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WEAKLY ALMOST PERIODICITY FOR ASYMPTOTIC NONEXPANSIVE SEMIGROUPS

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I. Introduction

Let C be a nonempty closed bounded convex subset of a real Banach space X. Then a one-parameter family $\Im = \{T(t) : t \geq 0\}$ of mappings from C into itself is said to be a strongly continuous asymptotically nonexpansive semigroup on C if $\Im$ satisfies the following conditions;

(i) $T(0)x = x$(the identity on C) ;

(ii) $T(t+s)x = T(t)T(s)x$ for all x in C and $t, s \geq 0$;

(iii) for each t, there exists $k(t)$ such that $\| T(t)x - T(t)y \| \leq (1 + k(t)) \| x - y \|$ for $x, y \in C$ with $\lim_{t\to\infty} k(t) = 0$ and

(iv) for each x in C, the mapping $T(t)x$ is continuous for $t \in [0, \infty)$.

Note that if $k(t) = 0$ for all $t \geq 0$ then $\Im = \{T(t) : t \geq 0\}$ is a (continuous) nonexpansive semigroup on X. We set $F(\Im) = \{x \in C : T(t)x = x$ for all $t \geq 0\}$, and $w_w(u) = \{y \in X : \exists 0 \leq t_n \to \infty$ such that $(u(t_n))_n$ weakly converges to $y\}$. We let $\gamma(u) = \{u(t) : t \geq 0\}$ and w-cl$\gamma(u)$ denotes the weak closure of $\gamma(u)$.

For $J \in \{R, R^+\}$, we let $C_b(J, X)$ denote the usual Banach space of bounded continuous functions from J into X under the supremum norm, while $C_0(J, X)$ denotes the subspace consisting of those $f \in C_b(J, X)$ which vanish at infinity on J. Similarly, $C_b(J, X_w)$ will denote the space of all bounded, weakly continuous functions from J into X endowed with the topology of uniform convergence when X has its weak topology, and $C_0(J, X_w)$ denotes the subspace of functions which vanish at infinity with respect to the weak topology of X. Further, given a function $f: J \to X$ and $\omega \in J$, the ω-translate f_ω of f is defined by $f_\omega(t) = f(t + \omega)$, $t \in J$, and $H(f) = \{f_\omega : \omega \in J\}$ denote the set of all translates of f.

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DEFINITION 1.1. (a) A function $f \in C_b(R,X)$ [respectively, $f \in C_b(R^+,X)$] is said to be *almost periodic* (a.p.) [respectively, *asymptotically almost periodic* (a.a.p.)] if $H(f)$ is relatively compact in $C_b(R,X)$ [respectively, $C_b(R^+,X)$].[Fréchet[3]].

(b) A function $f \in C_b(R,X_w)$ [respectively, $f \in C_b(R^+,X_w)$] is said to be *weakly almost periodic* (w.a.p.) [respectively, *weakly asymptotically almost periodic* (w.a.a.p.)] if $x^* \circ f$ is almost periodic [respectively, asymptotically almost periodic] for all $x^* \in X^*$.

(c) For $J \in \{R, R^+\}$, a function $f \in C_b(J,X)$ is said to be *Eberlein-weakly almost periodic* (E.-w.a.p.) if $H(f)$ is *weakly* relatively compact in $C_b(J,X)$.[Eberlein[2]].

The spaces of X-valued functions defined in (a)-(c) of definition 1.1 will be denoted respectively by (a) $AP(R,X)$ $[AAP(R^+,X)]$,(b) $WAP(R,X)$ $[WAAP(R^+,X)]$,and (c)$W(J,X),J \in \{R,R^+\}$. We also let $W_0(J,X)$ denote subspace of $W(J,X)$ consisting of those $\varphi \in W(R^+,X)$ for which the zero function belongs to $w - clH(\varphi)$. The following are well known.

THEOREM 1.2. *[7][8] Let $u \in C_b(R^+,X)$. Then the following are equivalent:*

(i) $u \in W(R^+,X)$

(ii) there exist unique functions $g \in AP(R,X)$ and $\varphi \in W_0(R^+,X)$ such that $u = g\,|_{R+} + \varphi$.

(iii) given any sequences $((t_m,x_m^*))_m$ in $R^+ \times B_X^*$ and $(w_n)_n$ in R^+, $\lim_n \lim_m < u(t_m+w_n), x_m^* >= \lim_m \lim_n < u(t_m+w_n), x_m^* >$ whenever both iterated limits exist.

II. Almost-orbits of asymptotically nonexpansive semigroup

Bruck[2] introduced the notion of an almost-orbit of a nonexpansive mapping. Miyadera and Kobayashi[5] extended the notion of to the case of a nonexpansive semigroup on C and established the weak and strong converges of such an almost-orbit. Recently Takahashi and Zhang [11] established the weak convergence of an almost-orbit of an asymptotically nonexpansive semigroup on C in a Banach space.

DEFINITION 2.1. [11]. A continuous function $u: R^+ \to C$ is called an *almost − orbit* of $\Im = \{T(t) \mid t \geq 0\}$ if

$$\lim_{t\to\infty} \sup_{h\geq 0} \| u(t+h) - T(h)u(t) \| = 0. \tag{2.1}$$

Throughout the rest of this paper, we assume that the semigroup $\Im = \{T(t): t \geq 0\}$ is asymptotically nonexpansive semigroup on C.

The following lemmas have been proved by Takahashi and Zhang[11] and T.H.Kim[4].

LEMMA 2.2. *[4][11]. Suppose $u, v: R^+ \to C$ are almost-orbits of $\Im = \{T(t): t \geq 0\}$.*

Then (a) $\| u(t) - v(t) \|$ converges as $t \to \infty$, (b) for every $h \geq 0$, $\| u(t+h) - u(t) \|$ converges as $t \to \infty$, (c) $z \in F$, $\| u(t) - z \|$ converges as $t \to \infty$.

LEMMA 2.3. *[11]. Let X be uniformly convex and let $u: R^+ \to C$ be an almost-orbit of $\Im = \{T(t): t \geq 0\}$. Then $F(\Im) \neq \emptyset$ if and only if $\{u(t): t \geq t_0\}$ is bounded for some $t_0 \geq 0$.*

Now we prove the following lemma.

LEMMA 2.4. *If $u: R^+ \to C$ is an almost-orbit of $\Im = \{T(t): t \geq 0\}$, then u is uniformly continuous on $[0, \infty)$.*

Proof. We put $M = \sup_{t\geq 0}(1 + k(t))$. Let $\phi(t) = \sup_{s\geq 0} \| u(t+s) - T(s)u(t) \|$. Let $\epsilon > 0$ be arbitrary. Then we can choose $t_0 = t_0(\epsilon) > 0$ such that $\phi(t) < \frac{\epsilon}{3}$ for all $t \geq t_0$. We set $K = \min\{\frac{\epsilon}{3}, \frac{\epsilon}{3M}\}$. Then $K > 0$. Since u is uniformly continuous on $[0, t_0 + 1]$, there exists $\delta = \delta(\epsilon) > 0$ with $\delta < 1$ such that $\| u(t') - u(t) \| < K$ for $t, t' \in [0, t_0 + 1]$ with $| t - t' | < \delta$.

Now, let $0 \leq t' - t < \delta$. If $t \in [0, t_0]$ then $t' \in [0, t_0 + 1]$ and so $\| u(t') - u(t) \| < K < \epsilon$. If $t > t_0$, then

$$\begin{aligned}
&\| u(t') - u(t) \| \leq \| u(t') - T(t - t_0)u(t' - t + t_0) \| \\
&+ \| T(t - t_0)u(t' - t + t_0) - T(t - t_0)u(t_0) \| \\
&+ \| T(t - t_0)u(t_0) - u(t) \| \\
\leq &\phi(t' - t + t_0) + (1 + k(t - t_0)) \| u(t' - t + t_0) - u(t_0) \| + \phi(t) \\
\leq &\frac{\epsilon}{3} + M\frac{\epsilon}{3M} + \frac{\epsilon}{3} = \epsilon
\end{aligned}$$

Consequently, $\| u(t') - u(t) \| < \epsilon$ for $t, t' \in [0, \infty)$ with $| t - t' | < \delta$. That is, u is uniformly continuous on $[0, \infty)$.

THEOREM 2.5. *[6][7][8]. Assume that $\{T(t): t \geq 0\}$ is a strongly continuous semigroup of operators on a weakly closed subset C of X, and let $u : R^+ \to C$ be an almost-orbit of $\{T(t): t \geq 0\}$ such that*

(*) $T(h) |_{w-cl\gamma(u)}: w - cl\gamma(u) \to X$ is weak-to-weak continuous for each $h \in R^+$. Then $u \in W(R^+, X)$ if and only if there exist unique elements $y \in w_w(u)$ and $\varphi \in W_0(R^+, X)$ such that

(i) $u = T(\cdot)y + \varphi$

(ii) $T(\cdot)y$ is almost periodic.

III. Weakly almost periodicity for asymptotically nonexpansive semigroups.

We can now formulate the main result of this paper. The result is generalizations of Ruess and Summer [6]. We put $d = \sup_{x \in C} \| x \|$.

THEOREM 3.1. *Assume that X is uniformly convex, and let $u: R^+ \to C$ be an almost-orbit of $\Im = \{T(t) : t \geq 0\}$ such that*

(a) $\lim_{t \to \infty} \| u(t+h) - u(t) \| = \rho(h)$ exists uniformly in $h \in R^+$ and

(b) $T(h)_{|w-cl\gamma(u)}: w - cl\gamma(u) \to X$ is weak to weak continuous for every $h \geq 0$.

Then u is weakly almost periodic in the sense of Eberlein, and there exist unique elements $y \in w_w(u)$ and $\varphi \in W_0(R^+, X)$ such that

(i) $u = T(\cdot)y + \varphi$

(ii)$T(\cdot)y$ is almost periodic.

The condition (a) is called asymptotically isometic. The following proposition give the sufficient condition for u to be asymptotically isometic.

PROPOSITION 3.2. *Let $u: R^+ \to C$ be an almost-orbit of $\Im = \{T(t) : t \geq 0\}$. Assume that u satisfies one of the following conditions:*

(1) There is a sequence $(t_n)_n$ such that $t_n \to \infty$ and $\lim_{n \to \infty} \| u(t_n + h) - u(t_n) \|$ exists uniformly in $h \geq 0$

(2) There is a sequence $(t_n)_n$ such that $t_n \to \infty$ and $\{u(t_n)\}$ is strongly convergent.

Then (*) $\lim_{t\to\infty} \| u(t+h)-u(t) \| = \rho(h)$ *exists uniformly in* $h \geq 0$.

Proof. Let $\epsilon > 0$ be arbitrary. From Lemma 2.2(b), since $\lim_{s\to\infty} \| u(s+h) - u(s) \| = \rho(h)$ for every $h \geq 0$, there exists $M = M(\epsilon, h) > 0$ such that

$$\rho(h) - \frac{\epsilon}{4} < \| u(s+h) - u(s) \| < \rho(h) + \frac{\epsilon}{4}$$

for every $s > M$ and $h \geq 0$. Let $\phi(t) = \sup_{s\geq 0} \| u(t+s) - T(s)u(t) \|$. Assume that (1) holds. Then there is an integer $N = N(\epsilon)$ such that $\phi(t) < \frac{\epsilon}{4}$ and $k(t) < \frac{\epsilon}{8d}$ for every $t \geq t_N$ and $\| u(t_N + h) - u(t_N) \| < \rho(h) + \frac{\epsilon}{4}$ for every $h \geq 0$.

Now we have for every $t \geq t_N$, $s > \max(t_N, M)$ and $h \geq 0$,

$$\begin{aligned}
&\rho(h) - \frac{\epsilon}{4} \\
&< \| u(t+s+h) - u(t+s) \| \\
&\leq \| u(t+s+h) - T(s)u(t+h) \| + \| T(s)u(t+h) - T(s)u(t) \| \\
&\quad + \| T(s)u(t) - u(t+s) \| \\
&\leq \phi(t+h) + (1 + k(s)) \| u(t+h) - u(t) \| + \phi(t) \\
&< \frac{\epsilon}{4} + \| u(t+h) - u(t) \| + \frac{\epsilon}{8d} \cdot 2d + \frac{\epsilon}{4} \\
&= \frac{3\epsilon}{4} + \| u(t+h) - u(t) \| .
\end{aligned}$$

Hence for every $t \geq 2t_N$ we obtain,

$$\begin{aligned}
&\rho(h) - \epsilon \\
&< \| u(t+h) - u(t) \| \\
&\leq \sup_{w\geq t_N} \| u(t_N + w + h) - u(t_N + w) \| \\
&\leq \sup_{w\geq t_N} \| u(t_N + w + h) - T(w)u(t_N + h) \| \\
&\quad + \sup_{w\geq t_N} \| T(w)u(t_N + h) - T(w)u(t_N) \| \\
&\quad + \sup_{w\geq t_N} \| T(w)u(t_N) - u(t_N + w) \|]
\end{aligned}$$

$$\begin{aligned}&\leq \phi(t_N+h)+ \| u(t_N+h)-u(t_N) \| \\ &\quad + \sup_{w\geq t_N} k(w) \| u(t_N+h)-u(t_N) \| + \phi(t_N) \\ &< \frac{\epsilon}{4}+\rho(h)+\frac{\epsilon}{8d}\cdot 2d+\frac{\epsilon}{4}=\epsilon+\rho(h).\end{aligned}$$

This shows that the equality (*) holds.

Next, assume that (2) holds. Then

$$\begin{aligned}&\sup_{h\geq 0} |\| u(t_n+h)-u(t_n) \| - \| u(t_m+h)-u(t_m) \|| \\ &\leq \sup_{h\geq 0} \| u(t_n+h)-u(t_m+h)+u(t_m)-u(t_n) \| \\ &\leq \sup_{h\geq 0}\{\| u(t_n+h)-T(h)u(t_n) \| + \| T(h)u(t_n)-T(h)u(t_m) \| \\ &\quad + \| T(h)u(t_m)-u(t_m+h) \| + \| u(t_m)-u(t_n) \|\} \\ &\leq \phi(t_n)+\phi(t_m)+[2+k(h)] \| u(t_m)-u(t_n) \| \to 0\end{aligned}$$

as $m,n\to\infty$.

Therefore (1) holds and hence the equality (*) is satisfied.

We get the following lemma by Bruck's inequality.

LEMMA 3.3. *Suppose C is a bounded closed convex subset of a uniformly convex Banach space X. Then there exists a strictly increasing continuous convex function $\gamma\colon [0,\infty)\to[0,\infty)$ with $\gamma(0)=0$ such that*

$$\begin{aligned}&\left\| T(t)\Big(\sum_{i=1}^n \lambda_i x_i\Big)-\sum_{i=1}^n \lambda_i T(t)x_i\right\| \\ &\leq (1+k(t))\gamma^{-1}\left(\max_{1\leq i,j\leq n}\left[\| x_i-x_j \| - \frac{1}{1+k(t)} \| T(t)x_i-T(t)x_j \|\right]\right)\end{aligned}$$

for any $t\geq 0$ and any $n\geq 1$, any $\lambda_1,\cdots,\lambda_n\geq 0$ with $\sum_{i=1}^n \lambda_i=1$ and any $x_1,\cdots,x_n\in C$.

Proof. Let $S(t)x=\frac{1}{1+k(t)}T(t)x$ for $x\in C$ and $t\geq 0$ then $\{S(t)\mid t\geq 0\}$ is nonexpansive semigroups on C into C. So applying Bruck's inequality [5] for $\{S(t)\mid t\geq 0\}$, we obtain the required conclusion.

THE PROOF OF THEOREM 3.1. Our proof emploies the method of Ruess and Summer [[6] Theorem 1.4]. We would first show that u is weakly almost periodic in the sense of Eberlein. For this, according to characterization of vector valued weakly almost periodic function in the sense of Eberlein [Theorem 1.2], given sequences $((\omega_n)_n, (t_m)_m) \subset R^+ \times R^+$, and $(x_m^*)_m \subset B_{X^*} = \{x^* \in X^* : \| x^* \| \leq 1\}$ such that the limits

$$\alpha = \lim_n \lim_m < (u(t_m + \omega_n), x_m^*) > \text{ and}$$

$$\beta = \lim_m \lim_n < u(t_m + \omega_n), x_m^* >$$

both exist, it will suffice to show that $\alpha = \beta$.

Case 1:$(\omega_n)_n$ is bounded.

From Lemma 2.4, u is uniformly continuous. The proof is similiar to the case 1 in [[6] Theorem 1.4].

Case 2:$(\omega_n)_n$ is unbounded.

We may assume that $0 \leq \omega_n \uparrow \infty$. Here, we note that $< T(t_m)u(\omega_n), x_m^* >$ is bounded for all $n \in N$. By going over to subsequences and using a diagonalization argument, we can assume that $\lim_m < T(t_m)u(\omega_n), x_m^* >$ exists for all $n \in N$, and also $\lim_n < T(t_m)u(\omega_n), x_m^* >$ exists for all $m \in N$. Furthermore, we can also assume that $(u(\omega_n))_n$ converges weakly to some $x_1 \in C$. Then we easily check that

$$\alpha = \lim_n \lim_m < T(t_m)u(\omega_n), x_m^* > \text{ and}$$

$$\beta = \lim_m \lim_n < T(t_m)u(\omega_n), x_m^* > .$$

First we consider that $(t_m)_m$ is bounded. we can thus assume that $t_m \to t_0 \in R$. By Banach-Alogue Theorem, given $(x_m^*)_m$ in B_{X^*}, there exist $x_0^* \in B_{X^*}$ and a subsequence $(x_{m_i}^*)_{m_i}$ of $(x_m^*)_m$ which is ω^*-convergent to x_0^*. For $\epsilon > 0$, by (b), we choose $n_0 \in N$ so that

$$|< T(h)u(\omega_n) - T(h)x_1, x_0^* >| < \epsilon \text{ for } n \geq n_0 \text{ and } h \geq 0.$$

Now $\beta = \lim_m \lim_n < T(t_m)u(\omega_n), x_m^* >= \lim_m < T(t_m)x_1, x_m^* >$.

Again setting $\alpha_n = \lim_{m_i} < T(t_{m_i})u(\omega_n), x_{m_i}^* >$ for $n \in N$, if $n \geq n_0$ we then have that

$$\begin{aligned} | \alpha_n - \beta | &= \lim_i |< T(t_{m_i})u(\omega_n) - T(t_{m_i})x_1, x^*_{m_i} >| \\ &= |< T(t_0)u(\omega_n) - T(t_0)x_1, x_0^* >| < \epsilon. \end{aligned}$$

Thus we have $\alpha = \beta$.

Secondly we consider that $(t_m)_m$ is unbounded. We can assume that $0 \leq t_m \uparrow \infty$. Since $(u(\omega_n))_n$ is weakly convergent to $x_1 \in C$, $x_1 = \bigcap_{n=1}^{\infty} clco\{u(\omega_k): k \geq n\}$. We are going to show that for any $\epsilon > 0$ there exist $n_1 = n_1(\epsilon)$ and $m_0 = m_0(\epsilon)$ in N such that

(1) $| \alpha_n - \alpha | < \epsilon$ for all $n \geq n_0$

(2) $y = \sum \lambda_j u(\omega_j) \in co\{u(\omega_l) \mid l \geq n_1\}$ with $\| y - x_1 \| < \frac{\epsilon}{M}$ and

(3) $\| T(t_m)y - \sum \lambda_j T(t_m)u(\omega_j) \| < \epsilon$ for any $\lambda_j \geq 0$ with $\sum \lambda_j \geq 0$ and $m > m_0$.

In fact, (1) and (2) are obvious. For (3), Above $\epsilon > 0$, let $\gamma(\cdot)$ and $k(\cdot)$ be functions stated in Lemma 3.3. Then we can choose $\delta > 0$ so that $\gamma^{-1}(\delta) < \frac{\epsilon}{M}$. Since $\lim_m k(t_m) = 0$, there exists $m_0 \in N$ such that $k(t_m) < \delta/d$ for all $m \geq m_0$. By the fact that $\lim_{t\to\infty} \| u(t+h) - u(t) \| = \rho(h)$ exists uniformly in $h \geq 0$,there exists $n_2 = n_2(\delta) \in N$ such that

$$|\| u(\omega_i) - u(\omega_j) \| - \rho(| \omega_i - \omega_j |) | < \frac{\delta}{8}$$

for $i, j \geq n_2$.

Hence

$$\begin{aligned} &\| u(\omega_i) - u(\omega_j) \| - \| u(\omega_i + t_m) - u(\omega_j + t_m) \| \\ \leq &\rho(| \omega_i - \omega_j |) + \frac{\delta}{8} - \| u(\omega_i + t_m) - u(\omega_j + t_m) \| \\ \leq &\frac{\delta}{4}. \end{aligned}$$

Moreover, u is an almost-orbit of $\Im = \{T(t): t \geq 0\}$, there exists $n_3 = n_3(\delta) \in N$ such that

$$\sup_m \| u(\omega_i + t_m) - T(t_m)u(\omega_i) \| < \frac{\delta}{4}$$

for all $i \geq n_3$.

Put $n_1 = \max\{n_2, n_3\}$. If $m \geq m_0$ and $i, j \geq n_1$, then

$$
\begin{aligned}
&\| u(\omega_i) - u(\omega_j) \| - \frac{1}{1 + k(t_m)} \| T(t_m)u(\omega_i) - T(t_m)u(\omega_j) \| \\
&\leq \| u(\omega_i) - u(\omega_j) \| - \| u(\omega_i + t_m) - u(\omega_j + t_m) \| \\
&\quad + \| u(\omega_i + t_m) - T(t_m)u(\omega_i) \| + \| u(\omega_j + t_m) - T(t_m)u(\omega_j) \| \\
&\quad + k(t_m) \| u(\omega_i) - u(\omega_j) \| \leq \delta.
\end{aligned}
$$

Thus by Lemma 3.3, $\| T(t_m) \sum \lambda_j u(\omega_j) - \sum \lambda_j T(t_m) u(\omega_j) \| < \epsilon$ for any $\lambda_j \geq 0$ with $\sum \lambda_j = 1$. This proves (3).

We have from (2),

(4) $\| T(t_m)y - T(t_m)x_1 \| \leq [1 + k(t_m)] \| y - x_1 \| < \epsilon$ for $m \in N$.

From (3) and (4), we have

$$
\begin{aligned}
&\left| < \sum \lambda_j T(t_m)u(\omega_j) - T(t_m)x_1, x_m^* > \right| \\
&\leq |< \sum \lambda_j T(t_m)u(\omega_j) - T(t_m)y, x_m^* >| \\
&\quad + |< T(t_m)y - T(t_m)x_1, x_m^* >| \\
&\leq \| \sum \lambda_j T(t_m)u(\omega_j) - T(t_m)y \| + \| T(t_m)y - T(t_m)x_1 \| \\
&\leq \epsilon + \epsilon = 2\epsilon \text{ for } m > m_0.
\end{aligned}
$$

This means that

(5) $| \sum \lambda_j \alpha_j - \beta | < 2\epsilon$.

We now conclude that from (1) and (5),

$$
\begin{aligned}
| \alpha - \beta | &\leq | \alpha - \sum \lambda_j \alpha_j | + | \sum \lambda_j \alpha_j - \beta | \\
&\leq \sum \lambda_j | \alpha - \alpha_j | + 2\epsilon \\
&\leq \epsilon + 2\epsilon = 3\epsilon.
\end{aligned}
$$

The fact that u decomposed as specified in Theorem 3.1 follows from [Theorem 2.5].

The proof is thus complete.

From Theorem 3.1 we obtain the following corollary.

COROLLARY. *[6]. Assume that X is uniformly convex, and let $u: R^+ \to C$ be an almost-orbit of a nonexpansive semigroup $\Im = \{T(t) : t \geq 0\}$ such that*

(a) $\lim_{t\to\infty} \| u(t+h) - u(t) \| = \rho(h)$ exists uniformly in $h \in R^+$.

Then u is weakly almost periodic in the sense of Eberlein, and there exist unique elements $y \in w_w(u)$ and $\varphi \in W_0(R^+, X)$ such that

(i) $u = T(\cdot)y + \varphi$

(ii) $T(\cdot)y$ is almost periodic.

A function $u : R^+ \to X$ is said to be weakly almost convergent to $y \in X$ if

$$w - \lim_{T\to\infty} \frac{1}{T} \int_0^T u(t+h)dt = y \quad \text{uniformly in } h \geq 0.$$

It is well known that if u is weakly almost convergent to $y \in X$ and if $w - \lim_{t\to\infty}(u(t+h) - u(t)) = 0$, then $u(t)$ converges weakly to y [5].

THEOREM 3.4. *Assume that X is a uniformly convex, and let $u: R^+ \to C$ be an almost-orbit of $\Im = \{T(t) : t \geq 0\}$ such that*

$$T(h) \mid w - cl\gamma(u) : w - cl\gamma(u) \to X$$
is weak-to-weak continuous for each $h > 0$.

If $u \in W(R^+, X)$ and if $w - \lim_{t\to\infty}(u(t+h) - u(t)) = 0$. Then $u \in WAAP(R^+, X)$.

Proof. Since $u \in W(R^+, X)$, $u(t)$ is weakly almost convergent to $y \in X$. So $u(t)$ converges weakly to $y \in X$. i.e. $\{y\} = w_w(u)$. From the fact that X is uniformly convex and $\Im = \{T(t) : t \geq 0\}$ is asymptotically nonexpansive semigroup on a closed bounded convex subset C of X, $\{y\} = w_w(u) \subset F(\Im)$. Now from Theorem 2.5, $u = T(\cdot)y + \varphi$ where $y \in w_w(u)$ and $\varphi \in W_0(R^+, X)$. This implies that $\varphi(t)$ converges weakly to 0, i.e. $\varphi \in C_0(R^+, X_w)$. Therefore $u \in WAAP(R^+, X)$ from [lemma 2.7 [9]].

REMARK. Condition $w - \lim_{t\to\infty}(u(t+h) - u(t)) = 0$ could be replaced by

$w - \lim_{t\to\infty}(T(t+h)x - T(t)x) = 0$ for each $x \in C$. In fact, $w - \lim_{t\to\infty}(T(t+h)x - T(t)x) = 0$ implies $w - \lim_{t\to\infty}(u(t+h) - u(t)) = 0$.

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References

1. R. E. Bruck, *A simple proof of the mean ergodic theorem for nonlinear contractions in Banach spaces*, Israel J.Math. **32** (1979), 107-116.
2. W. F. Eberlein, *Abstract ergodic theorems and weak almost periodic functions*, Trans. Amer. Math. Soc. **67** (1949), 217-240.
3. M. Fréchet, *Les functions asymptotiquement presque-périodiques*, Revue. Sci. **79** (1941), 341-354.
4. T. H. Kim, *Nonlinear ergodic theorems of almost-orbits of non-lipschitzian semigroups*, Kodai Math.J. **15** (1992), 296-309.
5. I. Miyadera and K. Kobayashi, *On the asymptotic behavior of almost-orbits of nonlinear contraction semigroups in Banach spaces*, Nonlinear Analysis **6** (1982), 349-365.
.2. W. M. Ruess and W. H. Summers, *Weakly almost peridicity and the strong ergondic limit Theorem for contraction semigroups*, Israel J. Math. **64** (1988), 139-157.
7. W. M. Ruess and W. H. Summers, *Weakly almost periodic semigroups of operators*, Pac.J. of Math. **143(1)** (1990), 175-193.
8. W. M. Ruess and W. H. Summers, *Almost periodicity properties of solutions to the nonlinear Cauchy problem in Banach spaces, Semigroup theory and evolution equations(the second international conference)*, Marcel Dekker, Inc. New York, Basel, Hongkon (1991), 421-439.
9. W. M. Ruess and W. H. Summers, *Weak asymptotic almost periodicity for semigroups of operators*, J.Math.Anal. and App. **164** (1992), 242-262.
10. W. M. Ruess and W. H. Summers, *Ergodic theorems for semigroups of operators*, Proc. A.M.S. **114(2)** (1992), 423-432.
11. W. Takahashi and P. J. Zhang, *Asymptotic behavior of almost-orbits of semigroups of Lipschitzian mappings in Banach spaces*, Kodai Math.J. **11** (1988), 129-140.
12. K. K. Tan and H. K. Yu, *An ergodic theorem for nonlinear semigroups of Lipschitzian mappings in Banach spaces*, Nonlinear Anal. **19(9)** (1992).

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