J. Korean Math. Soc. **31** (1994), No. 3, pp. 367–373

CAUCHY PROBLEM FOR THE EULER EQUATIONS OF A NONHOMOGENEOUS IDEAL INCOMPRESSIBLE FLUID

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1. Introduction

Let us consider the Cauchy problem

$$\begin{cases} \rho_t + v \cdot \nabla \rho = 0 \\ \rho\big[v_t + (v \cdot \nabla)v\big] + \nabla p = \rho f \\ \text{div } v = 0 \\ \rho|_{t=0} = \rho_0(x) \\ v|_{t=0} = v_0(x) \end{cases} \tag{1.1}$$

in $Q_T = \mathbb{R}^3 \times [0, T]$, where $f(x,t)$, $\rho_0(x)$ and $v_0(x)$ are given, while the density $\rho(x,t)$, the velocity vector $v(x,t) = \big(v^1(x,t), v^2(x,t), v^3(x,t)\big)$ and the pressure $p(x,t)$ are unknowns. The equations $(1.1)_1 - (1.1)_3$ describe the motion of a nonhomogeneous ideal incompressible fluid.

The aim of the present paper is to establish the unique solvability, local in time, of the problem (1.1). For the problem in a bounded domain with the boundary condition $v \cdot n = 0$, where n is the unit out normal to the boundary, that fact has been proved under appropriate assumptions by many authors, for instance, [1], [2], [3], [7], [8]. On the other hand, as far as we know, there is no investigation in an unbounded domain.

Our theorem is the following.

THEOREM. *Assume that*

$$\rho_0(x) - \bar{\rho} \in H^3(\mathbb{R}^3) \text{ for some positive constant } \bar{\rho}, \tag{1.2}$$

$$\inf \rho_0(x) \equiv m > 0 \text{ and } \sup \rho_0(x) \equiv M < \infty, \tag{1.3}$$

$$v_0(x) \in H^3(\mathbb{R}^3) \text{ and div } v_0 = 0 \tag{1.4}$$

Received April 9, 1993.

and

$$f(x,t) \in L^1(0,T : H^3(\mathbb{R}^3)) \text{ and } \operatorname{div} f \in L^1(Q_T). \tag{1.5}$$

Then there exists $T^* \in (0,T]$ such that the problem (1.1) has a unique solution (ρ, v, p) which satisfies

$$\begin{aligned}(\rho - \bar{\rho}, v, \nabla p) \in & L^\infty(0,T^* : H^3(\mathbb{R}^3)) \\ & \times L^\infty(0,T^* : H^3(\mathbb{R}^3)) \times L^1(0,T^* : H^3(\mathbb{R}^3)).\end{aligned} \tag{1.6}$$

Moreover, if $f(x,t) \in C^0(0,T : H^3(\mathbb{R}^3))$ and $\operatorname{div} f \in C^0(0,T : L^1(\mathbb{R}^3))$, then the solution (ρ, v, p) is classical.

In section 2, we solve three linear problems in sequence and get the estimates. In section 3, we first show the existence of a fixed point of some map, and then it is proved that this becomes the solution of (1.1).

2. Preliminaries

LEMMA 2.1. *For a given $v \in L^\infty(0,T : H^3(\mathbb{R}^3))$ with $\operatorname{div} v = 0$, the problem*

$$\begin{cases} \rho_t + v \cdot \nabla \rho = 0 \\ \rho|_{t=0} = \rho_0(x) \end{cases} \tag{2.1}$$

has a solution satisfying

$$m \le \rho \le M. \tag{2.2}$$

Moreover, if we put $\tilde{\rho} = \rho - \bar{\rho}$, then

$$\|\tilde{\rho}(t)\|_3 \le \|\tilde{\rho}_0\|_3 \exp\Big(c_1 \int_0^t \|v(s)\|_3 ds\Big), \tag{2.3}$$

where $\tilde{\rho}_0 = \rho_0 - \bar{\rho}, c_1$ is a positive constant depending only on imbedding theorems and $\|\cdot\|_k = \|\cdot\|_{H^k(\mathbb{R}^3)}$.

Proof. It is well-known that, using the classical method of characteristics, the solution of (2.1) is given by

$$\rho(x,t) = \rho_0(y(\tau, x, t)|_{\tau=0}), \tag{2.4}$$

where $y(\tau, x, t)$ is the solution of the Cauchy problem

$$\begin{cases} \frac{dy}{d\tau} = v(y, \tau) \\ y|_{\tau=t} = x. \end{cases} \tag{2.5}$$

Thus the first assertion has shown. Next we establish the estimate for $\tilde{\rho}$. It follows from $(1.1)_1$ and $(1.1)_4$ that $\tilde{\rho}$ satisfies the equation

$$\begin{cases} \tilde{\rho}_t + v \cdot \nabla \tilde{\rho} = 0 \\ \tilde{\rho}|_{t=0} = \tilde{\rho}_0(x). \end{cases} \tag{2.6}$$

Applying the operator $D^\alpha (= (\partial/\partial x_1)^{\alpha_1} (\partial/\partial x_2)^{\alpha_2} (\partial/\partial x_3)^{\alpha_3})$ to $(2.6)_1$, multiplying the result by $D^\alpha \tilde{\rho}$, integrating over $\mathbb{R}^3$ and adding in α with $|\alpha| \ (= \alpha_1 + \alpha_2 + \alpha_3) \leq 3$, then we have

$$\frac{d}{dt} \|\tilde{\rho}(t)\|_3^2 \leq c_1 \|v(t)\|_3 \|\tilde{\rho}(t)\|_3^2. \tag{2.7}$$

Hence, by Gronwall's inequality, it is easy to see that (2.3) holds. Q.E.D.

LEMMA 2.2. *Let ρ and v be in Lemma 2.1. Then the problem*

$$div\ (\rho^{-1} \nabla p) = -\sum_{i,j} v^i_{x_j} v^j_{x_i} + \ div\ f \equiv F \tag{2.8}$$

has a solution satisfying

$$\|\nabla p(t)\|_3 \tag{2.9}$$
$$\leq c_2 \Big(\sum_{j=0}^{9} \|\tilde{\rho}(t)\|_3^j \Big) \big(\|v(t)\|_3^2 + \|f(t)\|_3 + \|\ div\ f(t)\|_{L^1(\mathbb{R}^3)} \big),$$

where c_2 is a positive constant depending only on m, M, imbedding theorems and interpolation inequalities.

Proof. Thanks to (2.2), the solvability easily follows from the general theory of the elliptic partial differential equations of second order (cf. [5]), and so we can restrict ourselves to get the estimate (2.9).

Multiplying (2.8) by p and integrating over $\mathbb{R}^3$, then we obtain

$$\begin{aligned} M^{-1}\|\nabla p\|_0^2 &\le \|F\|_{L^{6/5}(\mathbb{R}^3)}\|p\|_{L^6(\mathbb{R}^3)} \\ &\le c_3\|F\|_{L^{6/5}(\mathbb{R}^3)}\|\nabla p\|_0 \\ &\le c_4\|F\|^{2/3}_{L^1(\mathbb{R}^3)}\|F\|_0^{1/3}\|\nabla p\|_0 \\ &\le c_4\left[(2/3)\|F\|_{L^1(\mathbb{R}^3)} + (1/3)\|F\|_0\right]\|\nabla p\|_0. \end{aligned} \tag{2.10}$$

Hence we get

$$\|\nabla p\|_0 \le c_5\big(\|F\|_{L^1(\mathbb{R}^3)} + \|F\|_0\big). \tag{2.11}$$

In order to accomplish our purpose, we use the following inequality (cf. [7]):

$$\|u\|_2 \le (3/2)^{1/2}\big(\|\Delta u\|_0 + \|u\|_0\big) \text{ for any } u \in H^2(\mathbb{R}^3). \tag{2.12}$$

Noting that (2.8) can be written in the form $\Delta p = \rho F + \rho^{-1}\nabla\rho\cdot\nabla p$, we get that for α with $|\alpha| = 2$

$$\|D^\alpha p\|_2 \le (3/2)^{1/2}\big(\|D^\alpha(\rho F + \rho^{-1}\nabla\rho\cdot\nabla p)\|_0 + \|D^\alpha p\|_0\big). \tag{2.13}$$

By the direct calculation, we obtain

$$\text{the right hand of (2.13)}$$

$$\le c_6\left[(M + \|\nabla\rho\|_2)\|F\|_2 + \Big(\sum_{j=0}^{3} m^{-j}\|\nabla\rho\|_2^j\Big)\|\nabla p\|_2\right]. \tag{2.14}$$

Now, from the interpolation inequality and Young's inequality, we have

$$\begin{aligned} A\|\nabla p\|_2 &\le A\|\nabla p\|_3^{2/3}\|\nabla p\|_0^{1/3} \\ &= A\|\nabla p\|_0 + A\|\nabla^2 p\|_2^{2/3}\|\nabla p\|_0^{1/3} \\ &\le \varepsilon\|\nabla^2 p\|_2 + (A + \varepsilon^{-2}A^3)\|\nabla p\|_0. \end{aligned} \tag{2.15}$$

Therefore, we find that

$$\|\nabla p\|_3 \le c_7\Big(\sum_{j=0}^{9}\|\nabla\rho\|_2^j\Big)\big(\|F\|_{L^1(\mathbb{R}^3)} + \|F\|_2\big). \tag{2.16}$$

On the other hand, it is easy to verify that

$$\|F\|_{L^1(\mathbb{R}^3)} + \|F\|_2 \le c_8\big(\|\operatorname{div} f\|_{L^1(\mathbb{R}^3)} + \|\operatorname{div} f\|_2 + \|\nabla v\|_2^2\big). \tag{2.17}$$

Consequently, the desired estimate is established. Q.E.D.

Moreover, similar to the above lemmas, we have

LEMMA 2.3. *Let ρ and v be in Lemma 2.1 and p in Lemma 2.2. Then the problem*

$$\begin{cases} w_t + (v \cdot \nabla)w = -\rho^{-1}\nabla p + f \\ w|_{t=0} = v_0(x) \end{cases} \tag{2.18}$$

has a solution satisfying

$$\begin{aligned} &\|w(t)\|_3 \\ \leq &\left[\|v_0\|_3 + \int_0^t \|f(s)\|_3 ds + \int_0^t \left(\sum_{j=0}^3 m^{-(j+1)}\|\tilde{\rho}(s)\|_3^j\right)\|\nabla p(s)\|_3 ds\right] \\ &\exp\left(c_1 \int_0^t \|v(s)\|_3 ds\right). \end{aligned} \tag{2.19}$$

3. Proof of Theorem

Let P be the orthogonal projector in $\{L^2(\mathbb{R}^3)\}^3$ onto H = the closure of $J = \{u \in \{C_0^\infty(\mathbb{R}^3)\}^3 : \text{div } u = 0\}$ in $\{L^2(\mathbb{R}^3)\}^3$. Then it is immediately follows from lemmas in section 2 that if $\|\tilde{\rho}_0\|_3 \leq A$ and $\sup_{0\leq t\leq T}\|v(t)\|_3 = \|v(t)\|_{3,\infty,T} \leq K$, then

$$\begin{aligned} &\|Pw(t)\|_3 \\ \leq &\|w(t)\|_3 \leq \left[\|v_0\|_3 + \|f\|_{L^1(0,T:H^3(\mathbb{R}^3))} + h(Ae^{c_1KT})\right. \\ &\left.(K^2T + \|f\|_{L^1(0,T:H^3(\mathbb{R}^3))} + \|\text{ div } f\|_{L^1(Q_T)})\right] e^{c_1KT}, \end{aligned} \tag{3.1}$$

where $h(r) = c_2\left(\sum_{j=0}^9 r^j\right)\left(\sum_{j=0}^3 m^{-(j+1)}r^j\right)$.

Therefore if we choose

$$\begin{aligned} 4K \geq &\|v_0\|_3 + \|f\|_{L^1(0,T:H^3(\mathbb{R}^3))} \\ &+ h(2A)\left(\|f\|_{L^1(0,T:H^3(\mathbb{R}^3))} + \|\text{ div } f\|_{L^1(Q_T)}\right) \end{aligned} \tag{3.2}$$

and define $S = \{v \in L^\infty(0,T;H^3(\mathbb{R}^3)) : Pv = v, \|v(t)\|_{3,\infty,T} \leq K\}$, then, from (3.1), the map $F : v \to Pw$ satisfies $F(S) \subset S$ for sufficiently small T, for example $T \leq T^* = \min\{[4Kh(2A)]^{-1}, (c_1K)^{-1}\log 2\}$. Let

$\{v_j\} \subset S$. Then we can verify that $\{Fv_j\}$ is the Cauchy sequence in $L^\infty(0,T;H^3(\mathbb{R}^3))$ with few difficulties. Hence, as F becomes a compact map, there exists a fixed point $v = Pw$ from Schauder's theorem. We have thus solved the problem

$$\begin{cases} \tilde{\rho}_t + Pw \cdot \nabla \tilde{\rho} = 0 \\ w_t + (Pw \cdot \nabla)w = -\rho^{-1}\nabla p + f \\ \operatorname{div}\left[\rho^{-1}\nabla p + (Pw \cdot \nabla)Pw - f\right] = 0 \\ \tilde{\rho}|_{t=0} = \tilde{\rho}_0(x) \\ w|_{t=0} = v_0(x) \end{cases} \tag{3.3}$$

This is equivalent to (1.1) if $Pw = w$.

Let us show that $Qw = (I - P)w = 0$. Apply the projection Q to $(3.3)_2$, then, due to $(3.3)_3$, we have

$$(Qw)_t + Q\left[(Pw \cdot \nabla)Qw\right] = 0. \tag{3.4}$$

Multiplying (3.4) by Qw and integrating over $\mathbb{R}^3$, we get

$$\frac{d}{dt}\int_{\mathbb{R}^3} |Qw|^2 dx + \int_{\mathbb{R}^3} Q\left[(Pw \cdot \nabla)Qw\right] \cdot Qw \, dx = 0. \tag{3.5}$$

Furthermore, from the definition of P, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^3} Q\left[(Pw \cdot \nabla)Qw\right] \cdot Qw \, dx \qquad (3.6) \\ =&\int_{\mathbb{R}^3} (Pw \cdot \nabla)Qw \cdot Qw \, dx - \int_{\mathbb{R}^3} P\left[(Pw \cdot \nabla)Qw\right] \cdot Qw \, dx \\ =&\frac{1}{2}\int_{\mathbb{R}^3} Pw \cdot \nabla |Qw|^2 dx = 0. \end{aligned}$$

Hence we find that $Qw = 0$, since $Qw|_{t=0} = Qv_0 = 0$.

Uniqueness is proved in [4].

This completes the proof.

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