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# HOLLOW MODULES AND CORANK RELATIVE TO A TORSION THEORY

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Let $\tau$ be a given hereditary torsion theory for left $R$-module category $R$-Mod. The class of all $\tau$-torsion left $R$-modules, denoted by $\mathcal{T}$ is closed under homomorphic images, submodules, direct sums and extensions. And the class of all $\tau$-torsionfree left $R$-modules, denoted by $\mathcal{F}$, is closed under submodules, injective hulls, direct products, and isomorphic copies ([3], Proposition 1.7 and 1.10).

Notation and terminology concerning (hereditary) torsion theories on $R$-Mod will follow [3]. In particular, if $\tau$ is a torsion theory on $R$-Mod, then a left $R$-submodule $N$ of $M$ is said to be $\tau$-closed ($\tau$-dense, resp.) submodule of $M$ if and only if $M/N$ is $\tau$-torsionfree ($\tau$-torsion, resp.). A module $M$ is called $\tau$-cocritical if $M \in \mathcal{F}$ and $M/N \in \mathcal{T}$ for each nonzero submodule $N$ of $M$. A left ideal $L$ of $R$ is $\tau$-critical if $R/L$ is $\tau$-cocritical.

The purpose of this paper is to investigate some properties of $\tau$-hollow modules and their finite direct sum. We define the irredundant number of this sum of $\tau$-hollow submodules of given an $R$-module $M$ as relative corank of $M$. We study systematically some properties on relative corank of $M$. Finally we give new class of modules that has a relative supplement in the sense of Page [6].

Following Porter [9], we say ideals $I$, $J$ are $\tau$-comaximal if $I+J$ is $\tau$-dense in $R$. Let $I_1, I_2, \cdots, I_n$ be ideals of $R$, they are pairwise $\tau$-comaximal in case $I_i + I_j$ is $\tau$-dense in $R$ whenever $i \neq j$. For example, if each $I_i$ is a maximal $\tau$-closed ideal of $R$ or each $I_i$ is a $\tau$-critical ideal, then these ideals are pairwise $\tau$-comaximal.

LEMMA 1. *([8]) Let $R$ be a commutative ring and $\{I_i | i = 1, 2, \cdots, n\}$ be pairwise $\tau$-comaximal ideals of $R$. Let $M$ be any left $R$-module, then we have the following:*

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*(1)* $I_i + \cap_{j\neq i} I_j$ *is* $\tau$*-dense in* $R$ *for each* $i = 1, 2, \cdots, n$.
*(2)* $I_i M + (\cap_{j\neq i} I_j)M$ *is* $\tau$*-dense in* $M$ *for each* $i = 1, 2, \cdots, n$.

Using the above lemma, we can improve one result of Porter ([9], ) and get the following:

THEOREM 2. . *([8]). Let* $R$ *be a commutative ring and* $\{I_i | i = 1, 2, \cdots, n\}$ *be a finite family of pairwise* $\tau$*-comaximal ideals in* $R$*. For any left* $R$*-module* $M$*, we have*
*(1)* $(\prod_{i=1}^{n} I_i)M \to (\cap_{i=1}^{n} I_i)M$ *is* $\tau$*-surjective and*
*(2)* $M \to \oplus_{i=1}^{n} M/I_i M$ *is* $\tau$*-surjective with kernel* $\cap_{i=1}^{n} I_i M$.

We examine $R$-submodules $\{I_i M | i = 1, 2, \cdots, n\}$ of $M$ in the above lemma and theorem, and consider the following concept in module theoretic sense.

DEFINITION. Let $M$ be a left $R$-module, a set of left $R$-submodules of $M$ $\{A_i | i = 1, 2, \cdots, n\}$ is called $\tau$*-coindependent* in $M$ if (i) each $A_i$ is not $\tau$-dense in $M$ and (ii) $A_i + \cap_{j\neq i} A_j$ is $\tau$-dense in $M$ for each $i = 1, 2, \cdots, n$.

A family $\{A_i | i \in I\}$ of submodules of an $R$-module $M$ is said to be $\tau$*-coindependent* if every finite subfamily is $\tau$-coindependent.

For example, given pairwise $\tau$-commaximal ideals $\{I_i | i = 1, 2, \cdots, n\}$, of a commutative ring $R$, consider left $R$-submodules $\{I_i M | i = 1, 2, \cdots, n\}$; then Lemma 1(2) shows that $\{I_i M | i = 1, 2, \cdots, n\}$ is $\tau$-coindependent in $M$.

For $A \subseteq M$, we define $\tau$-closure of $A$ in $M$ (denoted by $A^c$) by $A^c / A = \tau(M/A)$, and hence $A^c$ is $\tau$-closed in $M$.

At first we give a characterization of $\tau$-coindependency of a set of countably many submodules of $M$.

LEMMA 3. *Let* $\{A_i\}$ *be a countable family of non-*$\tau$*-dense submodules of* $M$*, then the following are equivalent. When we denote* $S_{i,n} = \cap_{j\neq i} A_j$ *for* $j = 1, 2, \cdots, n (j \neq i)$
*(a) For each* $n \geq 1, A_1, A_2, \cdots, A_n$ *are* $\tau$*-coindependent;*
*(b) For each* $n \geq 1, A_1^c, A_2^c, \cdots, A_n^c$ *are* $\tau$*-coindependent;*
*(c)* $A_1, A_2, A_3, \cdots$ *are* $\tau$*-coindenpendent;*
*(d)* $A_1^c, A_2^c, A_3^c, \cdots$ *are* $\tau$*-coindenpendent;*
*(e) For each* $n \geq 1$, $A_n + S_{n,n}$ *is* $\tau$*-dense in* $M$*;*
*(f) For each* $n \geq 1$, $A_n^c + S_{n,n}^c$ *is* $\tau$*-dense in* $M$*;*

*(g) For each $n \geq 1$, $\sum_{i=1}^{n} S_{i,n}$ is $\tau$-dense in $M$;*
*(h) For each $n \geq 1$, $\sum_{i=1}^{n} S_{i,n}^{c}$ is $\tau$-dense in $M$.*

*Proof.* From the definition, we can check easily the following (a) $\Leftrightarrow$ (c), (d) $\Rightarrow$ (b) and (a) $\Rightarrow$ (e).

(a) $\Rightarrow$ (b). Since each $A_i$ is not $\tau$-dense in $M$, $A_i^c \neq M$ (of course $A_i^c$ is not $\tau$-dense in $M$). $A_i^c + \cap_{j\neq i} A_j^c$ is clearly $\tau$-dense in $M$.

(b) $\Rightarrow$ (a). By the hypothesis $A_i^c \neq M$, each $A_i$ is not $\tau$-dense in $M$.

$$(A_i + \cap_{j\neq i} A_j)^c = A_i^c + (\cap_{j\neq i} A_j)^c = A_i^c + \cap_{j\neq i} A_j^c.$$

Thus $(A_i + \cap_{j\neq i} A_j)^c$ is $\tau$-dense in $M$, which means $A_i + \cap_{j\neq i} A_j = M$.

The equivalences of (c) $\Leftrightarrow$ (d), (e) $\Leftrightarrow$ (f) and (g) $\Leftrightarrow$ (h) are similar to (a) $\Leftrightarrow$ (b).

(e) $\Rightarrow$ (g). We prove this by induction on $n$. Assume that $A_n + S_{n,n}$ is $\tau$-dense in $M$ and $\sum_{i=1}^{n-1} S_{i,n-1}$ is $\tau$-dense in $M$. Since $S_{n,n} \subset S_{i,n-1}$, we have

$$\sum_{i=1}^{n-1} S_{i,n-1} = \sum_{i=1}^{n-1} (S_{i,n} + S_{n,n}) = \sum_{i=1}^{n} S_{i,n}$$

by the modular law. So $\sum_{i=1}^{n} S_{i,n}$ is $\tau$-dense in $M$.

(g) $\Rightarrow$ (e). Since $\sum_{i=1}^{n} S_{i,n} \subset A_i + S_{i,n}$, we have that $A_i + S_{i,n}$ is $\tau$-dense in $M$ for every $i = 1, 2, \cdots, n$.

For an example of a module which has countably many $\tau$-coindependent submodules, consider the left $\mathbf{Z}$-module $Z$ (the integer set) and usual torsion theory $\tau$ (i.e., $\mathcal{T} = \{0\}$ and $\mathcal{F} = \mathbf{Z}$-Mod). Then the set $\{p\mathbf{Z} | p = \text{all the prime numbers}\}$ forms a countable $\tau$-coindependent family of ${}_{\mathbf{Z}}\mathbf{Z}$.

Following [4] or [6], we give the following;

DEFINITION. (1) The submodule $N$ of $M$ is $\tau$-*small* in $M$ if for a submodule $L$ of $M$, such that $N + L$ is $\tau$-dense in $M$, then $L$ is $\tau$-dense in $M$.

(2) We call a left $R$-module $M$ $\tau$-*hollow* if every non-$\tau$-dense submodule of $M$ is $\tau$-small in $M$.

(3) If $U$ is a submodule of $M$, we say the submodule $X$ is $\tau$-*supplement* of $U$ in $M$ if $U + X$ is $\tau$-dense in $M$, but $U + Y$ is not

$\tau$-dense in $M$ for any proper submodule $Y$ of $X$. $L$ is called a $\tau$-supplement submodule if $L$ is a $\tau$-supplement of some submodule of $M$.

(4) $M$ is said to be $\tau$-supplement if for any two submodules $U$ and $X$ of $M$, with $U + X$ is $\tau$-dense in $M$, $X$ contains a $\tau$-supplement of $U$.

REMARK. Any $\tau$-torsion submodule of $M$ is always $\tau$-small in $M$. In particular if $M$ is $\tau$-torsion, then $M$ is automatically $\tau$-hollow. If $M$ is non-$\tau$-torsion, we can mod out $\tau$-torsion part, which is always $\tau$-hollow. Thus from now on we may consider $\tau$-torsionfree, $\tau$-hollow module $M$ mainly.

The following lemma provides a criterion to check when a submodule is a $\tau$-supplement. The idea of the proof modified from Miyashita's work.

LEMMA 4. *Let $U$ and $X$ be submodules of $M$. Then $X$ is a $\tau$-supplement of $U$ if and only if $X \cap U$ is $\tau$-small in $X$.*

*Proof.* Assume that $X$ is a $\tau$-supplement of $U$, and let $U \cap X + D$ be $\tau$-dense in $X$. Thus $U + U \cap X + D = U + D$ is $\tau$-dense in $M$. By the minimality of $X$ such that $U + X$ is $\tau$-dense in $M$, $D$ is $\tau$-dense in $X$. Hence $U \cap X$ is $\tau$-small in $X$. On the other hand, assume the condition and let $U + U'$ be $\tau$-dense in $M$ with $U' \leq X$. Then $U' + U \cap X$ is $\tau$-dense in $X$, since $U \cap X$ is $\tau$-small in $X$, $U'$ is $\tau$-dense in $X$. Therefore $X$ is a $\tau$-supplement of $U$.

Now we list some facts on $\tau$-small submodules which will be used freely.

LEMMA 5. *Let $A, B$ and $C$ be submodules of $M$*

*(1) If $A$ is $\tau$-small in $B$ and $B \leq C$, then $A$ is $\tau$-small in $C$;*

*(2) $A$ is $\tau$-small in $M$, $A \leq B$ and $B$ is $\tau$-direct summand of $M$, then $A$ is $\tau$-small in $B$;*

*(3) If $A$ is $\tau$-small in $M$ and $\varphi : M \to N$ is a homomorphism, then $\varphi(A)$ is $\tau$-small in $\varphi(M)$.*

*(4) $A$ is $\tau$-small in $B$ and $B$ is $\tau$-dense in $M$, then $A$ is $\tau$-small in $M$.*

*(5) $A$ is $\tau$-dense in $B$ and $B$ is $\tau$-small in $M$, then $A$ is $\tau$-small in $M$.*

*(6) $A$ is $\tau$-small in $M$ if and only if $A^c$ is $\tau$-small in $M$.*

Now we want to list some properties on $\tau$-hollow modules.

DEFINITION. Following Garcia, $M$ is said to be $\tau$-local if and only there exists a unique maximal proper $\tau$-closed submodule of $M$.

LEMMA 6. *If $M$ is a finitely generated $\tau$-hollow module, then either $J_\tau(M) = M$ or $M$ is $\tau$-local.*

*Proof.* Since $M$ is finitely generated, there exist $x_1, x_2, \cdots, x_n$ in $M$ such that $M = \sum_{i=1}^n Rx_i$. If each $Rx_i$ is non-$\tau$-dense in $M$ for $i = 1, 2, \cdots, n$, then $\sum_{i=1}^n Rx_i$ is $\tau$-small in $M$, which implies that $M = J_\tau(M)$. On the other hand, if $Rx_j$ is non-$\tau$-dense in $M$ for $j \in J \subseteq \{1, 2, \cdots, n\}$ then $\sum_J Rx_j = J_\tau(M)$ is maximal $\tau$-closed submodule of $M$; so by [3,E24.5] $M$ is $\tau$-local.

COROLLARY 7. *If $R$ is $\tau$-hollow as left $R$-module, then $R$ is $\tau$-local.*

LEMMA 8. *(1) If $M$ is $\tau$-hollow, then $M/K$ is $\tau$-hollow for any submodule $K$ of $M$.*

*(2) If $M/V$ is $\tau$-hollow and $V$ is $\tau$-small in $M$, then $M$ is $\tau$-hollow.*

*Proof.* (1) By the definition and Lemma 5(3).

(2) Suppose that $M$ is not $\tau$-hollow module. then there exists non-$\tau$-dense, non-$\tau$-small submodule $K$ of $M$. Since $V$ is $\tau$-small in $M$, $K + V$ is non-$\tau$-dense in $M$; so $\frac{K+V}{V}$ is non-$\tau$-dense in $\frac{M}{V}$. $K + V$ is non-$\tau$-small in $M$. Also, if $K + V + L$ is $\tau$-dense in $M$ for some $L \leq M$, $K + L$ is $\tau$-dense in $M$ because $V$ is $\tau$-small in $M$. Since $K$ is non-$\tau$-small in $M$, $L$ cannot be $\tau$-dense in $M$; i.e., $\frac{K+V}{V}$ is non-$\tau$-small in $\frac{M}{V}$, which contradicts that $M/V$ is $\tau$-hollow.

REMARK. We have an example : $\frac{M}{V}$ is $\tau$-hollow but may not $\tau$-cocritical. Let $R = Z$ and $M = Z(p^\infty)$ and $V = Z(2^i)$, then $\frac{M}{V}$ is $\tau$-hollow but not $\tau$-cocritical for given usual torsion theory $\tau$.

LEMMA 9. *If $X$ is $\tau$-hollow submodule and $\tau$-dense in $M$, then $M$ is $\tau$-hollow.*

*Proof.* Let $K$ be a non-$\tau$-dense submodule of $M$. Then $K \cap X$ is non-$\tau$-dense in $X$. Since $X$ is $\tau$-hollow, $K \cap X$ is a $\tau$-small submodule of $X$. Now by the Isomorphism Theorem $K$ is $\tau$-small in $K + X$. On the other hand $K + X$ is $\tau$-dense in $M$, by Lemma 5(4), $K$ is $\tau$-small in $M$. Thus we have the result.

LEMMA 10. *Let $\{A_i\}$ be a countable family of $\tau$-coindependent submodules of $M$ and $\{B_i\}$ be a family of submodules of $M$ such that $A_i \subset B_i$ and $B_i/A_i$ is $\tau$-small in $M/A_i$ for each $i$. Then for each $n \geq 1$, $\cap_{i=1}^n B_i / \cap_{i=1}^n A_i$ is $\tau$-small is $M/\cap_{i=1}^n A_i$.*

*Proof.* For each $n \geq 1$, $A_i + S_{i,n}$ is $\tau$-dense in $M$ for $i = 1, 2, \cdots, n$ by the Lemma.

$$\begin{aligned} B_i/A_i =& (A_i + (S_{i,n} \cap B_i))/A_i \\ \cong& (S_{i,n} \cap B_i)/\cap_{i=1}^n A_i \subseteq S_{i,n}/\cap_{i=1}^n A_i \end{aligned}$$

Thus $(S_{i,n} \cap B_i)/\cap_{i=1}^n A_i$ is $\tau$-small in $S_{i,n}/\cap_{i=1}^n A_i$ for every $i = 1, 2, \cdots, n$. It follows that $(\sum_{i=1}^n (S_{i,n} \cap B_i))/\cap_{i=1}^n A_i$ is $\tau$-small in $\sum_{i=1}^n (S_{i,n}/\cap_{i=1}^n A_i)$, which is $\tau$-dense in $M$. Thus $\sum_{i=1}^n (S_{i,n} \cap B_i) = \cap_{i=1}^n B_i$ implies the result.

Mainly, we study $R$-modules in which every family of $\tau$-coindependent submodules is finite. Examples of such modules include relative artinian modules, relative hollow modules and relative local modules. Also we can see in Proposition 16 a new class of modules satisfying this finiteness condition.

LEMMA 11. *Let $M$ be an $R$-module such that every family of $\tau$-coindependent submodules is finite. Then for every $\tau$-closed submodule $U$ of $M$, there exists a $\tau$-closed submodule $V$ of $M$ containing $U$ such that $M/V$ is $\tau$-hollow.*

*Proof.* If $M/U$ is not $\tau$-hollow, there exists a non-$\tau$-dense submodule $X_1/U$ of $M/U$ which is not $\tau$-small in $M/U$. Thus there exists $L_1/U \leq M/U$. $\frac{L_1+X_1}{U}$ is $\tau$-dense in $\frac{M}{U}$, but $\frac{L_1}{U}$ is not $\tau$-dense in $\frac{M}{U}$. If $M/L_1$ is not $\tau$-hollow, pick a $\tau$-closed submodule $L_2$ and $X_2$ of $M$ containing $L_1$ such that $L_2 + X_2$ is $\tau$-dense in $M$; just as before we pick $L_1$ and $X_1$. By induction pick at the $n$-th step, in case $M/L_{n-1}$ is not $\tau$-hollow, $\tau$-closed submodules $L_n$ and $X_n$ of $M$ such that $L_n + X_n$ is $\tau$-dense in $M$. To prove that this process must stop, it suffices to prove that $L_1, L_2, \cdots, L_n$ are $\tau$-coindependent for every $n \geq 1$. First we prove the following by induction on $n$,

$$\odot \qquad X_n + (L_1 \cap \cdots \cap L_n) \text{ is } \tau\text{-dense in } M.$$

Suppose that $X_{n-1} + (L_1 \cap \cdots \cap L_{n-1})$ is $\tau$-dense in $M$. Then

$$\begin{aligned} X_n + (L_1 \cap \cdots \cap L_n) =& X_n + X_{n-1} + (L_1 \cap \cdots \cap L_n) \\ =& X_n + (L_n \cap (X_{n-1} + (L_1 \cap \cdots \cap L_{n-1}))). \end{aligned}$$

Now by the induction hypothesis, $X_n + (L_1 \cap \cdots \cap L_n)$ is $\tau$-dense in $X_n + L_n$. Since $L_n + X_n$ is $\tau$-dense in $M$, we have the result. For every $i = 1, 2, \cdots, n$, we have

$$\begin{aligned} L_i + S_{i,n} \supset& L_i + (L_1 \cap \cdots \cap L_{i-1} \cap X_i) \\ =& L_i + X_{i-1} + (L_1 \cap \cdots \cap L_{i-1} \cap X_i) \\ =& L_i + X_i \cap ((L_1 \cap \cdots \cap L_{i-1}) \cap X_{i-1}) \end{aligned}$$

By $\odot$, $(L_1 \cap \cdots \cap L_{i-1}) + X_{i-1}$ is $\tau$-dense in $M$. Thus $L_i + X_i$ is $\tau$-dense in $M$ implies that $L_i + S_{i,n}$ is $\tau$-dense in $M$. Thus $\{L_1, L_2, \cdots, L_n\}$ are $\tau$-coindependent, and this proves the existence of a submodule $V$ containing $U$ such that $M/V$ is $\tau$-hollow.

A left $R$-module $M$ is called $\tau$-semicocritical if $M$ can be embedded in a finite direct sum of $\tau$-cocritical modules. The following seems a generalization of $\tau$-semicocritical module which is characterized in Teply ([12], Proposition 1.1).

THEOREM 12. *Let $M$ be a non-$\tau$-torsion $R$-module such that every family of $\tau$-coindependent submodule is finite. Then there exists an integer $n \geq 1$ with the following (*)*

*(*) $M$ has a family $U_1, U_2, \cdots, U_n$ of $\tau$-coindependent submodules such that $M/U_1$, $M/U_2, \cdots, M/U_n$ are $\tau$-hollow and $U_1 \cap U_2 \cap \cdots \cap U_n$ is $\tau$-small in $M$.*

*Proof.* By Lemma 11, $M$ has a $\tau$-closed submodule $U_1$ such that $M/U_1$ is $\tau$-hollow. If $U_1$ is not $\tau$-small in $M$, pick a $\tau$-closed submodule $V_1$ such that $U_1 + V_1$ is $\tau$-dense in $M$ and a $\tau$-closed submodule $U_2$ of $M$ containing $V_1$ such that $M/U_2$ is $\tau$-hollow. If $U_1 \cap U_2$ is not $\tau$-small in $M$, pick a $\tau$-closed submodule $V_2$ such that $(U_1 \cap U_2) + V_2$ is $\tau$-dense in $M$ and a $\tau$-closed submodule $U_3$ of $M$ containing $V_2$ such that $M/U_3$ is $\tau$-hollow. Proceed by induction to obtain at the $n$-th step: in case $U_1 \cap U_2 \cap \cdots \cap U_{n-1}$ is not $\tau$-small in $M$, choose a $\tau$-closed submodule

$V_{n-1}$ and $U_n$ such that $V_{n-1} \subset U_n, (U_1 \cap \cdots \cap U_{n-1}) + V_{n-1}$ is $\tau$-dense in $M$ and $M/U_n$ is $\tau$-hollow.

By Lemma 3, the family of $\tau$-closed submodules $U_1, U_2, \cdots, U_n$ is $\tau$-coindependent for every $n \geq 1$. Since $M$ does not have infinite families of $\tau$-coindependent submodules, the above process must stop at the $n$-th step for some $n \geq 1$.

DEFINITION. The number $n$ defined in above theorem is called *relative corank of $M$*and it shall be denoted by $C_\tau(M) = n$.

REMARK. If $C_\tau(M) = 2$, let $\{U_1, U_2\}$ be $\tau$-coindependent submodules satisfying the conditions in the definition. Then $U_1$ and $U_2$ are mutually $\tau$-supplement. For $U_1 + U_2$ is $\tau$-dense in $M$ and $U_1 \cap U_2$ is $\tau$-small in $U_1$ and $U_2$ respectively.

COROLLARY 13. *Let $C_\tau(M) = n$ and $N$ be a $\tau$-dense submodule of $M$, then $C_\tau(N) = n$.*

*Proof.* Let $\{U_1, U_2, \cdots, U_n\}$ be $\tau$-coindependent family in $M$ satisfying the condition (*). Let's consider a family $\{U_1 \cap N, U_2 \cap N, \cdots, U_n \cap N\}$ in $N$, we can check that this family satisfies the condition (*) on $N$.

At first we check the following elementary properties on $C_\tau(M) < \infty$.

LEMMA 14. *For a left $R$ module $M$, we have the following;*
*(1) $C_\tau(M) = 0$ if and only if $M \in \mathcal{T}$.*
*(2) $C_\tau(M) = 1$ if and only if $M$ is $\tau$-hollow.*
*(3) $N$ is $\tau$-direct summand of $M$, then $C_\tau(N) \leq C_\tau(M)$.*

*Proof.* (1) If $M \in \mathcal{T}$, then $M$ does not contain any $\tau$-closed submodules. So there is no $\tau$-coindependent submodule.

On the other hand if $M \notin \mathcal{T}$, then $\tau(M) \neq M$. By Lemma 11, there exists a $\tau$-closed submodule $V$ of $M$ such that $M/V$ is $\tau$-hollow. If $V$ is $\tau$-small in $M$, then $C_\tau(M) \geq 1$, contradicts the assumption $C_\tau(M) = 0$. If $V$ is not $\tau$-small in $M$, apply Lemma 6, there exists $K \supseteq V$ such that $M/K$ is $\tau$-cocritical, which means that $K$ is $\tau$-small in $M$, i.e. $C_\tau(M) \geq 1$. Thus we have the result by the contradiction.

(2) Suppose that $C_\tau(M) = 1$. Then by the definition, every family of $\tau$-coindependent subset consists only one element, say $\{V\}$, $M/V$

is $\tau$-hollow, $V$ is $\tau$-small in $M$. By the Lemma 8(2), $M$ is $\tau$-hollow. Conversely, we may assume that $M$ has finite $\tau$-corank, say $C_\tau(M) = n$. If $M$ is not $\tau$-hollow, then by the Lemma 11, there exists a non-$\tau$-small $\tau$-closed submodule $K$ in $M$ such that $M/K$ is $\tau$-hollow. So $C_\tau(M) \geq C_\tau(M/K) = 1$, which is contradicts the hypothesis $C_\tau(M) = 1$.

(3) There exists $N' \leq M$ such that $\tau(M) \leq N'$ and $N+N'$ is $\tau$-dense in $M$. By the Corollary 13, $C_\tau(N+N') = C_\tau(M)$ and $N' \geq \tau(M)$ by using the idea of the proof (2) in the above, we can see that $C_\tau(N') \geq 1$. Thus $C_\tau(N) \leq C_\tau(M)$.

PROPOSITION 15. *Let $M$ be a left $R$-module with relative corank $n$. We have the following:*

*(a) Every $\tau$-coindependent family of submodules of $M$ has at most $n$ elements.*

*(b) A submodule $U$ of $M$ is $\tau$-small in $M$ if and only if there exists a $\tau$-coindependent family $U_1, U_2, \cdots, U_n$ of submodules of $M$ such that $M/U_i$ is $\tau$-hollow for each $i = 1, 2, \cdots, n$ and $U \subset U_1 \cap U_2 \cap \cdots \cap U_n$.*

*Proof.* (a) Let $\{N_1, N_2, \cdots, N_k\}$ be a family of $\tau$-coindependent submodules of $M$. By Lemma 11, there exist $\{U_i\}$ such that $N_i \subset U_i$ and $M/U_i$ is $\tau$-hollow for each $i = 1, 2, \cdots, k$. If $U_1 \cap U_2 \cap \cdots \cap U_k$ is not $\tau$-small in $M$, by Theorem 12, it can be lengthened to be so; hence $k \leq n$.

(b) Since $C_\tau(M) = n$, let $\{V_1, V_2, \cdots, V_n\}$ satisfy condition (*) in Theorem 12.

($\Rightarrow$) Let $U$ be a $\tau$-small in $M$, and let $U_i = U + V_i$. Then $\{U_1, U_2, \cdots, U_n\}$ satisfies the condition.

($\Leftarrow$) Take $U = U_1 \cap U_2 \cap \cdots \cap U_n$.

PROPOSITION 16. *If a $\tau$-torsion free left $R$-module $M$ has a decomposition $M = N_1 \oplus N_2$, then $C_\tau(M) = C_\tau(N_1) + C_\tau(N_2)$, in case $C_\tau(M) < \infty$.*

*Proof.* We may denote that $C_\tau(N_1) = n$, $C_\tau(N_2) = m$. By definition we can choose $\{U_i\}$, $i = 1, 2, \cdots, n$ ($\{V_j\}, j = 1, 2, \cdots, m$ resp.) be non-$\tau$-dense submodules of $N_1$ ($N_2$ resp.) satisfying the condition (*) in Theorem 7. Consider a family $\{U_i + N_2, V_j + N_1\}$ where $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, m$.

$$(U_i + N_2) + (\cap_{k\neq i}(U_k + N_2) \cap (\cap_{j=1}^{m}(V_j + N_1)) \supseteq U_i + N_2 + \cap_{k\neq i} U_k.$$

Now by (*), we can see that $(U_i+N_2)+(\cap_{k\neq i}(U_k+N_2)\cap(\cap_{j=1}^{m}(V_j+N_1))$ is $\tau$-dense in $M$.

$$\frac{M}{U_i+N_2}=\frac{N_1\oplus N_2}{U_i+N_2}\cong\frac{N_1}{U_i}:\tau\text{-hollow}$$

thus $M/(U_i+N_2)$ is $\tau$-hollow. Similarly $M/(V_j+N_1)$ is $\tau$-hollow.

$$\begin{aligned}(\cap_{i=1}^{n}(U_i+N_2))\cap(\cap_{j=1}^{m}(V_j+N_1)) &=(\cap_{i=1}^{n}U_i+N_2)\cap(\cap_{j=1}^{m}V_j+N_1)\\ &=\cap_{i=1}^{n}U_i+\cap_{j=1}^{m}V_j\end{aligned}$$

which is $\tau$-small in $N_1+N_2$, so $(\cap_{i=1}^{n}(U_i+N_2))\cap(\cap_{j=1}^{m}(V_j+N_1))$ is $\tau$-small in $M$. Thus $C_\tau(M)=n+m$.

COROLLARY 17. *Let $N_1,N_2,\cdots,N_k$ be $\tau$-coindependent family of submodules of an $R$-module $M$ such that $\cap_{i=1}^{k}N_i=0$. Then*

$$C_\tau(M)=\sum_{i=1}^{k}C_\tau(M/N_i).$$

*Proof.* For any two $\tau$-coindependent submodules $N_1$ and $N_2$ of $M$, there exists an exact sequence,

$$0\to\frac{M}{N_1\cap N_2}\to\frac{M}{N_1}\oplus\frac{M}{N_2}\to\frac{M}{N_1+N_2}\to 0.$$

Thus $\frac{M}{N_1\cap N_2}$ is $\tau$-dense in $\frac{M}{N_1}\oplus\frac{M}{N_2}$. Since $\{N_1,N_2,\cdots,N_k\}$ is a $\tau$-coindependent family, inductively we have that $\frac{M}{\cap_{i=1}^{k}N_i}$ is $\tau$-dense in $\oplus_{i=1}^{k}\frac{M}{N_i}$. Since $\cap_{i=1}^{k}N_i=0$, we can say that $M$ is $\tau$-dense in $\oplus_{i=1}^{k}\frac{M}{N_i}$. By Corollary 13 and Proposition 16, we have the following;

$$C_\tau(M)=C_\tau(\oplus_{i=1}^{k}\frac{M}{N_i})=\sum_{i=1}^{k}C_\tau(\frac{M}{N_i}).$$

PROPOSITION 18. *$C_\tau(M) = n$ is equivalent to (**):*
*(**) There exist a family $H_1, H_2, \cdots, H_n$ of $\tau$-hollow $R$-modules and $f : M \to \oplus_{i=1}^n H_i$ such that $imf$ is $\tau$-dense in $\oplus_{i=1}^n H_i$ and $kerf$ is $\tau$-small in $M$.*

*Proof.* ($\Rightarrow$) Since $C_\tau(M) = n$, there exists $\tau$-coindependent family $\{U_1, U_2, \cdots, U_n\}$ satisfying (*). Consider an $R$-homomorphism $f : M \to \oplus_{i=1}^n \frac{M}{U_i}$, $\ker f = \cap_{i=1}^n U_i$ is $\tau$-small in $M$. $\operatorname{im} f \cong \frac{M}{\cap_{i=1}^n U_i}$ is $\tau$-dense in $\oplus_{i=1}^n \frac{M}{U_i}$ in the proof of Corollary 17.

($\Leftarrow$) By hypothesis $M/kerf$ is $\tau$-dense in $\oplus_{i=1}^n H_i$, where each $H_i$ is $\tau$-hollow module. By Lemma 14(1), and Proposition 16,

$$C_\tau(M) = C_\tau(M/kerf) = C_\tau(\oplus_{i=1}^n H_i) = \sum_{i=1}^n C_\tau(H_i) = n.$$

LEMMA 19. *If $C_\tau(M) = n < \infty$ and $K$ is $\tau$-supplemented in $M$, then $C_\tau(M) = C_\tau(K) + C_\tau(M/K)$.*

*Proof.* Since $K$ is $\tau$-supplemented, there exists a $\tau$-closed submodule $W$ in $M$ such that $K + W$ is $\tau$-dense in $M$ and $K \cap W$ is $\tau$-small in $K$; thus $K \cap W$ is $\tau$-small in $M$. Using the short exact sequence,

$$0 \to \frac{M}{K \cap W} \to \frac{M}{K} \oplus \frac{M}{W} \to \frac{M}{K + W} \to 0.$$

We have $\frac{M}{K \cap W} \cong \frac{M}{K} \oplus \frac{M}{W}$, and since $K \cap W$ is $\tau$-small in $M$, $C_\tau(M) = C_\tau(\frac{M}{K \cap W}) = C_\tau(\frac{M}{K}) + C_\tau(\frac{M}{W})$. And $\frac{K}{K \cap W}$ is isomorphic to $\frac{K+W}{W}$ which is $\tau$-dense in $\frac{M}{W}$. Thus we have $C_\tau(\frac{M}{W}) = C_\tau(\frac{K}{K \cap W}) = C_\tau(K)$, using the fact $K \cap W$ is $\tau$-small in $K$. We have the result $C_\tau(M) = C_\tau(K) + C_\tau(M/K)$.

PROPOSITION 20. *If $M \in \mathcal{F}$ and $C_\tau(M) = n$, then $M/J_r(M)$ is $\tau$-semicocritical.*

*Proof.* If $J_\tau(M) = 0$, since $C_\tau(M) = n$ there exists a family $\{U_1, U_2, \cdots, U_n\}$ $\tau$-coindependent submodules of $M$ such that $M/U_i$ is $\tau$-hollow for each $i = 1, 2, \cdots, n$ and $\cap_{i=1}^n U_i$ is $\tau$-small in $M$. By ([3], 24.3) $J_\tau(M) \supset \cap_{i=1}^n U_i$; so $\cap_{i=1}^n U_i = 0$. Consider an exact sequence

$$0 \to \frac{M}{\cap_{i=1}^n U_i} \to \oplus \frac{M}{U_i} \to \frac{M}{\sum_{i=1}^n M_i} \to 0.$$

We have $M \cong \oplus_{i=1}^{n} \frac{M}{U_i}$. Since $\frac{M}{U_i}$ is $\tau$-hollow, thus $\tau$-supplemented. So $M$ is $\tau$-supplemented; i.e., for any non-$\tau$-dense submodule $A$ of $M$, there exists $A'$ in $M$ such that $A + A'$ is $\tau$-dense in $M$ and $A \cap A'$ is $\tau$-small in $A'$. So $A \cap A'$ is $\tau$-small in $M$, which implies $A \cap A' = 0$. Consequently $A \oplus A'$ is $\tau$-dense in $M$. From the above, we can say that each $\tau$-closed submodule is $\tau$-direct summand of $M$. Thus $M$ is $\tau$-semisimple. Now we want to see that $M$ is $\tau$-artinian. Suppose that $M$ is not $\tau$-artinian, then $M$ contains a direct sum of infinitely many submodules; i.e., $\oplus_{i=1}^{\infty} N_i$ is $\tau$-dense in $M$. By the Lemma 14, $C_\tau(M) \geq \sum_{i=1}^{\infty} C_\tau(N_i)$. Since $N_i \notin \mathcal{T}$, $C_\tau(N_i) \geq 1$. By the [3, 24.10] $M$ is $\tau$-semicocritical.

We also consider the following problem.

PROPOSITION 21. *Let $C_\tau(M) = n < \infty$, and let $f : M \to M$ be epimorphism. Then we have the following statements:*
*(a) ker $f$ is $\tau$-small in $M$;*
*(b) if $M$ is projective $\tau$-torsionfree, then $f$ is isomorphism.*

*Proof.* (a) Note that $M/\ker f \cong M$, thus $C_\tau(M/\ker f) = C_\tau(M)$. By Proposition 15(6), ker $f$ is $\tau$-small in $M$.

(b) By the following short exact sequence and projectivity of $M$.

$$0 \to \ker f \to M \to M/\ker f \cong M \to 0.$$

ker $f$ is a direct summand of $M$. So $\ker f \oplus M_1 \cong M$, where $M_1 \cong M$. If $M \in \mathcal{F}$ and $\ker f \neq 0$, then $C_\tau(\ker f) \neq 0$ by Lemma 14(1), which is a contradiction. So $\ker f = 0$, which implies (b).

COROLLARY 22. *Let $\tau$ be a faithful torsion theory and let $R$ have finite $\tau$-corank. Then an epimorphism $f : R \to R$ is an isomorphism.*

DEFINITION. A left $R$-module $M$ has *property* (P) with respect to a given torsion theory $\tau$, if for any strictly descending chain of $\tau$-closed submodules $M \supset U_0 \supset U_1 \supset U_2 \supset \cdots$, there exists $i$ such that $U_j$ is $\tau$-small in $M$ for all $j \geq i$.

For example, if $K = R/J_\tau(R)$ is semisimple $\tau$-artinian, then ${}_K K$ has property (P) with respect to a torsion theory induced by $\bar{\tau}$ on $R/J_\tau(R)$.

LEMMA 23. *Let $M$ have property (P). If $N$ is a submodule of $M$ and $N+X$ is $\tau$-dense in $M$, then $X$ contains a $\tau$-supplement of $M$.*

*Proof.* If $N$ is a $\tau$-small submodule of $M$, then $X$ is $\tau$-dense in $M$. So the $\tau$-supplements of $N$ is any submodule of $X$. If $N$ is non-$\tau$-dense non-$\tau$-small submodule of $M$, we can find a non-$\tau$-dense in $M$ and $N+X$ is $\tau$-dense in $M$. If $X$ is a $\tau$-supplement of $N$, we are done; otherwise, we can obtain an $X_2$. Continuously we have the chain $X \supseteq X_1 \supseteq X_2 \supseteq \cdots$. By property (P), we have only finitely many non-$\tau$-small submodules. Then $X$ must contain a $\tau$-supplments of $N$. Finally if $N$ is $\tau$-dense in $M$ and $N+X$ is $\tau$-dense, then any submodule of $X$ is a $\tau$-supplement of $N$.

COROLLARY 24. *If $M$ has property (P) then every non-$\tau$-dense submodule of $M$ has a $\tau$-supplement.*

COROLLARY 25. *If $M$ is $\tau$-semicocritical, then every non-$\tau$-dense submodule of $M$ has a $\tau$-supplement.*

Now we want to give a relation between $M$ with finite relative corank and property (P).

PROPOSITION 26. *If $M$ has property (P), then $M$ has finite relative corank. On the other hand, if $M$ is $\tau$-supplemented, the converse is also true.*

*Proof.* ($\Rightarrow$) Suppose that $M$ has property (P) and $C_\tau(M)$ is not finite, then there exist a set of infinitely many $\tau$-coindependent submodules $\{X_1, X_2, X_3, \cdots\}$. By Lemma 3, we may consider each $X_i$ is $\tau$-closed in $M$, and make a strictly descending chain of $\tau$-closed submodules $M \supset X_1 \supset X_1 \cap X_2 \supset X_1 \cap X_2 \cap X_3 \supset \cdots$. By condition (P), there exists some $k$ such that $\cap_{i=1}^k X_i$ is $\tau$-small in $M$. By Lemma 3, $X_{k+1} + \cap_{i=1}^k X_i$ is $\tau$-dense in $M$ which implies that $X_{k+1}$ is $\tau$-dense in $M$. This contradicts that each $X_j$ is $\tau$-closed in $M$.

($\Leftarrow$) Let $\{U_1, U_2, \cdots, U_n\}$ be a family of $n$ $\tau$-coindependent submodules of $M$ such that $M/U_1, M/U_2, \cdots, M/U_n$ is $\tau$-hollow and $U_1 \cap U_2 \cap \cdots \cap U_n$ is $\tau$-small in $M$. For each $i = 1, 2, \cdots, n$, we know that $U_i + S_{i,n}$ is $\tau$-dense in $M$, where $S_{i,n} = \cap_{j \neq i} U_j$ and $M$ is $\tau$-supplemented. There exists $S_i (\leq S_{i,n})$, which is $\tau$-supplement of $U_i$ and $U_i + S_i$ is $\tau$-dense in $M$ and $U_i \cap S_i$ is $\tau$-small in $S_i$.

$$S_i + \cap_{i=1}^n U_i = S_i + U_i \cap S_{i,n} = S_{i,n} \cap (U_i + S_i)$$

which is $\tau$-dense in $S_{i,n} \cap M = S_{i,n}$. And by Lemma 3, $S_{i,n} + S_{2,n} + \cdots + S_{n,n}$ is $\tau$-dense in $M$; thus $S_1 + S_2 + \cdots + S_n + \cap_{j=1}^{n} U_i$ is $\tau$-dense in $M$. Since $\cap_{i=1}^{n} U_i$ is $\tau$-small in $M$, we have that $S_1 + S_2 + \cdots + S_n$ is $\tau$-dense in $M$.

Now we claim that each $S_i$ is $\tau$-hollow. $\frac{S_i}{S_i \cap U_i} \cong \frac{S_i + U_i}{U_i}$, which is $\tau$-dense in $\tau$-hollow module $\frac{M}{U_i}$ by the lemma 8(2), $\frac{S_i}{S_i \cap U_i}$ is $\tau$-hollow and $S_i \cap U_i$ is $\tau$-small in $S_i$ implies that $S_i$ is $\tau$-hollow.

If there exists $k < n$ such that $S_1 + S_2 + \cdots + S_k$ is $\tau$-dense in $M$, then $\frac{S_1 \oplus S_2 \oplus \cdots \oplus S_k}{K}$ is $\tau$-dense in $M$ for some $K \subseteq M$.

$$C_\tau(M) = C_\tau\left(\frac{S_1 \oplus S_2 \oplus \cdots \oplus S_k}{K}\right) \le C_\tau(S_1 \oplus S_2 \oplus \cdots \oplus S_k) = k$$

which contradicts $C_\tau(M) = n > k$. Thus the sume $S_1 + S_2 + \cdots + S_n$ is irredundant sum of $\tau$-hollows and $\tau$-dense in $M$ implies that $M$ satisfies the property (P).

If $M$ has property (P), then there is $n$ irredundant $\tau$-dense sum of $\tau$-hollow submodules $H_i$ of $M$.

Following [4], we list the following concepts.

DEFINITION. A left $R$-module $P$ is called strongly $\tau$-projective if $Q_\tau(P)$ is a projective object in $(R,\tau)$-Mod; concretely, for a given diagram

$$\begin{array}{ccc} & & P \\ & \swarrow & \downarrow \\ N & \xrightarrow[f]{} & N' \end{array}$$

where $N$ is a $\tau$-closed module, $N'$ is a $\tau$-torsionfree module and im$f$ is $\tau$-dense in $N'$, can be completed commutatively. For a given left $R$-module $M$, an $R$-homomorphism $\varepsilon : P \to M$ is called *$\tau$-projective cover of* $M$ if $P$ is strongly $\tau$-projective and im$\varepsilon$ is $\tau$-dense in $M$ and ker$\varepsilon$ is $\tau$-small in $P$. Usually we denote $(P, \varepsilon)$ as the $\tau$-projective cover of $M$.

LEMMA 27. *Let $N$ be a $\tau$-torsion submodule in $M$. Then $N$ is $\tau$-small in $M$.*

*Proof.* If $N$ is not $\tau$-small in $M$, then there exists $\tau$-closed submodule $B$ in $M$ such that $N + B$ is $\tau$-dense in $M$. $\frac{N+B}{B} \cong \frac{N}{N \cap B} \in \mathcal{T}$, since homomorphic image of torsion module $N$. On the other hand, $B$ is $\tau$-closed in $M$. Thus $\frac{N+B}{B} \in \mathcal{F}$, so we have that $N + B = B$; i.e., $N \leq B$ which contradicts that $B$ is $\tau$-closed in $M$.

LEMMA 28. *Let $\tau$ be a torsion theory on $R$-Mod. Let $A$ and $B$ be $\tau$-closed submodules of $M$ such that $Q_\tau(M) = Q_\tau(A) \oplus A_\tau(B)$. Then $A$ and $B$ are mutually $\tau$-supplement in $M$, (i.e., $A \cap B$ is $\tau$-small in $A$ and $B$ respectively and $A + B$ is $\tau$-dense in $M$.) If $M$ is strongly $\tau$-projective, then we have the converse also.*

*Proof.* Consider an exact sequence

$$0 \to \frac{A+B}{\tau(A+B)} \to \frac{M}{\tau(M)} \to \frac{M/\tau(M)}{(A+B)/\tau(A+B)} \to 0.$$

Apply the left exact functor $Q_\tau(-)$:

$$0 \to Q_\tau(A+B) \to Q_\tau(M) \to Q_\tau\Big(\frac{M}{A+B}\Big).$$

By the hypothesis $Q_\tau(M) = Q_\tau(A) \oplus Q_\tau(B)$, we have that $Q_\tau(A+B) = Q_\tau(M)$; i.e., $A + B$ is $\tau$-dense in $M$. If $x \in A \cap B \backslash \tau(A \cap B)$, then $x + \tau(A \cap B)$ is a non-zero element of $\frac{A \cap B}{\tau(A \cap B)}$ that is contained in $Q_\tau(A \cap B) \subseteq Q_\tau(A) \cap Q_\tau(B) = 0$, which is contradiction. Thus $A \cap B$ is $\tau$-torsion in $A$ and $B$ respectively. Applying the Lemma 27, we have that $A \cap B$ is $\tau$-small in $A$ and $B$ respectively.

For the converse part, consider a $\tau$-surjection $f : A \oplus B \to M$ given by $f(a, b) = a - b$. Then $\frac{A \oplus B}{kerf} \cong A + B$ is $\tau$-dense in $M$. And $kerf = (A \cap B) \oplus (A \cap B)$, which is $\tau$-small in $A \oplus B$, since $M$ is strongly $\tau$-projective $Q_\tau(kerf)$ is direct summand of $Q_\tau(M)$. So we have $\ker f \subset \tau(M)$. Thus $A \oplus B$ is $\tau$-dense in $M$; i.e., $Q_\tau(A \oplus B) = Q_\tau(M)$.

PROPOSITION 29. *If $M$ has property $(P)$ and is strongly $\tau$-projective, then the sum representation of $M$ in Proposition 26 is direct, and the direct summands $S_1, S_2, \cdots, S_n$ are unique up to isomorphism.*

*Proof.* We prove by induction on $n$ that the sum $S_1 + S_2 + \cdots + S_n$ is direct in $M$. It is clear from the construction of $S_1, S_2, \cdots, S_n$ in Proposition 26 that $S_i \cap V_i$ is $\tau$-small in $S_i$ for every $i = 1, 2, \cdots, n$, where $V_i = \sum_{j \neq i} S_j$. Also it is clear that $S_1 \cap V_1 \subset (S_2 \cap V_2) + \cdots + (S_n \cap V_n)$ which is $\tau$-small in $V_1$. Hence $S_1 \cap V_1$ is $\tau$-small in $V_1$. By Lemma 28, $S_1 \oplus V_1$ is $\tau$-dense in $M$. We can check that $V_1$ satisfies the condition $(P)$ and by the induction hypothesis $S_2 \oplus S_3 \oplus \cdots \oplus S_n$ is $\tau$-dense in $V_1$, thus $S_1 \oplus S_2 \oplus \cdots \oplus S_n$ is $\tau$-dense in $M$.

COROLLARY 30. *Let $P$ be a $\tau$-projective cover of a module $M$ $\tau$-corank $n$. Then $C_\tau(P) = n$.*

*Proof.* Consider a covering map $\varepsilon : P \to M$, then $\ker \varepsilon$ is $\tau$-small in $P$ and $im\varepsilon$ is $\tau$-dense in $M$. Applying the Isomorphism Theorem with Corollary 13 and Proposition 15(6), we have the following;

$$C_\tau(P) = C_\tau\left(\frac{P}{ker\varepsilon}\right) = C_\tau(im\varepsilon) = C_\tau(M) = n.$$

Recently, we found the following proposition which is essentially the same statement in [13]. But the method of the proof is different. So, we record it.

PROPOSITION 31. *([13, Theorem 14]). Let $A, B \leq M$ such that $A + B$ is $\tau$-dense in $M$ and let $(P, f)$ is $\tau$-projective cover of $M/A$. Then $A$ has a $\tau$-supplement $C$ in $B$.*

*Proof.* Let $f : P \to M/A$ be a $\tau$-projective cover. Define $g : B \to M/A$ by the mapping $g(b) = b + A$. By the strongly $\tau$-projectiveness of $P$, there exists a homomorphism $h : P \to B$ making the following diagram commutative

$$\begin{array}{ccc} & & B \\ & \overset{h}{\nearrow} & \downarrow \\ P & \underset{f}{\longrightarrow} & M/A \end{array}$$

Let $C = imh$. We claim that (i) $A + C$ is $\tau$-dense in $M$ and (ii) $A \cap C$ is $\tau$-small in $C$. Then by the Lemma , we can say that $C$ is a $\tau$-supplement of $A$ in $B$.

For (i), let $b \in B$, $x \in P$ such that $g(b) = f(x)$ then $h(x) - b \in kerg = A \cap B$. Hence $b \in C + (A \cap B)$ implies that $B = C + (A \cap B)$ $A + C = A + B$ is $\tau$-dense in $M$.

For (ii), let $D \leq C$ such that $A \cap C + D$ is $\tau$-dense in $C$, and let $D' = h^{-1}(D) \subset h^{-1}(B)$ if $D'$ is $\tau$-dense in $P$. Since the $\tau$-density is preserved by homomorphism, we are done. Let $x \in P$, $h(x) \in C$ and $A \cap C + D$ is $\tau$-dense in $C$. There exist a $\tau$-dense left ideal $I$ of $R$ such that $Ih(x) \subseteq A \cap C + D$; i.e. $h(Ix) \subseteq A \cap C + D$. Note that $A \cap C \subseteq B \cap A = kerg$. $h(Ix) \subseteq kerg + D$and so

$$\begin{aligned} Ix \subseteq h^{-1}(D) + h^{-1}(kerg) &= h^{-1}(D) + ker(g \cdot h) \\ &= h^{-1}(D) + kerf \end{aligned}$$

i.e., $h^{-1}(D) + kerf$ is $\tau$-dense in $P$.

REMARK. We denote $M^* = Hom_R(M, Q)$, where $Q$ is $\tau$-torsionfree injective cogenerator. Then for a given $\tau$-hollow module $H$ in $M$ we have an essential submodule $H^{\perp} = \{f \in M^* | f|_H = 0\}$ in $M^*$, where $M^*$ is considered as $T = End_R(Q)$-module, by Page [6].

Thus we can state the following result.

PROPOSITION 32. *$M$ has property (P) if and only if $M^*$ has the ascending chain conditions on essential submodules of $M^*$.*

COROLLARY 33. *$M$ has $\tau$-corank $n$ if and only if $M^*$ has uniform dimension $n$.*

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