ON THE GAUSS MAP OF QUADRIC HYPERSURFACES

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1. Introduction

Let $M^n$ be a connected hypersurface in Euclidean $(n + 1)$-space $E^{n+1}$, and let $G : M^n \rightarrow S^n(1) \subset E^{n+1}$ be its Gauss map. Then, according to a theorem of E.A. Ruh and J.Vilms [5], $M^n$ is a surface of constant mean curvature if and only if as a map from $M^n$ to $S^n(1)$, $G$ is harmonic, or equivalently, if and only if

$$\Delta G = \|dG\|^2 G,$$  \hspace{1cm} (1.1)

where $\Delta$ is the Laplace operator on $M^n$ corresponding to the induced metric on $M^n$ from $E^{n+1}$ and where $G$ is seen as a function from $M^n$ to $E^{n+1}$. A special case of (1.1) is given by

$$\Delta G = \lambda G, \ (\lambda \in \mathbb{R})$$  \hspace{1cm} (1.2)

that is, the case where the Gauss map $G : M^n \rightarrow E^{n+1}$ is an eigenfunction of the Laplacian $\Delta$ on $M^n$. And such hypersurfaces satisfying (1.2) were classified for some cases in [4].

On the other hand, F.Dillen, J. Pas and L. Verstraelen [3] proved that among the surfaces of revolution in $E^3$, the only ones whose Gauss map satisfy the condition

$$\Delta G = AG, \ (A \in \mathbb{R}^{3 	imes 3})$$  \hspace{1cm} (1.3)

are the planes, the spheres and the circular cylinders. And C. Baikoussis and D.E. Blair [1] recently proved that among the ruled surfaces

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in $E^3$, the only ones whose Gauss map satisfy (1.3) are the planes and the circular cylinders.

There are hyperplanes, hyperspheres and the cylinders over round spheres which satisfy the condition

$$\Delta G = AG, \quad (A \in R^{(n+1) \times (n+1)}). \quad (1.4)$$

And those examples are quadric hypersurfaces in $E^{n+1}$.

A question which arises now is: Are there any other quadric hypersurfaces in $E^{n+1}$ satisfying condition (1.4)?

In particular, we will prove the following:

**Theorem.** Among the quadric hypersurfaces in $E^{n+1}$, the only ones whose Gauss map satisfy (1.4) are the hyperplanes, the hyperspheres and the cylinders over round spheres.

Our proof of the above theorem essentially follows a reasoning which is given in [2], where B.Y. Chen, F.Dillen and H.Z. Song classified the quadric hypersurfaces of finite type.

2. Examples and preliminaries

(1) hyperplane. In this case $G$ is constant, so $\Delta G = 0$ and the hyperplane satisfies (1.4) with $A = 0$.

(2) sphere. Let $S^n(r)$ be the sphere with center 0 and radius $r$. If $x$ denotes the position vector field of $S^n(r)$, then the Gauss map $G$ is given by $\frac{1}{r}x$. Since $\Delta x = -nH$ and $H = -\frac{1}{r}G$, where $H$ is the mean curvature vector field on $S^n(r)$, we have $\Delta G = \frac{1}{r^2}G$. Hence we find that $S^n(r)$ satisfies (1.4) with $A = \text{diag} \left(1/r^2, \cdots, 1/r^2\right)$.

(3) cylinder over a round sphere. We consider the hypersurface $M = S^p(r) \times R^{n-p}$. Then as in the case of sphere, we have $\Delta G = AG$ with

$$A = \text{diag}(1/r^2, \cdots, 1/r^2, 0, \cdots, 0)(\text{with } n - p \text{ zeros }).$$
Let $M^n$ be a hypersurface in the Euclidean space $E^{n+1}$. We denote by $G$, $A$, $\sigma$ and $\alpha$ the Gauss map of $M^n$, the Weingarten map, the second fundamental form and the mean curvature of $M^n$ with respect to $G$ defined by $\alpha = \frac{1}{n} < \text{tr}(\sigma), G >$. Then we have the following ([4]):

$$\Delta G = n \nabla \alpha + |A|^2 G,$$

(2.1)

where $|A|^2$ is defined by $\text{tr}(A^2)$.

If $\Delta G = 0$, then $|A|^2 = \mu_1^2 + \cdots + \mu_n^2 = 0$, where $\mu_1, \cdots, \mu_n$ are principal curvature of $M^n$ with respect to $G$. Hence $M^n$ is totally geodesic and we obtain the following:

**Lemma 1.** The hyperplanes are the only hypersurfaces satisfying $\Delta G = 0$.

### 3. Quadric hypersurfaces

A subset $M$ of an $(n+1)$-dimensional Euclidean space $E^{n+1}$ is called a quadric hypersurface if it is the set of points $(x_1, \cdots, x_{n+1})$ satisfying the following equation of the second degree:

$$\sum_{i,j=1}^{n+1} a_{ij} x_i x_j + \sum_{i=1}^{n+1} b_i x_i + c = 0,$$

(3.1)

where $a_{ij}, b_i, c$ are all real numbers. Suppose that $M$ is not a hyperplane. Then $A$ is not a zero matrix and we may assume without loss of generality that the matrix $A = (a_{ij})$ is symmetric. By applying a coordinate transformation in $E^{n+1}$ if necessary, we may assume that (3.1) takes one of the following canonical forms:

(I) $\sum_{i=1}^{r} a_i x_i^2 + 2x_{r+1} = 0,$

(II) $\sum_{i=1}^{r} a_i x_i^2 + 1 = 0,$

(III) $\sum_{i=1}^{r} a_i x_i^2 = 0,$
where \((a_1, \cdots, a_r, 0, \cdots, 0)\) is proportional to the eigenvalues of the matrix \(A\) with \(a_1 a_2 \cdots a_r \neq 0\). In the cases where \(r = n\) in (I) and \(r = n + 1\) in (II) and (III) the hypersurface is called a properly \(n\)-dimensional quadric hypersurface, and in other cases, a quadric cylindrical hypersurface. In the case (I), the quadric cylindrical hypersurface is the product of an \((n-r)\)-dimensional linear subspace and a properly \(r\)-dimensional quadric hypersurface. In case (II) and (III), the quadric cylindrical hypersurface is the product of an \((n-r)\)-dimensional linear subspace and a properly \((r-1)\)-dimensional quadric hypersurface.

Now let \(M\) be a hypersurface in \(E^{n+1}\). We consider a parametrization
\[
x(u_1, \cdots, u_n) = (u_1, \cdots, u_n, v)
\]
where \(v = v(u_1, \cdots, u_n)\).

Denote \(\partial v/\partial u_i\) by \(v_i\). Then we have ([2])
\[
g_{ij} = \delta_{ij} + v_i v_j, \quad g^{ij} = \delta_{ij} - v_i v_j/g
\]
where
\[
g = \text{det}(g_{ij}) = 1 + \sum_{i=1}^{n} v_i^2, \quad (3.4)
\]
and \(g_{ij} = <\partial_i x, \partial_j x>\). The Laplacian \(\Delta\) of \(M\) is given by
\[
\Delta = -\sum_{i,j} \left( \frac{\partial_i g}{2g} g^{ij} + \partial_i g^{ij} \right) \partial_j - \sum_{i,j} g^{ij} \partial_i \partial_j. \quad (3.5)
\]
And the Gauss map \(G\) of \(M\) is given by
\[
G = (G_1, \cdots, G_n, G_{n+1}) = g^{-\frac{1}{2}}(-v_1, \cdots, -v_n, 1). \quad (3.6)
\]

If \(M\) is a properly \(n\)-dimensional quadric hypersurface, then \(M\) is one of the following three kinds:

(I) \(v = \frac{1}{2} \sum_{i=1}^{n} a_i u_i^2, \quad a_1 \cdots a_n \neq 0,\)

(II) \(v^2 = \sum_{i=1}^{n} a_i u_i^2 + c, \quad a_1 \cdots a_n c \neq 0,\)

(III) \(v^2 = \sum_{i=1}^{n} a_i u_i^2, \quad a_1 \cdots a_n \neq 0.\)
4. Proper quadric hypersurfaces of kind (I)

We consider the following parametrization:

\[ x = (u_1, \ldots, u_n, v), \quad v = \frac{1}{2} \sum_{i=1}^{n} a_i u_i^2, \quad a_1 \cdots a_n \neq 0. \]  

(4.1)

In this case, we have

\[ g_{ij} = \delta_{ij} + a_i a_j u_i u_j, \quad g^{ij} = \delta_{ij} - g^{-1} a_i a_j u_i u_j, \]  

(4.2)

\[ g = \det(g_{ij}) = 1 + \sum_i a_i^2 u_i^2, \]  

(4.3)

\[ \Delta = -g^{-2} \sum_i a_i^3 u_i^2 \sum_j a_j u_j \partial_j + g^{-1} \sum_i a_i \sum_j a_j u_j \partial_j - \sum_{i,j} g^{ij} \partial_i \partial_j \]  

(4.4)

and we have

\[ G = (G_1, \ldots, G_n, G_{n+1}) = g^{-\frac{1}{2}}(-a_1 u_1, \ldots, -a_n u_n, 1). \]  

(4.5)

**Lemma 2.** For each \( k = 1, \ldots, n \) we have

\[ \Delta G_k \]  

(4.6)

\[ = -a_k u_k g^{-\frac{1}{2}} \left\{ 4 \left( \sum_i a_i^3 u_i^2 \right)^2 - 2g \sum_i a_i^4 u_i^2 - g \left( \sum_i a_i \right) \sum_j a_j^3 u_j^2 \right. \]

\[ - 3g a_k \sum_i a_i^3 u_i^2 + g^2 \sum_i a_i^2 + g^2 a_k \sum_i a_i \right\}. \]

And we have

\[ \Delta G_{n+1} \]  

(4.7)

\[ = g^{-\frac{1}{2}} \left\{ 4 \left( \sum_i a_i^3 u_i^2 \right)^2 - g \sum_i a_i \sum_j a_j^3 u_j^2 - 2g \sum_i a_i^4 u_i^2 + g^2 \sum_i a_i^2 \right\}. \]

**Proof.** Note that the Gauss map \( G = (G_1, \ldots, G_n, G_{n+1}) \) is given by \( G_k = -a_k u_k g^{-\frac{1}{2}} \) for \( 1 \leq k \leq n \) and \( G_{n+1} = g^{-\frac{1}{2}} \). From (4.2) and
we may derive the above formula (4.6) and (4.7) by a straightforward computation.

Now suppose that $M$ satisfies the condition (1.4) with $A = (a_{ij})$, $1 \leq i, j \leq n + 1$. Then for each $k = 1, \ldots, n$ we have from (4.6) and (4.7)

\[ g^3 \left\{ \sum_j a_{kj} a_j u_j - a_{k(n+1)} \right\} = a_k u_k \{\ast\} \]  
\[ g^3 \left\{ -\sum_j a_{n+1j} a_j u_j + a_{n+1n+1} \right\} = \{\ast\ast\} \]  

where $\{\ast\}$ and $\{\ast\ast\}$ are the parentheses in the right side of (5.6) and (5.7), respectively. Note that $g$ is a polynomial in $u_1, \ldots, u_n$ of degree 2 and note that the left side of (4.8) is a polynomial in $u_1, \ldots, u_n$ of possible degree 0, 6 or 7 and the right side of (4.8) is a polynomial in $u_1, \ldots, u_n$ of degree less than or equal to 5. Hence we have

\[ a_{k\ell} = 0, \quad 1 \leq k \leq n, \quad 1 \leq \ell \leq n + 1. \]  

(4.10)

Similarly from (4.9) we have

\[ a_{n+1\ell} = 0, \quad 1 \leq \ell \leq n + 1. \]  

(4.11)

Thus (1.4), (4.10) and (4.11) show that $M$ satisfies the condition $\Delta G = 0$. Hence by Lemma 1, we see that $M$ is a hyperplane, which is not a quadric hypersurface of kind (I).

5. Proper quadric hypersurfaces of kind (II)

For each hypersurfaces we consider a parametrization

\[ x = (u_1, \ldots, u_n, v), \quad v^2 = a_1 u_1^2 + \cdots + a_n u_n^2 + c, \quad a_1 \cdots a_n c \neq 0. \]  

(5.1)

In this case, we have

\[ g_{ij} = \delta_{ij} + W^{-1} a_i a_j u_i u_j, \quad g^{ij} = \delta_{ij} - \tilde{g}^{-1} a_i a_j u_i u_j, \]  

(5.2)

\[ g = 1 + W^{-1} \sum_i a_i^2 u_i^2, \quad g^{-1} = 1 - \tilde{g}^{-1} \sum_i a_i^2 u_i^2. \]  

(5.3)
where
\[ W = v^2 = \sum_i a_i u_i^2, \quad \hat{g} = g W = c + \sum_i a_i (1 + a_i) u_i^2. \quad (5.4) \]

And the Gauss map \( G \) of \( M \) is given by
\[ G = (G_1, \cdots, G_n, G_{n+1}) = \hat{g}^{-\frac{1}{2}}(-a_1 u_1, \cdots, -a_n u_n, v). \quad (5.5) \]

As in Section 4, by a straightforward computation, we have the following:

**Lemma 3.** For each \( k = 1, \cdots, n \) we have
\[ \Delta G_k \]
\[ = a_k u_k \hat{g}^{-\frac{3}{2}} W^{-1} \left\{ 2\hat{g} WB - WCD - \hat{g} C E + CE^2 - a_k^2 \hat{g}^2 W \\
+ a_k \hat{g} WD + a_k \hat{g}^2 E - a_k \hat{g} E^2 + \hat{g} W \sum_j a_j^2 a_j^2 (1 + a_j) u_j^2 \\
- a_k \hat{g}^2 W a_k + 3\hat{g} W F - \hat{g}^2 W \sum_i a_i(1 + a_i) \\
- \hat{g}^2 W a_k(1 + a_k) - 3WC^2 + 2a_k \hat{g} WC \right\}. \quad (5.6) \]

And we have
\[ \Delta G_{n+1} \]
\[ = \hat{g}^{-\frac{3}{2}} W^{-\frac{3}{2}} \left\{ -2\hat{g} W^2 B + W^2 CD - \hat{g} WCE - WCE^2 \\
+ 2\hat{g}^2 WD - \hat{g} WED - 2\hat{g}^2 E^2 + \hat{g} E^3 - \hat{g} W^2 \sum_j a_i a_i^2 (1 + a_i) u_i^2 \\
+ \hat{g}^2 W \sum_i a_i a_i^2 u_i^2 + 3W^2 C^2 - 3\hat{g} W^2 F + \hat{g}^2 W^2 \sum_i a_i(1 + a_i) \\
+ 2\hat{g}^2 WC + \hat{g}^3 E - \hat{g}^3 W \sum_i a_i \right\}, \quad (5.7) \]
where
\[ B = \sum_i a_i^3 (1 + a_i) u_i^2, \quad C = \sum_i a_i^2 (1 + a_i) u_i^2, \quad D = \sum_i a_i^3 u_i^2, \quad (5.8) \]
\[ E = \sum_i a_i^2 u_i^2, \quad F = \sum_i a_i^2 (1 + a_i) u_i^2, \quad \alpha_i = \sum_{j \neq i} a_j. \]
Now suppose that $M$ satisfies the condition (1.4) with $A = (a_{ij})$, $1 \leq i, j \leq n + 1$. Then we obtain from (5.6) and (5.7)

\[
W \tilde{g}^{3} \left\{ a_{k} n_{1} W^{1} \frac{1}{2} - \sum_{\ell=1}^{n} a_{k \ell} a_{\ell} u_{\ell} \right\} = a_{k} u_{k} \{\ast\}, \quad k = 1, \cdots, n, \tag{5.9}
\]

\[
\tilde{g}^{3} \left\{ W^{\frac{3}{2}} \left( - \sum_{\ell=1}^{n} a_{n+1 \ell} a_{\ell} u_{\ell} \right) + a_{n+1} n_{1} W^{2} \right\} = \{\ast\ast\}, \tag{5.10}
\]

where $\{\ast\}$ and $\{\ast\ast\}$ are the parentheses in the right side of (5.6) and (5.7), respectively.

From (5.9) we see that $a_{k} n_{1} = 0$ for all $k = 1, \cdots, n$ and that if $a_{k \ell} \neq 0$ for some $1 \leq k, \ell \leq n$ then $\tilde{g}$ must be a constant, that is, $a_{i} = -1$ for all $i = 1, \cdots, n$. And from (5.10) we see that $a_{n+1 \ell} = 0$ for all $\ell = 1, \cdots, n$ and that if $a_{n+1 n_{1}} \neq 0$ then $\tilde{g}$ must be a constant, that is, $a_{i} = -1$ for all $i = 1, \cdots, n$.

Hence if $A$ is not a zero matrix, then $M$ is a sphere. And if $A = 0$, then by Lemma 1, $M$ is a hyperplane, which is not a quadric hypersurface of kind (II).

6. Proper quadric hypersurfaces of kind (III)

For such hypersurfaces we consider a parametrization

\[
x = (u_{1}, \cdots, u_{n}, v), \quad v^{2} = a_{1} u_{1}^{2} + \cdots + a_{n} u_{n}^{2}, \quad a_{1} \cdots a_{n} \neq 0. \tag{6.1}
\]

In Section 5 with $c = 0$, the nondegeneracy of $M$ implies that $\tilde{g} = \sum_{i=1}^{n} a_{i} (1 + a_{i}) u_{i}^{2}$ is a polynomial of degree 2, or equivalently, $a_{i} \neq -1$ for some $i = 1, \cdots, n$. And the formulae (5.9) and (5.10) are also valid with $c = 0$.

We now suppose that $M$ satisfies the condition (1.4). As in Section 5, we see that if $A \neq 0$, then we have $a_{i} = -1, \quad i = 1, \cdots, n$, which is a contradiction. And we see that if $A = 0$, then by Lemma 1, $M$ is a hyperplane, which is not a quadric hypersurface of kind (III).
7. Proof of theorem

Suppose that a quadric hypersurface $M$ satisfies the condition (1.4) and that $M$ is not a hyperplane. If $M$ is a quadric cylindrical hypersurface in $E^{n+1}$, then $M$ is the product of a proper quadric hypersurface $N^p$ in $E^{p+1}$ and a linear subspace $E^{n-p}$. Since $N$ also satisfies the condition (1.4) with a suitable square matrix, $N$ is a hypersphere $S^p(r)$ in $E^{p+1}$. This completes the proof of the theorem.

References


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