0,T] \) such that the problem (1.1) has a unique solution \( (\rho,v,p) \) which satisfies
\[
(\rho - \bar{\rho}, v, \nabla p) \in L^\infty(0,T^* : H^3(\mathbb{R}^3)) \times L^\infty(0,T^* : H^3(\mathbb{R}^3)) \times L^1(0,T^* : H^3(\mathbb{R}^3)).
\] (1.6)

Moreover, if \( f(x,t) \in C^0(0,T : H^3(\mathbb{R}^3)) \) and \( \text{div } f \in C^0(0,T : L^1(\mathbb{R}^3)) \), then the solution \( (\rho,v,p) \) is classical.

In section 2, we solve three linear problems in sequence and get the estimates. In section 3, we first show the existence of a fixed point of some map, and then it is proved that this becomes the solution of (1.1).

2. Preliminaries

**Lemma 2.1.** For a given \( v \in L^\infty(0,T : H^3(\mathbb{R}^3)) \) with \( \text{div } v = 0 \), the problem
\[
\begin{cases}
\rho_t + v \cdot \nabla \rho = 0 \\
\rho|_{t=0} = \rho_0(x)
\end{cases}
\] (2.1)

has a solution satisfying
\[ m \leq \rho \leq M. \] (2.2)

Moreover, if we put \( \tilde{\rho} = \rho - \bar{\rho} \), then
\[
\|\tilde{\rho}(t)\|_3 \leq \|\tilde{\rho}_0\|_3 \exp\left(c_1 \int_0^t \|v(s)\|_3 ds\right),
\] (2.3)

where \( \tilde{\rho}_0 = \rho_0 - \bar{\rho} \), \( c_1 \) is a positive constant depending only on imbedding theorems and \( \| \cdot \|_k = \| \cdot \|_{H^k(\mathbb{R}^3)} \).

**Proof.** It is well-known that, using the classical method of characteristics, the solution of (2.1) is given by
\[
\rho(x,t) = \rho_0(y(\tau,x,t)|_{\tau=0}),
\] (2.4)
where \( y(\tau, x, t) \) is the solution of the Cauchy problem

\[
\begin{align*}
\frac{dy}{d\tau} &= v(y, \tau) \\
y|_{\tau=t} &= x.
\end{align*}
\] (2.5)

Thus the first assertion has shown. Next we establish the estimate for \( \tilde{\rho} \). It follows from (1.1)_1 and (1.1)_4 that \( \tilde{\rho} \) satisfies the equation

\[
\begin{align*}
\tilde{\rho}_t + v \cdot \nabla \tilde{\rho} &= 0 \\
\tilde{\rho}|_{t=0} &= \tilde{\rho}_0(x).
\end{align*}
\] (2.6)

Applying the operator \( D^\alpha (= (\partial/\partial x_1)^{\alpha_1}(\partial/\partial x_2)^{\alpha_2}(\partial/\partial x_3)^{\alpha_3}) \) to (2.6)_1, multiplying the result by \( D^\alpha \tilde{\rho} \), integrating over \( \mathbb{R}^3 \) and adding in \( \alpha \) with \(|\alpha| (= \alpha_1 + \alpha_2 + \alpha_3) \leq 3\), then we have

\[
\frac{d}{dt} \|\tilde{\rho}(t)\|_3^2 \leq c_1 \|v(t)\|_3 \|\tilde{\rho}(t)\|_3^2.
\] (2.7)

Hence, by Gronwall’s inequality, it is easy to see that (2.3) holds. Q.E.D.

**Lemma 2.2.** Let \( \rho \) and \( v \) be in Lemma 2.1. Then the problem

\[
\text{div } (\rho^{-1} \nabla \rho) = - \sum_{i,j} v_{x_j}^i v_{x_i}^j + \text{div } f \equiv F
\] (2.8)

has a solution satisfying

\[
\|\nabla \rho(t)\|_3 \leq c_2 \left( \frac{1}{\sum_{j=0}^9 (\|\tilde{\rho}(t)\|_3^2) (\|v(t)\|_3^2 + \|f(t)\|_3 + \|\text{div } f(t)\|_{L^1(\mathbb{R}^3)}) \right),
\] (2.9)

where \( c_2 \) is a positive constant depending only on \( m, M, \) imbedding theorems and interpolation inequalities.

**Proof.** Thanks to (2.2), the solvability easily follows from the general theory of the elliptic partial differential equations of second order (cf. [5]), and so we can restrict ourselves to get the estimate (2.9).
Multiplying (2.8) by \( p \) and integrating over \( \mathbb{R}^3 \), then we obtain
\[
M^{-1} \| \nabla p \|_0^2 \leq \| F \|_{L^{6/5}(\mathbb{R}^3)} \| p \|_{L^6(\mathbb{R}^3)} \leq c_3 \| F \|_{L^{6/5}(\mathbb{R}^3)} \| \nabla p \|_0 \leq c_4 \| F \|_{L^1(\mathbb{R}^3)}^{2/3} \| F \|_0^{1/3} \| \nabla p \|_0 \leq c_4 \left( (2/3) \| F \|_{L^1(\mathbb{R}^3)} + (1/3) \| F \|_0 \right) \| \nabla p \|_0.
\]

Hence we get
\[
\| \nabla p \|_0 \leq c_5 \left( \| F \|_{L^1(\mathbb{R}^3)} + \| F \|_0 \right). \tag{2.11}
\]

In order to accomplish our purpose, we use the following inequality (cf. [7]):
\[
\| u \|_2 \leq (3/2)^{1/2} (\| \Delta u \|_0 + \| u \|_0) \text{ for any } u \in H^2(\mathbb{R}^3). \tag{2.12}
\]
Noting that (2.8) can be written in the form \( \Delta p = \rho F + \rho^{-1} \nabla \rho \cdot \nabla p \), we get that for \( \alpha \) with \(|\alpha| = 2\)
\[
\| D^\alpha p \|_2 \leq (3/2)^{1/2} \left( \| D^\alpha (\rho F + \rho^{-1} \nabla \rho \cdot \nabla p) \|_0 + \| D^\alpha p \|_0 \right). \tag{2.13}
\]
By the direct calculation, we obtain
\[
\text{the right hand of (2.13)}
\]
\[
\leq c_6 \left( (M + \| \nabla \rho \|_2) \| F \|_2 + \left( \sum_{j=0}^{3} m^{-j} \| \nabla \rho \|_2^j \right) \| \nabla p \|_2 \right). \tag{2.14}
\]
Now, from the interpolation inequality and Young's inequality, we have
\[
A \| \nabla p \|_2 \leq A \| \nabla p \|_3^{2/3} \| \nabla p \|_0^{1/3} = A \| \nabla p \|_0 + A \| \nabla^2 p \|_2^{2/3} \| \nabla p \|_0^{1/3} \leq \varepsilon \| \nabla^2 p \|_2 + (A + \varepsilon^{-2} A^3) \| \nabla p \|_0.
\]
Therefore, we find that
\[
\| \nabla p \|_3 \leq c_7 \left( \sum_{j=0}^{9} \| \nabla \rho \|_2^j \right) \left( \| F \|_{L^1(\mathbb{R}^3)} + \| F \|_2 \right). \tag{2.16}
\]
On the other hand, it is easy to verify that
\[
\| F \|_{L^1(\mathbb{R}^3)} + \| F \|_2 \leq c_8 \left( \| \text{div } f \|_{L^1(\mathbb{R}^3)} + \| \text{div } f \|_2 + \| \nabla v \|_2^2 \right). \tag{2.17}
\]
Consequently, the desired estimate is established. Q.E.D.

Moreover, similar to the above lemmas, we have
Lemma 2.3. Let \( \rho \) and \( v \) be in Lemma 2.1 and \( p \) in Lemma 2.2. Then the problem

\[
\begin{aligned}
    \begin{cases}
        w_t + (v \cdot \nabla)w &= -\rho^{-1} \nabla p + f \\
        w|_{t=0} &= v_0(x)
    \end{cases}
\end{aligned}
\] (2.18)

has a solution satisfying

\[
\|w(t)\|_3 
\leq \left[ \|v_0\|_3 + \int_0^t \|f(s)\|_3 ds + \int_0^t \left( \sum_{j=0}^3 m^{-j+1} \|\tilde{\rho}(s)\|_3^j \right) \|\nabla p(s)\|_3 ds \right] 
\exp \left( c_1 \int_0^t \|v(s)\|_3 ds \right). \tag{2.19}
\]

3. Proof of Theorem

Let \( P \) be the orthogonal projector in \( \{L^2(\mathbb{R}^3)\}^3 \) onto \( H = \) the closure of \( J = \{ u \in \{C_0^\infty(\mathbb{R}^3)\}^3 : \text{div} \ u = 0 \} \) in \( \{L^2(\mathbb{R}^3)\}^3 \). Then it is immediately follows from lemmas in section 2 that if \( \|\tilde{\rho}_0\|_3 \leq A \) and \( \sup_{0 \leq t \leq T} \|v(t)\|_3 = \|v(t)\|_{3,\infty,T} \leq K \), then

\[
\|Pw(t)\|_3 
\leq \|w(t)\|_3 
\leq \left[ \|v_0\|_3 + \|f\|_{L^1(0,T;H^3(\mathbb{R}^3))} + h(Ae^{c_1KT}) \right] 
\left( K^2T + \|f\|_{L^1(0,T;H^3(\mathbb{R}^3))} + \|\text{div} \ f\|_{L^1(Q_T)} \right) e^{c_1KT}, \tag{3.1}
\]

where \( h(r) = c_2 \left( \sum_{j=0}^9 r^j \right) \left( \sum_{j=0}^3 m^{-j+1} r^j \right) \).

Therefore if we choose

\[
4K \geq \|v_0\|_3 + \|f\|_{L^1(0,T;H^3(\mathbb{R}^3))} \tag{3.2}
\]

and define \( S = \{ v \in L^\infty(0,T;H^3(\mathbb{R}^3)) : Pv = v, \|v(t)\|_{3,\infty,T} \leq K \} \), then, from (3.1), the map \( F : v \to Pw \) satisfies \( F(S) \subset S \) for sufficiently small \( T \), for example \( T \leq T^* = \min\{[4K h(2A)]^{-1}, (c_1K)^{-1} \log 2 \} \). Let
\{v_j\} \subset S$. Then we can verify that \( \{Fv_j\} \) is the Cauchy sequence in \( L^\infty(0,T; H^3(\mathbb{R}^3)) \) with few difficulties. Hence, as \( F \) becomes a compact map, there exists a fixed point \( \upsilon = P\upsilon \) from Schauder's theorem. We have thus solved the problem

\[
\begin{align*}
\tilde{\rho}_t + P\upsilon \cdot \nabla \tilde{\rho} &= 0, \\
w_t + (P\upsilon \cdot \nabla)w &= -\rho^{-1}\nabla p + f, \\
\text{div} \left[ \rho^{-1}\nabla p + (P\upsilon \cdot \nabla)P\upsilon - f \right] &= 0, \\
\tilde{\rho}|_{t=0} &= \tilde{\rho}_0(\upsilon), \\
w|_{t=0} &= \upsilon_0(\upsilon)
\end{align*}
\tag{3.3}
\]

This is equivalent to (1.1) if \( P \upsilon = \upsilon \).

Let us show that \( Q\upsilon = (I - P)\upsilon = 0 \). Apply the projection \( Q \) to (3.3)_2, then, due to (3.3)_3, we have

\[
(Q\upsilon)_t + Q[(P\upsilon \cdot \nabla)Q\upsilon] = 0. \tag{3.4}
\]

Multiplying (3.4) by \( Q\upsilon \) and integrating over \( \mathbb{R}^3 \), we get

\[
\frac{d}{dt} \int_{\mathbb{R}^3} |Q\upsilon|^2 dx + \int_{\mathbb{R}^3} Q[(P\upsilon \cdot \nabla)Q\upsilon] \cdot Q\upsilon dx = 0. \tag{3.5}
\]

Furthermore, from the definition of \( P \), we obtain

\[
\begin{align*}
\int_{\mathbb{R}^3} Q[(P\upsilon \cdot \nabla)Q\upsilon] \cdot Q\upsilon dx &= \int_{\mathbb{R}^3} (P\upsilon \cdot \nabla)Q\upsilon \cdot Q\upsilon dx - \int_{\mathbb{R}^3} P[(P\upsilon \cdot \nabla)Q\upsilon] \cdot Q\upsilon dx \\
&= \frac{1}{2} \int_{\mathbb{R}^3} P\upsilon \cdot \nabla |Q\upsilon|^2 dx = 0.
\end{align*}
\tag{3.6}
\]

Hence we find that \( Q\upsilon = 0 \), since \( Q\upsilon|_{t=0} = Q\upsilon_0 = 0 \).

Uniqueness is proved in [4].

This completes the proof.
Cauchy problem for the Euler equations

References


Department of Mathematics
Faculty of Education
Hirosaki University
Hirosaki 036, Japan