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THE INTERMEDIATE SOLUTION OF QUASILINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS

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1. Introduction

We study the existence of an intermediate solution of nonlinear elliptic boundary value problems (BVP) of the form

$$(BVP) \qquad \begin{cases} \Delta u = f(x,u,\nabla u), & \text{in } \Omega \\ Bu(x) = \phi(x), & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbf{R}^n$, $n \geq 1$, and $\partial\Omega \in C^{2,\alpha}$, $(0 < \alpha < 1)$, Δ is the Laplacian operator, $\nabla u = (D_1 u, D_2 u, \cdots, D_n u)$ denotes the gradient of u and

$$Bu(x) = p(x)u(x) + q(x)\frac{du}{d\nu}(x),$$

where $\frac{du}{d\nu}$ denotes the outward normal derivative of u on $\partial\Omega$.

Suppose now, that $\bar{v}$, $\hat{v}$ and $\bar{w}$, $\hat{w}$ are two pairs of subsolutions and supersolutions in the class $C^2(\bar{\Omega})$ or in the usual Sobolev space $W^{2,p}(\Omega)$, $p > n$ of (BVP) such that $\bar{v}(x) \leq \hat{v}(x)$, $\bar{w}(x) \leq \hat{w}(x)$, $\bar{v}(x) \leq \hat{w}(x)$ for all $x \in \bar{\Omega}$ and $\hat{v}(x_0) < \bar{w}(x_0)$ for some $x_0 \in \bar{\Omega}$. Then there is a solution in the order interval $[\bar{v}, \hat{v}] = \{ u \in C(\bar{\Omega}) : \bar{v}(x) \leq u(x) \leq \hat{v}(x), x \in \bar{\Omega} \}$ and a solution in $[\bar{w}, \hat{w}]$. And furthermore Amann [1] or Amann and Crandall [3] showed that there exists an intermediate solution in the set $[\bar{v}, \hat{w}] \setminus ([\bar{v}, \hat{v}] \cup [\bar{w}, \hat{w}])$ under additional conditions.

The existence of a solution given a pair of quasisubsolution and quasisupersolution of (BVP), $\bar{v}$ and $\hat{v}$, with $\bar{v}(x) \leq \hat{v}(x)$ for all $x \in \bar{\Omega}$, is well known (see [9]). Since these functions may have singular points

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in the interior of Ω, there arises the question, does there also exist an intermediate solution if there are pairs of quasisubsolutions and quasisupersolutions as in the preceding paragraph ?

Suppose now, in addition, that f is independent of ∇u and that there are pairs of quasisubsolutions and quasisupersolutions as in the preceding paragraph. Ko [6] proved the existence of an intermediate solution in $[\bar{v}, \hat{w}] \setminus ([\bar{v}, \hat{v}] \cup [\bar{w}, \hat{w}])$ under additional conditions. Hence there also arises the question, does there exist an intermediate solution if f depends nonlinearly on ∇u ?

The author is able to solve the above problem which is the existence of an intermediate solution for (BVP) using Maximum Principles and the theorem on existence of several fixed points (see pp241, [4]). The multiplicity result is a generalization of Theorem(1.6) in [1] or Theorem 2 in [3].

As a simple application of our results, we prove the existence of several positive ordered solutions in a class of singularly perturbed quasilinear elliptic Dirichlet boundary value problems with small positive parameter.

Throughout this paper we assume that $p, q \in C^{1,\alpha}(\partial\Omega)$ are nonnegative real valued functions which either $q(x) = 0$ for all $x \in \partial\Omega$ or $q(x) > 0$ for all $x \in \partial\Omega$, and f satisfies the following conditions:

$0 < \alpha < 1$,

(1) $f : \bar{\Omega} \times \mathbf{I} \times \mathbf{R}^n \to \mathbf{R}$ is a α-Hölder continuous function, such that $f(\cdot, \xi, \eta) \in C^\alpha(\bar{\Omega})$ and such that $\dfrac{\partial f}{\partial \xi}$ and $\dfrac{\partial f}{\partial \eta}$ are continuous where (x, ξ, η) denotes a generic point of $\bar{\Omega} \times \mathbf{I} \times \mathbf{R}^n$ and $\mathbf{I}$ is a fixed bounded and closed interval in $\mathbf{R}$.

(2) There exists a continuous function $c : \mathbf{R}_+ \to \mathbf{R}_+ = [0, \infty)$ such that

$$|f(x, \xi, \eta)| \leq c(\rho)(1 + |\eta|^2)$$

for every $\rho \geq 0$ and $(x, \xi, \eta) \in \bar{\Omega} \times [-\rho, \rho] \times \mathbf{R}^n$.

(3) $\phi \in C^{2,\alpha}(\bar{\Omega})$ and for the Dirichlet problem case, $\phi(x) \in \mathbf{I}$ for all $x \in \partial\Omega$.

By a solution of (BVP) we mean a function $u : \bar{\Omega} \to \mathbf{I}$ such that $u \in C^2(\bar{\Omega})$ and u satisfies (BVP) pointwise.

2. Main results

First of all, we state definitions of a quasisubsolution and a quasisupersolution of (BVP).

DEFINITIONS. A function $w : \bar{\Omega} \to \mathbf{R}$ is a quasisupersolution of (BVP) in $\bar{\Omega}$, if for any $x_0 \in \bar{\Omega}$, there exist a neighborhood N of x_0 and a finite number of functions $w_k \in C^2(N)$, $k = 1, 2, \cdots, p$ such that

$$(I) \qquad w(x) = \min_{1 \le k \le p} w_k(x),$$

for all $x \in N$, where p may depend on x_0, and

$$(II) \qquad \Delta w_k(x) \le f(x, w_k(x), \nabla w_k(x)),$$

for all $x \in N \cap \Omega$ and $k = 1, 2, \cdots, p$. Furthermore, if $x_0 \in \partial\Omega$,

$$(III) \qquad p(x_0) w_k(x_0) + q(x_0) \frac{dw_k}{d\nu}(x_0) \ge \phi(x_0),$$

for all k.

A quasisubsolution $w : \bar{\Omega} \to \mathbf{R}$ is defined similarly, replacing min by max in (I) and reversing the inequalities (II) and (III).

To state the theorem for the existence of an intermediate solution of (BVP), we need the following notations: Let $u, v : \bar{\Omega} \to \mathbf{R}$ be functions. Then we write $u \le v$ if $u(x) \le v(x)$ for all $x \in \bar{\Omega}$, and $u < v$ if $u \le v$ but $u \ne v$. By $[u, v]$ we mean the order interval between u and v, that is,

$$[u, v] = \{ w : \bar{\Omega} \to \mathbf{R} : u \le w \le v \} .$$

The following theorem is the main result.

THEOREM 1. *Let f satisfy (1) and (2) and ϕ satisfy (3). Suppose that $\bar{v}_j$ is a quasisubsolution and $\hat{v}_j$ is a quasisupersolution of (BVP) for $j = 1, 2$ such that $\bar{v}_1 \le \hat{v}_1$, $\bar{v}_2 \le \hat{v}_2$, $\bar{v}_1 \le \hat{v}_2$ and $\hat{v}_1(x_0) < \bar{v}_2(x_0)$ for some $x_0 \in \bar{\Omega}$. Assume moreover that $\hat{v}_1$ and $\bar{v}_2$ are not solutions of (BVP) and $[\bar{v}_1(x), \hat{v}_2(x)] \subset \mathbf{I}$ for all $x \in \bar{\Omega}$. Then (BVP) has at least three distinct solutions u_j such that $\bar{v}_1 \le u_1 < u_0 < u_2 \le \hat{v}_2$, $u_j \in [\bar{v}_j, \hat{v}_j]$ for $j = 1, 2$ and $u_0 \in [\bar{v}_1, \hat{v}_2] \setminus ([\bar{v}_1, \hat{v}_1] \cup [\bar{v}_2, \hat{v}_2])$.*

Theorem 1 is a generalization of Theorem(1.6) in [1] or Theorem 2 in [3] and follows at once from the next proposition.

PROPOSITION. *Let the hypotheses of Theorem 1 hold and let* $h : \mathbf{R}^n \to \mathbf{R}^n$ *be defined by* $h(x) = (h_1(x), h_2(x), \cdots, h_n(x))$ *and bounded and of class* C^1 *such that each partial derivatives for* h_i *is bounded on* $\mathbf{R}^n$. *Then the following elliptic boundary value problem*

$$(BVP_h) \qquad \begin{cases} \Delta u = f(x, u, h(\nabla u)) & in \quad \Omega \\ Bu = \phi & on \quad \partial\Omega \end{cases}$$

has at least three distinct solutions u_0, u_1, u_2 *such that* $\bar{v}_1 \leq u_1 < u_0 < u_2 \leq \hat{v}_2$, $u_j \in [\bar{v}_j, \hat{v}_j]$ *for* $j = 1, 2$, *and* $u_0 \in [\bar{v}_1, \hat{v}_2] \setminus ([\bar{v}_1, \hat{v}_1] \cup [\bar{v}_2, \hat{v}_2])$.

To prove Proposition, we first convert (BVP_h) into an operator equation. Choose $\lambda > 0$ large enough so that $\dfrac{\partial f}{\partial \xi}(x, \xi, h(\eta)) + \lambda > 0$ for all $(x, \xi, \eta) \in \bar{\Omega} \times \mathbf{I} \times \mathbf{R}^n$. For any $g \in [\bar{v}_1, \hat{v}_2] \cap C^\alpha(\bar{\Omega})$ we assume that λ satisfies

$$f(x, c_1, h(0)) + \lambda(c_1 - g(x)) < 0 < f(x, c_2, h(0)) + \lambda(c_2 - g(x))$$

for some constants c_1, c_2 with $c_1 < 0 < c_2$ and for all $x \in \bar{\Omega}$. Furthermore, if $Bu = u$, then $\phi : \bar{\Omega} \to [c_1, c_2]$, and if not, $p(x)c_1 \leq \phi(x) \leq p(x)c_2$ for all $x \in \partial\bar{\Omega}$. Then it is known that there is a solution $u \in C^{2,\alpha}(\bar{\Omega})$ of the following boundary value problem

$$\begin{cases} \Delta u = f(x, u, h(\nabla u)) + \lambda(u - g) & \text{in} \quad \Omega \\ Bu = \phi \quad \text{on} \quad \partial\Omega. \end{cases}$$

This solution is denoted by $u = Tg$ below.

LEMMA 1. *A function* $u \in [\bar{v}_1, \hat{v}_2] \cap C^\alpha(\bar{\Omega})$ *is a solution of* (BVP_h) *if and only if* $u = Tu$.

LEMMA 2. *Let* $\bar{v}_j$ *and* $\hat{v}_j$ *be a quasisubsolution and a quasisupersolution of* (BVP_h), *respectively. Then*

$$\bar{v}_j \leq T\bar{v}_j \quad and \quad T\hat{v}_j \leq \hat{v}_j\,.$$

Proof. To show $T\hat{v}_j(x) \leq \hat{v}_j(x)$ for all $x \in \bar{\Omega}$, suppose that there is a point $x_0 \in \bar{\Omega}$ such that $\hat{v}_j(x_0) < T\hat{v}_j(x_0)$. Let $a = T\hat{v}_j(\hat{x}) - \hat{v}_j(\hat{x})$ be positive maximum value of $T\hat{v}_j - \hat{v}_j$. Then there exists a neighborhood $U_{\hat{x}}$ of $\hat{x}$ such that $\hat{x} \in U_{\hat{x}}$ and $0 < T\hat{v}_j(x) - \hat{v}_j(x) \le a$ for all $x \in U_{\hat{x}}$.

Case 1. $\hat{x} \in \Omega$.

By the definition of a quasisupersolution, there exist a neighborhood $N_{\hat{x}}$ and a finite number of functions $w_k \in C^2(N_{\hat{x}})$, $k = 1, 2, \cdots, p$ such that $\hat{x} \in N_{\hat{x}} \subset U_{\hat{x}}$ and

$$\hat{v}_j(x) = \min_{1 \le k \le p} w_k(x)$$

for all $x \in N_{\hat{x}} \cap \Omega$. Let $\hat{v}_j(\hat{x}) = w_k(\hat{x})$ for some k. Then $0 < T\hat{v}_j(x) - w_k(x) \le a$ for all $x \in N_{\hat{x}} \cap \Omega$ and $T\hat{v}_j - w_k$ has the positive maximum value a at $\hat{x}$ in the neighborhood $N_{\hat{x}} \cap \Omega$.

On the other hand, in $N_{\hat{x}} \cap \Omega$, by Mean Value Theorem,

$$\begin{aligned}
&\Delta(w_k - T\hat{v}_j)(x) \\
\le& f(x, w_k(x), h(\nabla w_k(x))) - f(x, T\hat{v}_j(x), h(\nabla T\hat{v}_j(x))) \\
&- \lambda(T\hat{v}_j - \hat{v}_j)(x) \\
\le& [f_\xi(x, \xi^*(x), h(\nabla w_k(x))) + \lambda](w_k - T\hat{v}_j)(x) \\
&+ f_\eta(x, T\hat{v}_j(x), h(\eta^*(x))) \cdot dh \cdot \nabla(w_k - T\hat{v}_j)(x).
\end{aligned}$$

Since $f_\eta \cdot dh$ is bounded on $\bar{\Omega} \times \mathbf{I} \times \mathbf{R}^n$, we can choose a bounded function $b : \mathbf{R}^n \to \mathbf{R}^n$ such that

$$f_\eta(x, T\hat{v}_j(x), h(\eta^*(x))) \cdot dh \cdot \nabla(w_k - T\hat{v}_j)(x) \le b(x) \cdot \nabla(w_k - T\hat{v}_j)(x)$$

for all $x \in N_{\hat{x}} \cap \Omega$. Hence, on $N_{\hat{x}} \cap \Omega$,

$$\Delta(w_k - T\hat{v}_j)(x) - b(x) \cdot \nabla(w_k - T\hat{v}_j)(x) \le 0.$$

By Maximum Principles, $T\hat{v}_j(x) - w_k(x) = a$ for all $x \in N_{\hat{x}} \cap \Omega$, whence $T\hat{v}_j(x) = \hat{v}_j(x) + a$ for all $x \in N_{\hat{x}} \cap \Omega$. By the continuation of the method on the boundary of $N_{\hat{x}} \cap \Omega$, we can conclude that $T\hat{v}_j(x) = \hat{v}_j(x) + a$ for all $x \in \bar{\Omega}$. And so, for any $x \in \Omega$,

$$\begin{aligned}
&\Delta\hat{v}_j(x) \\
=& \Delta T\hat{v}_j(x) \\
=& f(x, \hat{v}_j(x), h(\nabla\hat{v}_j(x))) + (f_\xi(x, \xi^*(x), h(\nabla\hat{v}_j(x))) + \lambda)a,
\end{aligned}$$

where $\xi^*(x)$ lies between $\hat{v}_j(x)$ and $\hat{v}_j(x)+a$. Hence, for any $x \in \Omega$,

$$\Delta \hat{v}_j(x) \geq f(x, \hat{v}_j(x), h(\nabla \hat{v}_j(x)))\,.$$

Since $\Delta \hat{v}_j(x) \leq f(x, \hat{v}_j(x), h(\nabla \hat{v}_j(x))$ locally in Ω, so

$$[f_\xi(x, \xi^*(x), h(\nabla \hat{v}_j(x))) + \lambda]a \leq 0\,.$$

This leads to a contradiction for $a > 0$.

Case 2. $\hat{x} \in \partial\Omega$.

Since $T\hat{v}_j(\hat{x}) = w_k(\hat{x}) + a$ and $0 < T\hat{v}_j(x) - w_k(x) < a$ for all $x \in N_{\hat{x}} \cap \Omega$, so

$$\frac{dT\hat{v}_j(\hat{x})}{d\nu} \geq \frac{dw_k(\hat{x})}{d\nu}\,.$$

If $p(\hat{x}) > 0$, then

$$\begin{aligned}\phi(\hat{x}) &= p(\hat{x})T\hat{v}_j(\hat{x}) + q(\hat{x})\frac{dT\hat{v}_j(\hat{x})}{d\nu} \\ &\geq p(\hat{x})[w_k(\hat{x}) + a] + q(\hat{x})\frac{dw_k(\hat{x})}{d\nu} \\ &\geq \phi(\hat{x}) + p(\hat{x})a\,.\end{aligned}$$

This leads to a contradiction for $p(\hat{x})a > 0$.

Let $p(\hat{x}) = 0$. Then $q(\hat{x}) > 0$. If

$$\frac{dT\hat{v}_j(\hat{x})}{d\nu} > \frac{dw_k(\hat{x})}{d\nu}\,,$$

then

$$\phi(\hat{x}) = q(\hat{x})\frac{dT\hat{v}_j(\hat{x})}{d\nu} > q(\hat{x})\frac{dw_k(\hat{x})}{d\nu} \geq \phi(\hat{x})\,.$$

This also leads to a contradiction. Let

$$\frac{dT\hat{v}_j(\hat{x})}{d\nu} = \frac{dw_k(\hat{x})}{d\nu}\,.$$

For all $x \in N_{\hat{x}} \cap \Omega$,

$$\begin{aligned}&\Delta(T\hat{v}_j - w_k - a)(x) \\ \geq& f(x, T\hat{v}_j(x), h(\nabla T\hat{v}_j(x))) - f(x, w_k(x), h(\nabla w_k(x))) \\ &+ \lambda(T\hat{v}_j - \hat{v}_j)(x)\,.\end{aligned}$$

By the Mean Value Theorem and choosing suitable bounded function $b : \mathbf{R}^n \to \mathbf{R}^n$ as before, we can also show that

$$\Delta(T\hat{v}_j - w_k - a)(x) - b(x) \cdot \nabla(T\hat{v}_j - w_k - a)(x) \geq 0 .$$

Since $T\hat{v}_j - w_k - a$ has the zero maximum value of the boundary point $\hat{x}$ in $N_{\hat{x}} \cap \Omega$, by Maximum Principles, for all $x \in N_{\hat{x}} \cap \Omega$,

$$T\hat{v}_j(x) - w_k(x) = a .$$

This implies that $T\hat{v}_j - w_k$ has the positive maximum value a at an interior point of Ω. By Case 1, this also leads to a contradiction. Therefore, $T\hat{v}_j \leq \hat{v}_j$.

Similarly, we can show that $T\bar{v}_j \geq \bar{v}_j$. ■

LEMMA 3. *Let $\bar{v}_j$ and $\hat{v}_j$ be a quasisubsolution and a quasisuperso-lution of (BVP_h), respectively. Then T is an increasing operator from $[\bar{v}_j, \hat{v}_j] \cap C^\alpha(\bar{\Omega})$ into itself, i.e. if $u \leq v$, then $Tu \leq Tv$.*

Proof. Since $[\bar{v}_j, \hat{v}_j] \cap C^\alpha(\bar{\Omega})$ is a bounded interval in $C(\bar{\Omega})$, if we choose two constants c_1 and c_2 such that $c_1 < 0 < c_2$, $c_1 - g < 0$, $c_2 - g > 0$ for all $g \in [\bar{v}_j, \hat{v}_j] \cap C^\alpha(\bar{\Omega})$, then there is $u \in [c_1, c_2] \cap C^\alpha(\bar{\Omega})$ such that $u = Tg$.

We first show that T is well-defined on $[\bar{v}_j, \hat{v}_j] \cap C^\alpha(\bar{\Omega})$. Let $u = Tg$ and $v = Tg$. Then

$$\Delta(u-v)(x) = f(x, u(x), h(\nabla u(x))) - f(x, v(x), h(\nabla v(x))) + \lambda(u-v)(x) .$$

If we choose some bounded functions $b_i : \mathbf{R}^n \to \mathbf{R}^n$, $i = 1, 2$ so that

$$\Delta(u-v)(x) - b_1(x) \cdot \nabla(u-v)(x) \leq 0 ,$$

and

$$\Delta(u-v)(x) - b_2(x) \cdot \nabla(u-v)(x) \geq 0 ,$$

for all $x \in \bar{\Omega}$, then by Maximum Principles, $u = v$.

We secondly prove that T is increasing on $[\bar{v}_j, \hat{v}_j] \cap C^{\alpha}(\bar{\Omega})$. Let $g_1, g_2 \in [\bar{v}_j, \hat{v}_j] \cap C^{\alpha}(\bar{\Omega})$ and $g_1 \leq g_2$. Then

$$\begin{aligned}&\Delta(Tg_2 - Tg_1)(x)\\ =&[f(x, Tg_2(x), h(\nabla Tg_2(x))) - f(x, Tg_2(x), h(\nabla Tg_1(x)))]\\ &+ [f(x, Tg_2(x), h(\nabla Tg_1(x))) - f(x, Tg_1(x), h(\nabla Tg_1(x)))]\\ &+ \lambda(Tg_2 - Tg_1)(x) + \lambda(g_1 - g_2)(x).\end{aligned}$$

We then choose two bounded functions $b : \mathbf{R}^n \to \mathbf{R}^n$ and $\beta : \mathbf{R}^n \to \mathbf{R}$ such that $-\lambda < \beta(x) < \lambda$ for all $x \in \bar{\Omega}$ and

$$\begin{aligned}&\Delta(Tg_2 - Tg_1)(x) - b(x) \cdot \nabla(Tg_2 - Tg_1)(x)\\ &- (\beta(x) + \lambda)(Tg_2 - Tg_1)(x) \leq \lambda(g_1 - g_2)(x) \leq 0\end{aligned}$$

for all $x \in \bar{\Omega}$. By Maximum Principles, $(Tg_2 - Tg_1)(x) \geq 0$, $x \in \bar{\Omega}$.

By Lemma 2 and that T is increasing, we note that $u = Tg \in [\bar{v}_j, \hat{v}_j] \cap C^{\alpha}(\bar{\Omega})$. ■

LEMMA 4. *Let $\bar{v}_j$ and $\hat{v}_j$ be a quasisubsolution and a quasisupersolution of (BVP_h), respectively. Then T is continuous and compact from $[\bar{v}_j, \hat{v}_j] \cap C^{\alpha}(\bar{\Omega})$ into itself.*

Proof. We first show that T is continuous. Consider a sequence $\{g_n\}$ in $[\bar{v}_j, \hat{v}_j] \cap C^{\alpha}(\bar{\Omega})$ and suppose $\lim_{n\to\infty} g_n = g$ in $[\bar{v}_1, \hat{v}_2] \cap C^{\alpha}(\bar{\Omega})$ and

$$\lim_{n\to\infty} Tg_n = y$$

in $C^{2,\alpha}(\bar{\Omega})$. Then $\lim_{n\to\infty} \Delta Tg_n = \Delta y$ and $\lim_{n\to\infty} \nabla g_n = \nabla y$ in $C(\bar{\Omega})$. Hence

$$\Delta y(x) = f(x, y(x), h(\nabla y(x))) + \lambda(y - g)(x)$$

for all $x \in \Omega$ and $By(x) = \phi(x)$ for all $x \in \partial\Omega$. By the uniqueness of solutions corresponding g, $Tg = y$. By the Closed Graph Theorem, T is continuous on $[\bar{v}_j, \hat{v}_j] \cap C^{\alpha}(\bar{\Omega})$.

We note that $C^{2,\alpha}(\bar{\Omega})$ is compactly embedded in $C^{\alpha}(\bar{\Omega})$. Hence T is compact on $[\bar{v}_j, \hat{v}_j] \cap C^{\alpha}(\bar{\Omega})$. ■

To extend the operator T to $[\bar{v}_j, \hat{v}_j] \cap C(\bar{\Omega})$ continuously, we will use the following theorem. It can be found in Amann and Crandall [3].

THEOREM 2. *Let f satisfy $(f2)$. Then there is an increasing function $\gamma : [0,\infty) \to [0,\infty)$ such that if u is a solution of (BVP_h) then*

$$\|u\|_{W^{2,p}(\Omega)} \le \gamma(\|u\|_{C(\bar{\Omega})}) .$$

Moreover, γ depends only on Δ, B, Ω, n, p, and c.

Since $C^{\alpha}(\bar{\Omega})$ is dense in $C(\bar{\Omega})$ and T is a continuous increasing compact operator from $[\bar{v}_j, \hat{v}_j] \cap C^{\alpha}(\bar{\Omega})$ into itself, we can extend T to $[\bar{v}_j, \hat{v}_j] \cap C(\bar{\Omega})$ continuously and compactly. To show that this is possible, let $u \in [\bar{v}_j, \hat{v}_j] \cap C(\bar{\Omega})$. Then there exists a monotone sequence $\{u_n\}$ in $[\bar{v}_j, \hat{v}_j] \cap C^{\alpha}(\bar{\Omega})$ so that $u_n \to u$ in $C(\bar{\Omega})$ as $n \to \infty$. Since $\{Tu_n\}$ is bounded in $C(\bar{\Omega})$, by Theorem 2, $\{Tu_n\}$ is bounded in $W^{2,p}(\Omega)$, and if $p > n$, then $\{Tu_n\}$ is bounded in $C^{1,\alpha}(\bar{\Omega})$. By the Mean Value Theorem, $\{Tu_n\}$ is equicontinuous on $C(\bar{\Omega})$. By Ascoli-Azela Theorem, $\{Tu_n\}$ has a convergent subsequence in $C(\bar{\Omega})$. Since $\{Tu_n\}$ is monotone, we can define Tu by

$$Tu = \lim_{n\to\infty} Tu_n .$$

Since Tu_n is bounded in $C^{1,\alpha}(\bar{\Omega})$, so $Tu \in C^{\alpha}(\bar{\Omega})$. Therefore, we view T as a continuous extension to an operator (denoted again by T) mapping $[\bar{v}_j, \hat{v}_j] \cap C(\bar{\Omega})$ into $[\bar{v}_j, \hat{v}_j] \cap C^{\alpha}(\bar{\Omega})$. Since the imbedding of $C^{\alpha}(\bar{\Omega})$ in $C(\bar{\Omega})$ is compact it follows that the operator T maps $[\bar{v}_j, \hat{v}_j] \cap C(\bar{\Omega})$ compactly into $[\bar{v}_j, \hat{v}_j] \cap C(\bar{\Omega})$.

To complete the proof of Proposition, we need the special ordered Banach space $C_e(\bar{\Omega})$ whose positive cone is normal and has nonempty interior. In defining $C_e(\bar{\Omega})$, $e \in C(\bar{\Omega})$, $e(x) \ge 0$ for all $x \in \bar{\Omega}$, $e(x) \ne 0$. Let $C_e(\bar{\Omega})$ be the set of all functions $u \in C(\bar{\Omega})$ so that

$$-ce(x) \le u(x) \le ce(x)$$

for some constant $c \ge 0$ and for all $x \in \bar{\Omega}$. If $u \in C_e(\bar{\Omega})$, we define the norm

$$\|u\|_e = \inf\{c > 0 : -ce(x) \le u(x) \le ce(x),\ x \in \bar{\Omega}\} .$$

It can be shown that the Minkowski functional $\|\cdot\|_e$ is a norm on $C_e(\bar{\Omega})$. Furthermore, $C_e(\bar{\Omega})$ is a Banach space with respect to the norm.(see [2])

Now we state the theorem which will be use in proving the existence of an intermediate solution of (BVP_h) for Dirichlet boundary condition. The main idea of the proof for the following theorem can be found in Amann [2].

THEOREM 3. *Let e be the unique solution of the boundary value problem*

$$\begin{cases} \Delta e(x) = -1, & x \in \Omega \\ e(x) = 0, & x \in \partial\Omega \end{cases}$$

and T be the operator induced by the boundary value problem

$$\begin{cases} \Delta Tu = f(x, Tu, h(\nabla Tu)) + \lambda(Tu - u), \; x \in \Omega \\ Tu(x) = 0, \quad x \in \partial\Omega \end{cases}$$

with a quasisubsolution $\bar{v}_j$ and a quasisupersolution $\hat{v}_j$ of (BVP_h) so that $\bar{v}_j < \hat{v}_j$. Then $C_e(\bar{\Omega})$ is continuously imbedded in $C(\bar{\Omega})$ and T is a compact operator from $[\bar{v}_j, \hat{v}_j] \cap C(\bar{\Omega})$ into $C_e(\bar{\Omega})$.

Proof. By the previous statements, T maps $[\bar{v}_j, \hat{v}_j] \cap \{u \in C(\bar{\Omega}) : u|_{\partial\Omega} = 0\}$ compactly into $[\bar{v}_j, \hat{v}_j] \cap \{u \in C^1(\bar{\Omega}) : u|_{\partial\Omega} = 0\}$. Therefore it suffices to show that $\{u \in C^1(\bar{\Omega}) : u|_{\partial\Omega} = 0\}$ is continuously imbedded in $C_e(\bar{\Omega})$. We follows the proof in [2, Theorem 4.2]. Since, by the Maximum Principle, on every compact subset of Ω, e is bounded below by a positive constant and since, for every $x \in \partial\Omega$, $\frac{\partial e}{\partial \nu} < 0$, it follows by continuity that, for every $u \in C_0^1(\bar{\Omega})$, there exist α, $\beta > 0$ with

$$-\alpha e \leq u \leq \beta e\,,$$

i.e. $C_0^1(\bar{\Omega})$ is a subset of $C_e(\bar{\Omega})$. Since convergence in the norm of $C_e(\bar{\Omega})$ as well as in the norm of $C_0^1(\bar{\Omega})$ implies pointwise convergence, it is easily seen that the injective map from $C_0^1(\bar{\Omega})$ into $C_e(\bar{\Omega})$ is a closed linear operator. Hence, by the Closed Graph Theorem, $C_0^1(\bar{\Omega})$ is continuously imbedded in $C_e(\bar{\Omega})$ and the statement follows. ■

Now, we obtain the conclusion.

THEOREM 4. *Let $\bar{v}_j$ and $\hat{v}_j$ be a quasisubsolution and a quasisupersolution of (BVP_h), respectively. Then the operator T induced by the problem (BVP_h) is continuous, increasing and compact from $[\bar{v}_j, \hat{v}_j] \cap C(\bar{\Omega})$ into itself.*

Finally, to prove Proposition, we will use the following theorem of existence of several fixed points, and we can find it in Deimling [4] or Amann [2].

THEOREM 5. *Let X be a Banach space: $S \subset X$ a retract and $T : S \to S$ compact; S_1, S_2 nonempty disjoint retracts of S; $E_j \subset S_j$ open in S for $j = 1,2$. Suppose that $T(S_j) \subset S_j$ and $Fix(T) \cap (S_j \setminus E_j) = \emptyset$ for $j = 1,2$, where $Fix(T) = \{u \in S : Tu = u\}$. Then T has fixed points $u_j \in E_j$ and a third fixed point $u_0 \in S \setminus (S_1 \cup S_2)$.*

Proof of Proposition.
Case 1. $q(x) > 0$ for all $x \in \partial\Omega$.
Let

$$O_1 = \{u \in C(\bar{\Omega}) : u(x) < \hat{v}_1(x),\ x \in \bar{\Omega}\},$$

and

$$O_2 = \{u \in C(\bar{\Omega}) : u(x) > \bar{v}_2(x),\ x \in \bar{\Omega}\},$$

$S = [\bar{v}_1, \hat{v}_2] \cap C(\bar{\Omega})$, $S_1 = [\bar{v}_1, \hat{v}_1] \cap C(\bar{\Omega})$, $S_2 = [\bar{v}_2, \hat{v}_2] \cap C(\bar{\Omega})$, $E_1 = S \cap O_1$, and $E_2 = S \cap O_2$. Then E_1 and E_2 are open in S. From Lemma 2,3,4, and Theorem 4, $T : S \to S$ is compact. Clearly, S_1 and S_2 are disjoint retracts of S, $E_j \subset S_j$, $T(S_j) \subset S_j$ for $j = 1,2$. To show that $\text{Fix}(T) \cap (S_j \setminus E_j) = \emptyset$, we assume that there is $u \in \text{Fix}(T) \cap (S_j \setminus E_j)$ for some j. Then $u \in S_j \setminus E_j$ and $Tu = u$.

Let $j = 1$. We note that u is a solution of (BVP_h). Since $u \in S_1 \setminus E_1$, so $\bar{v}_1 \le u \le \hat{v}_1$ and there is a point $x_0 \in \bar{\Omega}$ such that $u(x_0) = \hat{v}_1(x_0)$. By the definition of a quasisupersolution, let

$$\hat{v}_1(x) = \min_{1 \le k \le p} w_k(x)$$

on some neighborhood U_{x_0} of x_0 and let $u(x_0) = w_k(x_0)$ for some k. For all $x \in U_{x_0} \cap \Omega$, we can show that

$$\begin{aligned}&\Delta(u - w_k)(x) - b(x) \cdot \nabla(u - w_k)(x) - \lambda(u - w_k)(x)\\ \ge&[-G_\xi(x, \xi^*(x), h(\nabla u(x))) + \lambda](w_k - u)(x) \ge 0,\end{aligned}$$

where $b : \mathbf{R}^n \to \mathbf{R}^n$ is some bounded function. Since $\hat{v}_1$ is not a solution of (BVP_h), by Maximum Principles, $u(x) = w_k(x)$ for all $x \in U_{x_0} \cap \Omega$ and $x_0 \in \partial\Omega$. Since $u - w_k$ has a zero maximum value at the boundary point x_0, either $u(x) = w_k(x)$ and $x \in U_{x_0} \cap \bar{\Omega}$ or $\frac{dw_k}{d\nu}(x_0) < \frac{du}{d\nu}(x_0)$. We note that both cases lead to a contradiction. Consequently, $\mathrm{Fix}(T) \cap (S_1 \setminus E_1) = \emptyset$.

Similarly, we can show that $\mathrm{Fix}(T) \cap (S_2 \setminus E_2) = \emptyset$. Therefore, by Theorem 5, T has at least three distinct fixed points u_0, u_1, u_2 such that $u_j \in [\bar{v}_j, \hat{v}_j]$ for $j = 1, 2$, and especially note that

$$u_0 \in [\bar{v}_1, \hat{v}_2] \setminus [\bar{v}_1, \hat{v}_1] \cup [\bar{v}_2, \hat{v}_2].$$

Case 2. $q(x) = 0$ for all $x \in \partial\Omega$.

We assume that the Dirichlet boundary condition for (BVP_h), i.e. $Bu = u = \phi = 0$ on $\partial\Omega$. Let

$$S = C_e(\bar{\Omega}) \cap [\bar{v}_1, \hat{v}_2]$$

and

$$S_j = C_e(\bar{\Omega}) \cap [\bar{v}_j, \hat{v}_j]$$

for $j = 1, 2$. We note that $T : S \to S$ is compact; $S_j \subset S$ and nonempty; $T(S) \subset S$, $T(S_j) \subset S_j$ for $j = 1, 2$. Since S, S_1 and S_2 are convex in $C_e(\bar{\Omega})$, these are retracts of S, and clearly $S_1 \cap S_2 = \emptyset$. Let

$$E_1 = S_1 \cap \{u \in C_e(\bar{\Omega}) : u(x) < \hat{v}_1(x), x \in \Omega\}$$

and

$$E_2 = S_2 \cap \{u \in C_e(\bar{\Omega}) : u(x) > \bar{v}_2(x), x \in \Omega\}.$$

We show that E_1 and E_2 are open in S. Let $v \in E_1$. Then $v(x) < \hat{v}_1(x)$ for all $x \in \Omega$, and there is constant $c \geq 0$ such that

$$-ce(x) \leq v(x) \leq ce(x)$$

for all $x \in \bar{\Omega}$. Then we can choose $\beta \geq 0$ so that $\beta \leq c$ and $v(x) + \beta e(x) < \hat{v}_1(x)$ for all $x \in \Omega$. Let $B(v, \beta)$ be the open ball in S with respect to the norm $\| \cdot \|_e$, with center v and radius β. Then for any $u \in B(v, \beta)$,

$$-\beta e(x) \leq u(x) - v(x) \leq \beta e(x)$$

for all $x \in \bar{\Omega}$. Hence $u(x) \leq \beta e(x) + v(x) < \hat{v}_1(x)$ for all $x \in \Omega$. Hence $u \in E_1$. Therefore, $B(v, \beta) \subset E_1$.

Similarly, we can show that E_2 is also open in S.

Next, we show that $\mathrm{Fix}(T) \cap (S_j \setminus E_j) = \emptyset$, $j = 1, 2$. Suppose that there is $u \in \mathrm{Fix}(T) \cap (S_j \setminus E_j)$ for some j. Then $u \in S_j \setminus E_j$ and $Tu = u$.

Let $j = 1$. We note that u is a solution of (BVP_h). Since $u \in S_1 \setminus E_1$, $\bar{v}_1 \leq u \leq \hat{v}_1$ and there is a point $x_0 \in \Omega$ such that $u(x_0) = \hat{v}_1(x_0)$. By Maximum Principles and the definition of a quasisupersolution of (BVP_h), we can show that there is a neighborhood U_{x_0} of x_0 such that $u(x) = \hat{v}_1(x)$ for all $x \in U_{x_0} \cap \Omega$. By the continuation of this method on the boundary of the neighborhood $U_{x_0} \cap \Omega$, we can conclude that $u(x) = \hat{v}_1(x)$ for all $x \in \bar{\Omega}$. This implies that $\hat{v}_1$ is a solution of the (BVP_h). This leads to a contradiction because $\hat{v}_1$ is not a solution of (BVP_h).

Similarly, we can prove that $\mathrm{Fix}(T) \cap (S_2 \setminus E_2) = \emptyset$. Therefore, T satisfies all conditions of Theorem 5. So T has at least three distinct fixed points u_0, u_1, u_2 such that $u_j \in [\bar{v}_j, \hat{v}_j]$, $j = 1, 2$, and note that

$$u_0 \in [\bar{v}_1, \hat{v}_2] \setminus [\bar{v}_1, \hat{v}_1] \cup [\bar{v}_2, \hat{v}_2] . \quad \blacksquare$$

To prove the main theorem, we will use the following well known theorem and it can be found in [7]:

THEOREM 6. *Let f satisfy the condition $(f2)$. For every constants $P > 0$ there exists a constant $Q > 0$ such that: if u is a solution of*

$$\Delta u = f(x, u, \nabla u), \quad x \in \Omega ,$$

$u \in C^2(\bar{\Omega})$, $|u(x)| \leq P$ for all $x \in \bar{\Omega}$, then $|\nabla u(x)| \leq Q$ for all $x \in \bar{\Omega}$. The constant Q only depends on P and the bounding function c.

Proof of Theorem 1.

Since we seek solutions of (BVP_h) on the order interval $[\bar{v}_1, \hat{v}_2] \cap C(\bar{\Omega})$, we can choose $Q_0 > 0$ such that if u is a solution of $\Delta u =$

$f(x,u,\nabla u)$, for all $x \in \Omega$ and $\bar{v}_1 \le u \le \hat{v}_2$, then $|\nabla u(x)| \le Q_0$ for all $x \in \bar{\Omega}$. Since $\bar{\Omega}$ is compact, we let

$$\bar{Q}_j = \sup_{x\in\bar{\Omega}}\{\text{any directional derivatives of } \bar{v}_j \text{ at } x\} < \infty$$

and

$$\hat{Q}_j = \sup_{x\in\bar{\Omega}}\{\text{any directional derivatives of } \hat{v}_j \text{ at } x\} < \infty$$

for $j = 1,2$. Furthermore, let $Q = \max\{Q_0, \bar{Q}_1, \hat{Q}_1, \bar{Q}_2, \hat{Q}_2\}$. Then we choose a bounded smooth function $h : \mathbf{R}^n \to \mathbf{R}^n$ such that $h(\eta) = \eta$ if $|\eta| < Q+1$ and its differential dh is bounded on $\mathbf{R}^n$. To get the main result, we solve the following boundary value problem

$$\begin{cases} \Delta u = f(x,u,h(\nabla u)), & x \in \Omega \\ Bu(x) = \phi(x), & x \in \partial\Omega\,. \end{cases}$$

Hence, Proposition implies the proof. ■

REMARK. The above theorem is valid if we replace Δ by a uniformly elliptic operator

$$L = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}(x)D^{ij} + \sum_{i=1}^{n} A_i(x)D^i + A_0(x)\,,$$

where the coefficients of L and B are smooth.

3. A simple application

In this section we apply the main theorem to obtain the several positive ordered solutions in a class of singularly perturbed quasilinear elliptic Dirichlet problem. Let Ω be a bounded open domain in $\mathbf{R}^n$ with $\partial\Omega \in C^{2,\alpha}$. We look for positive classical solutions of

$$(BVP_\epsilon) \qquad \begin{cases} \epsilon^2\Delta u + g(x)|\nabla u|^2 + f(u) = 0 & \text{in} \quad \Omega \\ u = 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

where ϵ is a small positive parameter, $g : \bar{\Omega} \to \mathbf{R}$ is α-Hölder continuous, and $f : [0, \infty) \to \mathbf{R}$ is required to be C^1. More assumptions on f:

(1) There exist exactly $N(\geq 2)$ positive numbers a_i such that $a_1 < a_2 < \cdots < a_N$, $f(a_i) = 0$, and $f'(a_i) < 0$.
(2) There exists an n-dimensional open subdomain Ω_0 of Ω such that $\partial\Omega_0 \in C^{2,\alpha}$, $g(x) \geq 0$ for all $x \in \bar{\Omega}_0$
(3)
$$\int_r^{a_i} f(u)du > 0$$
for all $r \in [0, a_i)$ and $i = 1, 2, \cdots, N$.

The result of this application is the following multiplicity theorem.

THEOREM 7. *With assumptions* (1), (2), *and* (3), *there exists* $\epsilon_0 > 0$ *such that for all* ϵ *with* $0 < \epsilon \leq \epsilon_0$, (BVP_ϵ) *has* $2N - 1$ *ordered positive solutions* $u_i(x;\epsilon)$, $u_{i+\frac{1}{2}}(x;\epsilon)$, $u_{i+1}(x;\epsilon)$ *for all* $x \in \bar{\Omega}$ *and* $u_i(x;\epsilon) \to a_i$ *as* $\epsilon \to 0$ *uniformly on every compact subsets of* Ω_0.

Proof. We solve the following boundary value problem

$$\begin{cases} \epsilon^2 \Delta u + f(u) = 0 & \text{in} \quad \Omega_0 \\ u = 0 & \text{on} \quad \partial\Omega_0 \,. \end{cases}$$

By the results of Dancer[10] and Ko[6], there is $\epsilon_0 > 0$ so that for all ϵ with $0 < \epsilon \leq \epsilon_0$, the above problem has $2N - 1$ ordered positive solutions $v_i(x;\epsilon)$, $v_{i+\frac{1}{2}}(x;\epsilon)$, $v_{i+1}(x;\epsilon)$, $(i = 1, 2, \cdots, N)$, such that $v_i(x;\epsilon) \leq a_i$, $v_i(x;\epsilon) \leq v_{i+\frac{1}{2}}(x;\epsilon) \leq v_{i+1}(x\ \epsilon)$ for all $x_0 \in \bar{\Omega}_0$ and $v_i(x;\epsilon) \to a_i$ as $\epsilon \to 0$ uniformly on every compact subsets of Ω_0. Let

$$\bar{u}_i(x;\epsilon) = \begin{cases} v_i(x;\epsilon) & \text{if} \quad x \in \bar{\Omega}_0 \\ 0 \quad \text{if} \quad \bar{\Omega} \setminus \bar{\Omega}_0 \end{cases}$$

for $i = 1, 2, \cdots, N$. Since $g(x) \geq 0$ on Ω_0, so $\bar{u}_i(x;\epsilon)$, a_i and $\bar{u}_{i+1}(x;\epsilon)$, a_{i+1} are two pairs of a quasisubsolution and a quasisupersolution of (BVP_ϵ) for $0 < \epsilon \leq \epsilon_0$, respectively and satisfy all conditions in the main theorem. This completes the proof.

REMARK 1. In the Theorem 7, if we replace the condition $f \in C^1$ by the assumption that $f \in C^\alpha$ and there is a positive number r_0 such that

$$f(a_i - r) > 0 \quad \text{and} \quad f(a_i + r) < 0$$

for all r with $0 < r \le r_0$, we can also obtain the same multiplicity result by choosing a function $\hat{f} \in C^1$ so that $\sup\{|f(u) - \hat{f}(u)| : u \in [0, \infty)\}$ is sufficiently small.

REMARK 2. If $f(0) > 0$, with assumption $f'(a_i) \le 0$ for $i = 1, 2, \cdots, N$, (BVP_ϵ) has $2N-1$ ordered positive solutions as the ones of Theorem 7 by choosing a function $f_i \in C^1$ such that $\sup\{|f(u) - f_i(u)| : u \in [0, a_i]\}$ is sufficiently small and $f_i'(a_i) < 0$.

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