J. Korean Math. Soc. 31 (1994), No. 3, pp. 521–537

CONDITIONAL FEYNMAN INTEGRALS INVOLVING INDEFINITE QUADRATIC FORM

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1. Introduction

We consider the Schrödinger equation of quantum mechanics

$$i\hbar\frac{\partial}{\partial t}\Gamma(t,\vec{\eta}) = -\frac{\hbar}{2m}\Delta(t,\vec{\eta}) + V(\vec{\eta})\Gamma(t,\vec{\eta}) \qquad (1.1)$$
$$\Gamma(0,\vec{\eta}) = \psi(\vec{\eta}), \quad \vec{\eta}\in \mathbf{R}^n$$

where Δ is the Laplacian on $\mathbf{R}^n$, $\hbar$ is Plank's constant and V is a suitable potential. Let $K(t,\vec{\eta},0,\vec{\xi})$ denote the fundamental solution to Schrödinger equation (1.1), i.e.,

$$\Gamma(t,\vec{\eta}) = \int_{\mathbf{R}^n} K(t,\vec{\eta},0,\vec{\xi})\psi(\vec{\xi})d\vec{\xi}.$$

According to Feynman [9], $K(t,\vec{\eta},0,\vec{\xi})$ is given by the formal path integral :

$$K(t,\vec{\eta},0,\vec{\xi}) = \int_{C_{\vec{\xi},\vec{\eta}}[0,t]} \exp\left\{\frac{i}{\hbar}S(x)\right\}\mathcal{D}(x), \qquad (1.2)$$

where $C_{\vec{\xi},\vec{\eta}}[0,t]$ is the space of all paths x with $x(0)=\vec{\xi}$ and $x(t)=\vec{\eta}$, $\mathcal{D}(x)$ is a uniform "measure" which does not exist, and $S(x)$ is the action integral associateed with the path x; i.e.,

$$S(x) = \int_0^t \left[\frac{m}{2}\left(\frac{dx}{ds}\right)^2 - V(x(s))\right]ds.$$

The basic problem of quantum mechanics is to find the solution $\Gamma(t,\vec{\eta})$ or the fundamental solution $K(t,\vec{\eta},0,\vec{\xi})$ to Eq. (1.1).

Received October 18, 1992. Revised December 20, 1993.
Partially supported by BSRIP, MOE and KOSEF, 1993.

In [10] Gelfand and Yaglom made an attempt to give sense to the formal integral in Eq.(1.2) by introducing a Wiener measure with complex variance parameter. Unfortunately their attempt was failed as pointed out by Cameron in [3,p.126].

There has been several rigorous approaches to Eq.(1.2) to provide the fundamental solution to Eq.(1.1)(see for examples, [12],[16],[18]).

In [5,7] we introduced the concept of the conditional Feynman integral and established formulas for the conditional Feynman integral for the Fresnel class and then use them to provide the fundamental solution to the Schrödinger equation for a class of potentials.

In this paper we establish the existence of the conditional Feynman integral for a wider class of functions than the Fresnel class and then use them to obtain the fundamental solution to the Schrödinger equation for the anharmonic oscillator.

2. Definition and preliminaries

Let H be a real separable infinite dimensional Hilbert space with inner product $< \cdot, \cdot >$ and norm $|\cdot|^2 =< \cdot, \cdot >$. Let $\|\cdot\|$ be a measurable norm on H with respect to the Gaussian cylinder set measure m on H (see [11,17]). Let B denote the completion of H with respect to $\|\cdot\|$. Let i denote the natural injection from H into B. The adjoint operator i^* of i is one-to-one and maps B^* continuously onto a dense subset of H^*. By identifying H and H^* and B^* with i^*B^*, we have a triple $B^* \subset H = H^* \subset B$ and $< x, y >= (x, y)$ for all x in H and y in B^*, where $(\cdot,\cdot)$ denotes the $B^* - B$ pairing. By a well known result of Gross, $m \circ i^{-1}$ has a unique countably additive extension μ to the Borel σ-algebra $\mathcal{B}(B)$ of B. The triple (H, B, μ) is called an *abstract Wiener space.*

Let $\{e_j; j \geq 1\}$ be a complete orthonormal set in H such that e_j's are in B^* which is the dual space of B. For each $h \in H$ and $x \in B$, let

$$(h, x)^\sim = \begin{cases} \lim_{n\to\infty} \sum_{j=1}^{n} < e_j, h > (e_j, x) \ , & \text{if the limit exist,} \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

Then it is well known that for each $h(\neq 0)$ in H, $(h, \cdot)^\sim$ is a Gaussian random variable on B with mean zero and variance $|h|^2$ and that if

$\{h_1, h_2, \cdots, h_n\}$ is an orthogonal set in H, then the random variable $(h_j, x)^{\sim}$'s are independent. Further, we see that $(h, \lambda x)^{\sim} = (\lambda h, x)^{\sim} = \lambda(h, x)^{\sim}$ for all $\lambda > 0$.

Let $\mathbf{R}^n$, $\mathbf{C}$ and $\mathbf{C}^+$ denote, respectively, the n-dimensional Euclidean space, the complex numbers and the complex numbers with positive real part. Let $M(H)$ be the class of all $\mathbf{C}$-valued Borel measures on H with bounded variation. Then $M(H)$ is a Banach algebra under the total variation norm and with convolution as multiplication. Given two $\mathbf{C}$-valued measurable functions F and G on $B \times B$, F is said to be equal to G s-almost surely(s-a.s.) if for each $\alpha, \beta > 0$, $\mu \times \mu\{(x_1, x_2) \in B \times B : F(\alpha x_1, \beta x_2) \neq G(\alpha x_1, \beta x_2)\} = 0$. For a measurable function F on $B \times B$, let $[F]$ denote the equivalence class of functions which are equal to F s-a.e(see [4]).

Let A_1 and A_2 be two bounded non-negative self-adjoint operators on H. Let $\mathcal{F}_{A_1, A_2}$ be the space of all s-equivalence classes of functions F which for some $\sigma \in M(H)$ have the form

$$F(x_1, x_2) = \int_H \exp\Big\{i\big[(A_1^{\frac{1}{2}} h, x_1)^{\sim} + (A_2^{\frac{1}{2}} h, x_2)^{\sim}\big]\Big\} d\sigma(h). \tag{2.2}$$

As is customary we will identify a function with its s-equivalence class and think of $\mathcal{F}_{A_1, A_2}$ as a class of functions on B rather than as a class of equivalence classes. It is known [15] that $\mathcal{F}_{A_1, A_2}$ forms a Banach algebra. Let $F(x_1, x_2)$ be a $\mathbf{C}$-valued measurable function on $B \times B$ such that the integral

$$J(\lambda_1, \lambda_2) = \int_{B \times B} F(\lambda_1^{-\frac{1}{2}} x_1, \lambda_2^{-\frac{1}{2}} x_2) d(\mu \times \mu)(x_1, x_2)$$

exists as a finite number for all $\lambda_1 > 0, \lambda_2 > 0$. If there exists a function $J^*(z_1, z_2)$, analytic in (z_1, z_2) on $\mathbf{C}^+ \times \mathbf{C}^+$ such that $J^*(\lambda_1, \lambda_2) = J(\lambda_1, \lambda_2)$ for all $\lambda_1 > 0, \lambda_2 > 0$, then $J^*(z_1, z_2)$ is defined to be the analytic Wiener integral of F over $B \times B$ with parameter (z_1, z_2), and for $(z_1, z_2) \in \mathbf{C}^+ \times \mathbf{C}^+$ we write

$$E^{anw_{z_1, z_2}}[F] = J^*(z_1, z_2).$$

Let $(q_1, q_2) \in \mathbf{R}^2$ and let F be a $\mathbf{C}$-valued measurable function such that $E^{anw_{z_1, z_2}}[F]$ exists for all $(z_1, z_2) \in \mathbf{C}^+ \times \mathbf{C}^+$. If the following limit exists, we call it the analytic Feynman integral of F over $B \times B$ with parameter (q_1, q_2), and we write

$$E^{anf_{q_1,q_2}}[F] = \lim_{\substack{z_1 \to -iq_1 \\ z_2 \to -iq_2}} E^{anw_{z_1,z_2}}[F],$$

where z_j approaches $-iq_j$ through $\mathbf{C}^+$ for each j=1,2.

It was shown in [15] that for $F \in \mathcal{F}_{A_1,A_2}$ given by (2.2),

$$E^{anw_{z_1,z_2}}[F] = \int_H \exp\Big\{-\frac{1}{2}\sum_{m=1}^{2} z_m^{-1}(A_m h, h)\Big\} d\sigma(h) \tag{2.3}$$

and

$$E^{anf_{q_1,q_2}}[F] = \int_H \exp\Big\{-\frac{i}{2}\sum_{m=1}^{2} q_m^{-1}(A_m h, h)\Big\} d\sigma(h) \tag{2.4}$$

for each real $q_1 \neq 0$ and $q_2 \neq 0$.

Let $X = (X_1, X_2)$ be an $\mathbf{R}^{n+m} (\equiv \mathbf{R}^n \times \mathbf{R}^m)$-valued measurable function and F a $\mathbf{C}$-valued integrable function on $(B \times B, \mathcal{B}(B \times B), \mu \times \mu)$. Let $\sigma(X)$ denote the σ -algebra generated by X. Then by the definition of the conditional expectation of F given $\sigma(X)$, written $E[F|X]$, is an $\mathbf{R}^{n+m}$-valued $\sigma(X)$-measurable function on $B \times B$ such that

$$\int_E F \; d(\mu \times \mu) = \int_E E[F|X] \; d(\mu \times \mu), \quad \text{for} \quad E \in \sigma(X).$$

It is well known that there exists a Borel measuable and $(\mu \times \mu)_X$-integrable function ψ on $(\mathbf{R}^{n+m}, \mathcal{B}(R^{n+m}))$ such that $E[F|X] = \psi \circ X$, where $\mathcal{B}(\mathbf{R}^{n+m})$ denotes the Borel σ -algebra of $\mathbf{R}^{n+m}$ and $(\mu \times \mu)_X$ is the probability distribution of X defined by $(\mu \times \mu)_X(A) = (\mu \times \mu)(X^{-1}(A))$, for $A \in \mathcal{B}(\mathbf{R}^{n+m})$. The function $\psi(\vec{\xi})$, $\vec{\xi} \in \mathbf{R}^{n+m}$ is unique up to Borel null sets in $\mathbf{R}^{n+m}$. The function $\psi(\vec{\xi})$, written $E[F|X = \vec{\xi}]$, will be called the *conditional abstract Wiener integral* of F given X.

DEFINITION 2.1. Let $X = (X_1, X_2)$ be an $\mathbf{R}^{n+m}$-valued measurable function and let F be a $\mathbf{C}$-valued measurable function on $B \times B$ such that the integral

$$\int_{B\times B} F(\lambda_1{}^{-\frac{1}{2}}x_1, \lambda_2{}^{-\frac{1}{2}}x_2)d(\mu \times \mu)(x_1, x_2)$$

exists as a finite number for all $\lambda_1, \lambda_2 > 0$. For $\lambda_1, \lambda_2 > 0$, let

$$J_{\lambda_1,\lambda_2}(\vec{\eta_1}, \vec{\eta_2}) = E\left[F(\lambda_1{}^{-\frac{1}{2}}\cdot, \lambda_2{}^{-\frac{1}{2}}\cdot)|X(\lambda_1{}^{-\frac{1}{2}}\cdot, \lambda_2{}^{-\frac{1}{2}}\cdot) = (\vec{\eta_1}, \vec{\eta_2})\right]$$

denote the conditional abstract Wiener integral of $F(\lambda_1{}^{-\frac{1}{2}}\cdot, \lambda_2{}^{-\frac{1}{2}}\cdot)$ given $X(\lambda_1{}^{-\frac{1}{2}}\cdot, \lambda_2{}^{-\frac{1}{2}}\cdot)$. If for a.e. $(\vec{\eta_1}, \vec{\eta_2}) \in \mathbf{R}^{n+m}$, there exists a function $J^*_{z_1,z_2}(\vec{\eta_1}, \vec{\eta_2})$, analytic on $\mathbf{C}^+ \times \mathbf{C}^+$ such that $J^*_{\lambda_1,\lambda_2}(\vec{\eta_1}, \vec{\eta_2}) = J_{\lambda_1,\lambda_2}(\vec{\eta_1}, \vec{\eta_2})$ for all $\lambda_1, \lambda_2 > 0$, then $J^*_{z_1,z_2}$ is defined to be the conditional analytic Wiener integral of F over $B \times B$ given X with parameter (z_1, z_2) and for $(z_1, z_2) \in \mathbf{C}^+ \times \mathbf{C}^+$, we write

$$E^{anw_{z_1,z_2}}\left[F|X = (\vec{\eta_1}, \vec{\eta_2})\right] = J^*_{z_1,z_2}(\vec{\eta_1}, \vec{\eta_2})$$

If for fixed $(q_1, q_2) \in \mathbf{R}^2$, the limit

$$\lim_{\substack{z_1 \to -iq_1 \\ z_2 \to -iq_2}} E^{anw_{z_1,z_2}}\left[F|X = (\vec{\eta_1}, \vec{\eta_2})\right]$$

exists for *a.e.* $(\vec{\eta_1}, \vec{\eta_2}) \in \mathbf{R}^{n+m}$, where λ_j approaches $-iq_j$ through $\mathbf{C}^+$ for each j=1,2, then we will denote the value of this limit by $E^{anf_{q_1,q_2}}[F|X]$ and call it the conditional analytic Feynman integral of F over $B \times B$ given X with parameter (q_1, q_2).

3. Conditional Feynman integrals of functions in $\mathcal{F}_{A_1,A_2}$.

In this section we establish the existence of the conditional analytic Feynman integral for all functions in $\mathcal{F}_{A_1,A_2}$.

Let X be an $\mathbf{R}^{n+m}$-valued random variable on $B \times B$ such that

$$X(x_1, x_2) = (X_1(x_1), X_2(x_2)) \tag{3.1}$$

where

$$X_1(x_1) = ((g_1, x_1)^\sim, \cdots, (g_n, x_1)^\sim), \ \ X_2(x_2) = ((h_1, x_2)^\sim, \cdots, (h_m, x_2)^\sim)$$

and $\{g_1, \cdots, g_n, h_1, \cdots, h_m\}$ are orthonormal subsets of H.

LEMMA 3.1. *Let F_1 and F_2 be random variables on $(B, \mathcal{B}(B), \mu)$ and let X be as in (3.1). Assume that $E[F_1]$, $E[F_2]$ and $E[F_1 \cdot F_2]$ exist, then we have , for $(\vec{\eta}, \vec{\xi}) \in \mathbf{R}^{n+m}$*

$$\begin{aligned} &E\big[F_1(x_1)F_2(x_2)|X(x_1,x_2) = (\vec{\eta}, \vec{\xi})\big] \\ &\quad = E\big[F_1(x_1)|X_1(x_1) = \vec{\eta}\big] \cdot E\big[F_2(x_2)|X_2(x_2) = \vec{\xi}\big]. \end{aligned} \tag{3.2}$$

Proof. By the definition of the conditional expectation, we have

$$\begin{aligned} &\int_{B\times B} F_1(x_1)F_2(x_2)d(\mu \times \mu)(x_1,x_2) \\ &= \int_{\mathbf{R}^{n+m}} E[F_1(x_1)F_2(x_2)|X = (\vec{\eta}, \vec{\xi})]d(\mu \times \mu)_{(X_1,X_2)}(\vec{\eta}, \vec{\xi}), \end{aligned} \tag{3.3}$$

where $(\mu \times \mu)_{(X_1,X_2)}(E) = (\mu \times \mu)(X_1, X_2)^{-1}(E)$ for $E \in \mathcal{B}(\mathbf{R}^{n+m})$. Since X_1 and X_2 are independent, $(\mu \times \mu)_{(X_1,X_2)} = \mu_{X_1} \times \mu_{X_2}$, and hence we obtain that

$$\begin{aligned} &\int_{B\times B} F_1(x_1)F_2(x_2)d(\mu \times \mu)(x_1,x_2) \\ &= \int_B F_1(x_1)d\mu(x_1) \cdot \int_B F_2(x_2)d\mu(x_2) \\ &= \int_{\mathbf{R}^n} E[F_1(x_1)|X_1 = \vec{\eta}]d\mu_{X_1}(\vec{\eta}) \cdot \int_{\mathbf{R}^m} E[F_2(x_2)|X_2 = \vec{\xi}]d\mu_{X_2}(\vec{\xi}) \\ &= \int_{\mathbf{R}^{n+m}} E[F_1(x_1)|X_1 = \vec{\eta}] \cdot E[F_2(x_2)|X_2 = \vec{\xi}]d(\mu_{X_1} \times \mu_{X_2})(\vec{\eta}, \vec{\xi}) \\ &= \int_{\mathbf{R}^{n+m}} E[F_1(x_1)|X_1 = \vec{\eta}] \cdot E[F_2(x_2)|X_2 = \vec{\xi}]d(\mu \times \mu)_{(X_1,X_2)}(\vec{\eta}, \vec{\xi}). \end{aligned} \tag{3.4}$$

Therefore, by (3.3) and (3.4) we have the desired result (3.2).

THEOREM 3.2. *Let $F \in \mathcal{F}_{A_1,A_2}$ and let $X(x_1,x_2)$ be as in (3.1). Then for all $\lambda_1, \lambda_2 \in C^+$, the conditional analytic Wiener integral over $B \times B$, $E^{anw_{\lambda_1,\lambda_2}}[F|X]$ exists, and for all $(\vec{\eta}, \vec{\xi}) \in \mathbf{R}^{n+m}$, is given by* *the formula*

$$E^{anw_{\lambda_1,\lambda_2}}[F|X=(\vec{\eta},\vec{\xi})] \tag{3.5}$$
$$=\int_H \exp\{i < X(A_1^{\frac{1}{2}}h, A_2^{\frac{1}{2}}h),(\vec{\eta},\vec{\xi}) >\}$$
$$\times \exp\Big\{-\frac{1}{2}\sum_{j=1}^{2}\lambda_j^{-1}\big(|A_j^{\frac{1}{2}}h|^2-|X_j(A_j^{\frac{1}{2}}h)|^2\big)\Big\}d\sigma(h),$$

where $X(g,h)=(X_1(g),X_2(h))=(<g_1,g>,\cdots,<g_n,g>,<h_1,h>,\cdots,<h_m,h>)$. *Furthermore, the conditional analytic Feynman integral* $E^{anf_{q_1,q_2}}[F|X]$ *exists for all* $q_1,q_2\neq 0$ *and for all* $(\vec{\eta},\vec{\xi})\in \mathbf{R}^{n+m}$ *is given by the formula*

$$E^{anf_{q_1,q_2}}[F|X=(\vec{\eta},\vec{\xi})] \tag{3.6}$$
$$=\int_H \exp\{i < X(A_1^{\frac{1}{2}}h, A_2^{\frac{1}{2}}h),(\vec{\eta},\vec{\xi}) >\}$$
$$\times \exp\Big\{-\frac{i}{2}\sum_{j=1}^{2}q_j^{-1}\big(|A_j^{\frac{1}{2}}h|^2-|X_j(A_j^{\frac{1}{2}}h)|^2\big)\Big\}d\sigma(h).$$

Proof. For a fixed $h\in H$, $A_1^{\frac{1}{2}}h$ and $A_2^{\frac{1}{2}}h$ can be written as

$$A_1^{\frac{1}{2}}h=\sum_{j=1}^{n}<g_j,A_1^{\frac{1}{2}}h>g_j+p_1,\quad p_1\in[g_1,\cdots,g_n]^{\perp} \tag{3.7}$$
$$A_2^{\frac{1}{2}}h=\sum_{k=1}^{m}<h_k,A_2^{\frac{1}{2}}h>h_k+p_2,\quad p_2\in[h_1,\cdots,h_m]^{\perp}$$

where $[C]^{\perp}$ is the orthogonal complement of the subspace of H generated by C. Since the random variables $(p_j,\tilde{x})$ and $X_j(x)$ are independent for j=1,2, it follows by the Fubini's Theorem, Lemma 3.1 and (3.7) we have, for all $\lambda_1,\lambda_2\in\mathbf{R}^+$

$$E\left[F(\lambda_1^{-\frac{1}{2}}\cdot,\lambda_2^{-\frac{1}{2}}\cdot)|(X(\lambda_1^{-\frac{1}{2}}\cdot,\lambda_2^{-\frac{1}{2}}\cdot))\right]$$
$$=E\Big[\int_H \exp\Big\{i(A_1^{\frac{1}{2}}h,\lambda_1^{-\frac{1}{2}}x_1\tilde{)}+i(A_2^{\frac{1}{2}}h,\lambda_2^{-\frac{1}{2}}x_2\tilde{)}\Big\}d\sigma(h)$$
$$|(X_1(\lambda_1^{-\frac{1}{2}}\cdot),X_2(\lambda_2^{-\frac{1}{2}}\cdot))\Big]$$

$$= \int_H E\Big[\exp\Big\{i\big(A_1^{\frac{1}{2}}h, \lambda_1^{-\frac{1}{2}}x_1\tilde{)}\big) + i\big(A_2^{\frac{1}{2}}h, \lambda_2^{-\frac{1}{2}}x_2\tilde{)}\big)\Big\}$$

$$|(X_1(\lambda_1^{-\frac{1}{2}}\cdot),(X_2(\lambda_2^{-\frac{1}{2}}\cdot))\Big]d\sigma(h)$$

$$= \int_H E\Big[\exp\Big\{i\sum_{j=1}^{n} < g_j, A_1^{\frac{1}{2}}h > (g_j, \lambda_1^{-\frac{1}{2}}x_1\tilde{)}$$

$$+ i(p_1, \lambda_1^{-\frac{1}{2}}x_1\tilde{)}\Big\}|X_1(\lambda_1^{-\frac{1}{2}}\cdot)\Big]$$

$$\times E\Big[\exp\Big\{i\sum_{k=1}^{m} < h_k, A_1^{\frac{1}{2}}h > (h_k, \lambda_2^{-\frac{1}{2}}x_2\tilde{)}$$

$$+ i(p_2, \lambda_2^{-\frac{1}{2}}x_2\tilde{)}\Big\}|\ X_2(\lambda_2^{-\frac{1}{2}}\cdot)\Big]d\sigma(h)$$

$$= \int_H E\Big[\exp\Big\{i\sum_{j=1}^{n} < g_j, A_1^{\frac{1}{2}}h > (g_j, \lambda_1^{-\frac{1}{2}}x_1\tilde{)}|\ X_1(\lambda_1^{-\frac{1}{2}}\cdot)\Big]$$

$$\times E\Big[\exp\Big\{i\sum_{k=1}^{m} < h_k, A_2^{\frac{1}{2}}h > (h_k, \lambda_2^{-\frac{1}{2}}x_2\tilde{)}|\ X_2(\lambda_2^{-\frac{1}{2}}\cdot)\Big]$$

$$\times E\Big[\exp\{i(p_1, \lambda_1^{-\frac{1}{2}}x_1\tilde{)}\}\Big] \cdot E\Big[\exp\{i(p_2, \lambda_2^{-\frac{1}{2}}x_1\tilde{)}\}\Big]d\sigma(h).$$

Since $E[\exp\{i\lambda_j^{-\frac{1}{2}}(p_j, x_j\tilde{)}\}] = \exp\{-(1/2\lambda_j)|p_j|^2\}$, we have

$$E\Big[F(\lambda_1^{-\frac{1}{2}}\cdot, \lambda_2^{-\frac{1}{2}}\cdot)|(X_1(\lambda_1^{-\frac{1}{2}}\cdot), X_2(\lambda_2^{-\frac{1}{2}}\cdot)) = (\vec{\eta}, \vec{\xi})\Big] \tag{3.8}$$

$$= \int_H \Big[\exp\{i < X_1(A_1^{\frac{1}{2}}h), \vec{\eta} >\} \cdot \exp\{i < X_2(A_2^{\frac{1}{2}}h), \vec{\xi} >\}$$

$$\times \exp\Big\{-\frac{1}{2\lambda_1}|p_1|^2 - \frac{1}{2\lambda_2}|p_2|^2\Big\}\Big]d\sigma(h).$$

Since $|p_j|^2 = |A_j^{\frac{1}{2}}h|^2 - |X_j(A_j^{\frac{1}{2}}h)|^2$ for $j = 1, 2$, (3.8) is equal to

$$\int_H \Big[\exp\{i < X(A_1^{\frac{1}{2}}h, A_2^{\frac{1}{2}}h), (\vec{\eta}, \vec{\xi}) >\}$$

$$\times \exp\Big\{-\frac{1}{2}\sum_{j=1}^{2} \lambda_j^{-1}(|A_j^{\frac{1}{2}}h|^2 - |X_j(A_j^{\frac{1}{2}}h)|^2)\Big\}\Big] d\sigma(h).$$

But $|A_j^{\frac{1}{2}}h|^2 - |X_j(A_j^{\frac{1}{2}}h)|^2 \geq 0$, $j = 1, 2$, for all $h \in H$ and $\sigma \in \mathbf{M}(H)$, the last integrand is analytic function of (λ_1, λ_2) throught $\mathbf{C}^+ \times \mathbf{C}^+$ (by Morea's theorem) and is continuous of (λ_1, λ_2) for $\mathrm{Re}\lambda_j \geq 0, \lambda_j \neq 0$, j=1, 2. Hence we establish the equations (3.5) and (3.6) as desired.

REMARK. Let A be a bounded self-adjoint operator on H. Then A can be written as $A = A^+ - A^-$ where A^+ and A^- are each bounded and non-negative self-adjoint. Take $A_1 = A^+$ and $A_2 = A^-$ in (2.2). When $A^- = 0$ and A^+ is the identity in Theorem 3.2, we obtain results for the Fresnel class (see, [7]).

In our next theorem, we will need the following summation procedure (see.[14])

$$\overline{\int}_{\mathbf{R}^n} f(\vec{\eta})d\vec{\eta} = \lim_{A\to\infty} \int_{\mathbf{R}^n} f(\vec{\eta})\exp\Big\{-\frac{|\vec{\eta}|^2}{2A}\Big\}d\vec{\eta} \tag{3.9}$$

whenever the expression on the right exists. Of course if $f \in L^1(\mathbf{R}^n)$, it is clear by using the dominated convergence theorem that

$$\overline{\int}_{\mathbf{R}^n} f(\vec{\eta})d\vec{\eta} = \int_{\mathbf{R}^n} f(\vec{\eta})d\vec{\eta}.$$

THEOREM 3.3. *Let F and X be as in Theorem 3.2. Then for all* $(\lambda_1, \lambda_2) \in \mathbf{C}^+ \times \mathbf{C}^+$,

$$\int_{\mathbf{R}^{n+m}} \Big(\frac{\lambda_1}{2\pi}\Big)^{\frac{n}{2}} \Big(\frac{\lambda_2}{2\pi}\Big)^{\frac{m}{2}} \exp\Big\{-\frac{1}{2}(\lambda_1|\vec{\eta}|^2 + \lambda_2|\vec{\xi}|^2)\Big\} \times E^{anw_{\lambda_1,\lambda_2}}[F|X = (\vec{\eta}, \vec{\xi})]\, d\vec{\eta}\, d\vec{\xi} \tag{3.10}$$

$$= E^{anw_{\lambda_1,\lambda_2}}[F]$$

and for all reals $q_1, q_2 \neq 0$,

$$\overline{\int}_{\mathbf{R}^{n+m}} \left(\frac{q_1}{2\pi i}\right)^{\frac{n}{2}} \left(\frac{q_2}{2\pi i}\right)^{\frac{m}{2}} \exp\left\{\frac{iq_1}{2}|\vec{\eta}|^2 + \frac{iq_2}{2}|\vec{\xi}|^2)\right\} \times E^{anf_{q_1,q_2}}\left[F|X = (\vec{\eta}, \vec{\xi})\right] d\vec{\eta} d\vec{\xi} = E^{anf_{q_1,q_2}}[F]. \tag{3.11}$$

Proof. We will only show that (3.11) holds since the proof of (3.10) is similar. Let $q_1, q_2 \neq 0$. Then by using (3.10), the Fubini theorem and (3.6)

$$\begin{aligned}
&\overline{\int}_{\mathbf{R}^{n+m}} \left(\frac{q_1}{2\pi i}\right)^{\frac{n}{2}} \left(\frac{q_2}{2\pi i}\right)^{\frac{m}{2}} \exp\left\{\frac{iq_1}{2}|\vec{\eta}|^2 + \frac{iq_2}{2}|\vec{\xi}|^2\right\} \\
&\qquad \times E^{anf_{q_1,q_2}}\left[F|X = (\vec{\eta}, \vec{\xi})\right] d\vec{\eta}\, d\vec{\xi} \\
&= \lim_{A\to\infty} \int_{\mathbf{R}^{n+m}} \left(\frac{q_1}{2\pi i}\right)^{\frac{n}{2}} \left(\frac{q_2}{2\pi i}\right)^{\frac{m}{2}} \exp\left\{\frac{iq_1}{2}|\vec{\eta}|^2 + \frac{iq_2}{2}|\vec{\xi}|^2\right\} \\
&\qquad \times \exp\left\{-\frac{|\vec{\eta}|^2 + |\vec{\xi}|^2}{2A}\right\} E^{anf_{q_1,q_2}}\left[F|X = (\vec{\eta}, \vec{\xi})\right] d\vec{\eta}\, d\vec{\xi} \\
&= \lim_{A\to\infty} \int_{\mathbf{R}^{n+m}} \Bigg[\left(\frac{q_1}{2\pi i}\right)^{\frac{n}{2}} \left(\frac{q_2}{2\pi i}\right)^{\frac{m}{2}} \exp\left\{\frac{iq_1}{2}|\vec{\eta}|^2 + \frac{iq_2}{2}|\vec{\xi}|^2)\right\} \\
&\qquad \times \exp\left\{-\frac{|\vec{\eta}|^2 + |\vec{\xi}|^2}{2A}\right\} \int_H \exp\left\{i < X(A_1^{\frac{1}{2}}h, A_2^{\frac{1}{2}}h), (\vec{\eta}, \vec{\xi}) >\right\} \\
&\qquad \times \exp\left\{-\frac{i}{2}\sum_{j=1}^{2} q_j^{-1}(|A_j^{\frac{1}{2}}h|^2 - |X_j(A_j^{\frac{1}{2}}h)|^2)\right\} d\sigma(h)\Bigg] d\vec{\eta}\, d\vec{\xi} \\
&= \lim_{A\to\infty} \int_H \Bigg[\exp\left\{-\frac{i}{2}\sum_{j=1}^{2} q_j^{-1}(|A_j^{\frac{1}{2}}h|^2 - |X_j(A_j^{\frac{1}{2}}h)|^2)\right\} \cdot \left(\frac{q_1}{2\pi i}\right)^{\frac{n}{2}} \left(\frac{q_2}{2\pi i}\right)^{\frac{m}{2}} \\
&\qquad \times \int_{\mathbf{R}^n} \exp\left\{-\frac{1}{2}(\frac{1 - Aiq_1}{A})|\vec{\eta}|^2 + i < X_1(A_1^{\frac{1}{2}}h), \vec{\eta} >\right\} d\vec{\eta} \\
&\qquad \times \int_{\mathbf{R}^m} \exp\left\{-\frac{1}{2}(\frac{1 - Aiq_2}{A})|\vec{\xi}|^2 + i < X_2(A_2^{\frac{1}{2}}h), \vec{\xi} >\right\} d\vec{\xi}\Bigg] d\sigma(h) \\
&= \lim_{A\to\infty} \int_H \Bigg[\exp\left\{-\frac{i}{2}\sum_{j=1}^{2} q_j^{-1}(|A_j^{\frac{1}{2}}h|^2 - |X_j(A_j^{\frac{1}{2}}h)|^2)\right\}
\end{aligned}$$

$$\times \Big(\frac{q_1}{2\pi i}\Big)^{\frac{n}{2}} \Big(\frac{2\pi A}{1-Aiq_1}\Big)^{\frac{n}{2}} \exp\Big\{-\frac{A|X_1(A_1^{\frac{1}{2}}h)|^2}{2(1-Aiq_1)}\Big\}$$

$$\times \Big(\frac{q_2}{2\pi i}\Big)^{\frac{m}{2}} \Big(\frac{2\pi A}{1-Aiq_2}\Big)^{\frac{m}{2}} \exp\Big\{-\frac{A|X_2(A_2^{\frac{1}{2}}h)|^2}{2(1-Aiq_2)}\Big\}\Big] d\sigma(h)$$

$$= \int_H \Big[\exp\Big\{-\frac{i}{2q_1} < A_1 h, h > -\frac{i}{2q_2} < A_2 h, h >\Big\}\Big] d\sigma(h)$$

$$= E^{anf_{q_1,q_2}}[F].$$

4. An application

In this section we use the concept of conditional analytic Feynman integral to provide a fundamental solution to the Schrödinger equation for the anharmonic oscillator.

THEOREM 4.1. *Let F and X be as in Theorem 3.2. Let ψ be the function defined on $\mathbf{R}^{n+m}$ by*

$$\psi(\vec{\eta}_1, \vec{\eta}_2) = \int_{\mathbf{R}^{n+m}} \exp\{i < (\vec{y}_1, \vec{y}_2), (\vec{\eta}_1, \vec{\eta}_2) >\} d\phi(\vec{y}_1, \vec{y}_2), \tag{4.1}$$

where ϕ is a complex Borel measure on $\mathbf{R}^{n+m}$ with bounded variation and let

$$G(x_1, x_2) \equiv G_{(\vec{\eta}_1, \vec{\eta}_2)}(x_1, x_2) = F(x_1, x_2)\psi(X(x_1, x_2) + (\vec{\eta}_1 . \vec{\eta}_2)).$$

Then for all $q_1, q_2 \neq 0$, we have that

$$\begin{aligned} &E^{anf_{q_1,q_2}}[G] \\ &= \int_H \Big[\exp\Big\{-\frac{i}{2}\sum_{j=1}^{2} q_j^{-1}(|A_j^{\frac{1}{2}}h|^2 - |X_j(A_j^{\frac{1}{2}}h)|^2)\Big\} \\ &\quad \times \int_{\mathbf{R}^{n+m}} \exp\Big\{i < (\vec{y}_1, \vec{y}_2), (\vec{\eta}_1, \vec{\eta}_2) > \\ &\quad - \frac{i}{2}\sum_{j=1}^{2} q_j^{-1}|X_j(A_j^{\frac{1}{2}}h) + \vec{y}_j|^2\Big\} d\phi(\vec{y}_1, \vec{y}_2)\Big] d\sigma(h). \end{aligned} \tag{4.2}$$

In addition, we have the alternative expression

$$E^{anf_{q_1,q_2}}[G] \tag{4.3}$$
$$= \int_{\mathbf{R}^{n+m}} E^{anf_{q_1,q_2}}\left[F|X = (\vec{\xi_1},\vec{\xi_2}) - (\vec{\eta_1},\vec{\eta_1})\right]$$
$$\times \left(\frac{q_1}{2\pi i}\right)^{\frac{n}{2}} \left(\frac{q_2}{2\pi i}\right)^{\frac{m}{2}} \exp\left\{\frac{i}{2}\sum_{j=1}^{2} q_j|\vec{\xi_j} - \vec{\eta_j}|^2\right\}\psi(\vec{\xi_1},\vec{\xi_2})d\vec{\xi_1}d\vec{\xi_2}.$$

Proof. By using Proposition 1 of [19], Theorem 3.2 and the Fubini theorem, we obtain for $\lambda_1, \lambda_2 > 0$,

$$\int_{B\times B} G(\lambda_1{}^{-\frac{1}{2}}x_1, \lambda_2{}^{-\frac{1}{2}}x_2)d(\mu\times\mu)(x_1,x_2)$$
$$= \int_{\mathbf{R}^{n+m}} E\left[F(\lambda_1{}^{-\frac{1}{2}}\cdot, \lambda_2{}^{-\frac{1}{2}}\cdot)|X(\lambda_1{}^{-\frac{1}{2}}\cdot, \lambda_2^{-\frac{1}{2}}\cdot) + (\vec{\eta_1},\vec{\eta_2}) = (\vec{\xi_1},\vec{\xi_2})\right]$$
$$\times \psi(\vec{\xi_1},\vec{\xi_2})\left(\frac{\lambda_1}{2\pi}\right)^{\frac{n}{2}}\left(\frac{\lambda_2}{2\pi}\right)^{\frac{m}{2}}$$
$$\exp\left\{-\frac{1}{2}\sum_{j=1}^{2}\lambda_j|\vec{\xi_j}-\vec{\eta_j}|^2\right\}d\vec{\xi_1}d\vec{\xi_2}$$
$$= \int_{\mathbf{R}^{n+m}}\left[\int_H \left[\exp\left\{i < X(A_1{}^{\frac{1}{2}}h, A_2{}^{\frac{1}{2}}h), (\vec{\xi_1},\vec{\xi_2}) - (\vec{\eta_1},\vec{\eta_2}) >\right\}\right.\right.$$
$$\left.\times \exp\left\{-\frac{1}{2}\sum_{j=1}^{2}\lambda_j^{-1}(|A_j{}^{\frac{1}{2}}h|^2 - |X_j(A_j{}^{\frac{1}{2}}h)|^2)\right\}\right]d\sigma(h)$$
$$\times \left(\frac{\lambda_1}{2\pi}\right)^{\frac{n}{2}}\left(\frac{\lambda_2}{2\pi}\right)^{\frac{m}{2}}\exp\left\{-\frac{1}{2}\sum_{j=1}^{2}\lambda_j|\vec{\xi_j}-\vec{\eta_j}|^2\right\}$$
$$\left.\times \int_{\mathbf{R}^{n+m}}\exp\{i < (\vec{y_1},\vec{y_2}),(\vec{\xi_1},\vec{\xi_2}) >\}d\phi(\vec{y_1},\vec{y_2})\right]d\vec{\xi_1}d\vec{\xi_2}$$
$$= \int_H\left[\int_{\mathbf{R}^{n+m}}\exp\left\{-\frac{1}{2}\sum_{j=1}^{2}\lambda_j^{-1}(|A_j{}^{\frac{1}{2}}h|^2 - |X_j(A_j{}^{\frac{1}{2}}h)|^2)\right\}\right.$$
$$\times\left(\frac{\lambda_1}{2\pi}\right)^{\frac{n}{2}}\int_{\mathbf{R}^n}\exp\left\{i < X_1(A_1{}^{\frac{1}{2}}h), \vec{\xi_1} - \vec{\eta_1} >\right.$$

$$- \frac{\lambda_1|\vec{\xi_1} - \vec{\eta_1}|^2}{2} + i < \vec{y_1}, \vec{\xi_1} > \Big\} d\vec{\xi_1}$$

$$\times \Big(\frac{\lambda_2}{2\pi}\Big)^{\frac{m}{2}} \int_{\mathbf{R}^m} \exp\Big\{ i < X_2(A_2{}^{\frac{1}{2}} h), \vec{\xi_2} - \vec{\eta_2} >$$

$$- \frac{\lambda_2|\vec{\xi_2} - \vec{\eta_2}|^2}{2} + i < \vec{y_2}, \vec{\xi_2} > \Big\} d\vec{\xi_2} d\phi(\vec{y_1}, \vec{y_2}) \Big] d\sigma(h)$$

$$= \int_H \Big[\exp\Big\{ -\frac{1}{2} \sum_{j=1}^{2} \lambda_j^{-1} (|A_j{}^{\frac{1}{2}} h|^2 - |X_j(A_j{}^{\frac{1}{2}} h)|^2) \Big\}$$

$$\times \int_{\mathbf{R}^{n+m}} \exp\Big\{ i \sum_{j=1}^{2} < \vec{\eta_j}, \vec{y_j} >$$

$$- \frac{1}{2} \sum_{j=1}^{2} \lambda_j^{-1} |X_j(A_j{}^{\frac{1}{2}} h) + \vec{y_j}|^2 \Big\} d\phi(\vec{y_1}, \vec{y_2}) \Big] d\sigma(h)$$

$$= \int_H \Big[\exp\Big\{ -\frac{1}{2} \sum_{j=1}^{2} \lambda_j^{-1} (|A_j{}^{\frac{1}{2}} h|^2 - |X_j(A_j{}^{\frac{1}{2}} h)|^2) \Big\}$$

$$\times \int_{\mathbf{R}^{n+m}} \exp\Big\{ i < (\vec{\eta_1}, \vec{\eta_2}), (\vec{y_1}, \vec{y_2}) >$$

$$- \frac{1}{2} \sum_{j=1}^{2} \lambda_j^{-1} |X_j(A_j{}^{\frac{1}{2}} h) + \vec{y_j}|^2 \Big\} d\phi(\vec{y_1}, \vec{y_2}) \Big] d\sigma(h)$$

for all $\lambda_1, \lambda_2 \in \mathbf{C}^+$. We note that $|A_j{}^{\frac{1}{2}} h|^2 - |X_j(A_j{}^{\frac{1}{2}} h)|^2 \geq 0$ for each j=1,2 and the last integrand is continuous in (λ_1, λ_2) for $Re\lambda_j \geq 0, \lambda_j \neq 0$, j=1,2 and hence $E^{anf_{q_1,q_2}}[G]$ exists and is given by (4.2).

To obtain the alternative expression (4.3),

$$\overline{\int}_{\mathbf{R}^{n+m}} E^{anf_{q_1,q_2}}[F|X = (\vec{\xi_1}, \vec{\xi_2}) - (\vec{\eta_1}, \vec{\eta_1})] \Big(\frac{q_1}{2\pi i}\Big)^{\frac{n}{2}} \Big(\frac{q_2}{2\pi i}\Big)^{\frac{m}{2}}$$

$$\times \exp\Big\{ \frac{i}{2} \sum_{j=1}^{2} q_j |\vec{\xi_j} - \vec{\eta_j}|^2 \Big\} \psi(\vec{\xi_1}, \vec{\xi_2}) d\vec{\xi_1} d\vec{\xi_2}$$

$$= \lim_{A \to \infty} \int_{\mathbf{R}^{n+m}} E^{anf_{q_1,q_2}}[F|X = (\vec{\xi_1}, \vec{\xi_2}) - (\vec{\eta_1}, \vec{\eta_1})] \Big(\frac{q_1}{2\pi i}\Big)^{\frac{n}{2}} \Big(\frac{q_2}{2\pi i}\Big)^{\frac{m}{2}}$$

$$\times \exp\Big\{\frac{1}{2}\sum_{j=1}^{2}\big(iq_j|\vec{\xi_j}-\vec{\eta_j}|^2-\frac{|\vec{\xi_j}|^2}{A}\big)\Big\}\psi(\vec{\xi_1},\vec{\xi_2})d\vec{\xi_1}d\vec{\xi_2}$$

$$= \lim_{A\to\infty}\int_{\mathbf{R}^{n+m}}\Big[\int_H \exp\{i < X(A_1{}^{\frac{1}{2}}h, A_2{}^{\frac{1}{2}}h),(\vec{\xi_1},\vec{\xi_2})-(\vec{\eta_1},\vec{\eta_2}) >\}$$

$$\times \exp\Big\{-\frac{i}{2}\sum_{j=1}^{2}q_j^{-1}(|A_j{}^{\frac{1}{2}}h|^2-|X_j(A_j{}^{\frac{1}{2}}h)|^2)\Big\}$$

$$\times \Big(\frac{q_1}{2\pi i}\Big)^{\frac{n}{2}}\Big(\frac{q_2}{2\pi i}\Big)^{\frac{m}{2}}\exp\Big\{\frac{1}{2}\sum_{j=1}^{2}\big(iq_j|\vec{\xi_j}-\vec{\eta_j}|^2-\frac{|\vec{\xi_j}|^2}{A}\big)\Big\}$$

$$\times \int_{\mathbf{R}^{n+m}}\exp\{i < (\vec{\xi_1},\vec{\xi_2}),(\vec{y_1},\vec{y_2}) >\}d\phi(\vec{y_1},\vec{y_2})d\sigma(h)\Big]d\vec{\xi_1}d\vec{\xi_2}$$

$$= \lim_{A\to\infty}\int_H\Big[\exp\Big\{-\frac{i}{2}\sum_{j=1}^{2}q_j^{-1}(|A_j{}^{\frac{1}{2}}h|^2-|X_j(A_j{}^{\frac{1}{2}}h)|^2)\Big\}$$

$$\times \int_{\mathbf{R}^{n+m}}\Big[\Big(\frac{q_1}{2\pi i}\Big)^{\frac{n}{2}}\int_{\mathbf{R}^n}\exp\{i < X_1(A_1{}^{\frac{1}{2}}h),\vec{\xi_1}-\vec{\eta_1} >\}$$

$$\times \exp\Big\{\frac{iq_1}{2}|\vec{\xi_1}-\vec{\eta_1}|^2-\frac{|\vec{\xi_1}|^2}{2A}+i < \vec{\xi_1},\vec{y_1} >\Big\}d\vec{\xi_1}\Big]$$

$$\times \Big[\big(\frac{q_2}{2\pi i}\big)^{\frac{m}{2}}\int_{\mathbf{R}^m}\exp\{i < X_2(A_2{}^{\frac{1}{2}}h),\vec{\xi_2}-\vec{\eta_2} >\}$$

$$\times \exp\Big\{\frac{iq_2}{2}|\vec{\xi_2}-\vec{\eta_2}|^2-\frac{|\vec{\xi_2}|^2}{2A}+i < \vec{\xi_2},\vec{y_2} >\Big\}d\vec{\xi_2}\Big]$$

$$d\phi(\vec{y_1},\vec{y_2})\Big]d\sigma(h)$$

$$= \lim_{A\to\infty}\int_H\Big[\exp\Big\{-\frac{i}{2}\sum_{j=1}^{2}q_j^{-1}(|A_j{}^{\frac{1}{2}}h|^2-|X_j(A_j{}^{\frac{1}{2}}h)|^2)\Big\}$$

$$\times \int_{\mathbf{R}^{n+m}}\Big[\exp\Big\{i\sum_{j=1}^{2} < \vec{\eta_j},\vec{y_j} >\Big\}\exp\Big\{-\sum_{j-1}^{2}\frac{1}{2A}|\vec{\eta_j}|^2\Big\}$$

$$\times \Big(\frac{q_1}{2\pi i}\Big)^{\frac{n}{2}}\Big(\frac{2\pi A}{1-Aiq_1}\Big)^{\frac{n}{2}}\Big(\frac{q_2}{2\pi i}\Big)^{\frac{m}{2}}\Big(\frac{2\pi A}{1-Aiq_2}\Big)^{\frac{m}{2}}$$

$$\times \exp\Big\{-\sum_{j=1}^{2}\frac{A|X_j(A_j{}^{\frac{1}{2}}h)+\vec{y_j}+\frac{1}{A}\vec{\eta_j}|^2}{2(1-Aiq_j)}\Big\}\Big]d\phi(\vec{y_1},\vec{y_2})\Big]d\sigma(h)$$

$$= \int_H \Big[\exp\Big\{ -\frac{i}{2} \sum_{j=1}^{2} q_j^{-1} (|A_j^{\frac{1}{2}} h|^2 - |X_j(A_j^{\frac{1}{2}} h)|^2) \Big\}$$

$$\times \int_{\mathbf{R}^{n+m}} \exp\Big\{ i < (\vec{y_1}, \vec{y_2}), (\vec{\eta_1}, \vec{\eta_2}) >$$

$$- \frac{i}{2} \sum_{j=1}^{2} q_j^{-1} |X_j(A_j^{\frac{1}{2}} h) + \vec{y_j}|^2 \Big\} d\phi(\vec{y_1}, \vec{y_2}) d\sigma(h)$$

$$= E^{anf_{q_1,q_2}}[G].$$

This completes the proof of (4.3).

EXAMPLE. Let $t > 0$ be fixed and let H be the real seperable Hilbert space of paths $\gamma : [0,t] \longrightarrow \mathbf{R}^{n+m}$ such that $\gamma(t) = 0$ and $\gamma(s) = -\int_s^t \gamma'(u)du$ with $\gamma' \in L^2([0,t])$. Define the inner product on H by

$$(\gamma_1, \gamma_2) = \int_0^t \gamma_1'(u) \cdot \gamma_2'(u) du,$$

where $\cdot$ stands for the inner product in $\mathbf{R}^{n+m}$. Let V be a real valued function of the form (4.1) and ψ be as in (4.1) with $\psi \in L^2(\mathbf{R}^{n+m})$.

Let $a = (a_{ij})$ be an $(n+m) \times (n+m)$ real matrix which is symmetric and positive definite such that

$$\det[\cos(a^{1/2} t)] \neq 0, \tag{4.4}$$

Then the indefinite quadratic form on H defined by

$$< \gamma_1, \gamma_2 > = \int_0^t \gamma_1'(u) \cdot \gamma_2'(u) - a\gamma_1(u) \cdot \gamma_2(u) du$$

determines uniquely a self-adjoint operator $\tilde{A}$ on H such that $(\tilde{A}\gamma_1, \gamma_2) = < \gamma_1, \gamma_2 >$ and (4.4) guarantees that $\tilde{A}$ is invertible. Let $\tilde{A}^{-1} = A$.

Let $\beta : [0,t] \longrightarrow \mathbf{R}^{n+m}$ be any absolutely continuous function such that $\int_0^t |\beta'(u)|^2 du < \infty$ and $\beta(t) = \vec{x}$. Let

$$f(r) = \exp\Big\{ \frac{i}{2} < \beta, \beta > + i < \gamma, \beta > - i \int_0^t V(\gamma(u) + \beta(u)) du \Big\}.$$

Then it is easy to see that f is the Fourier transform of some $\sigma \in M(H)$. We can proceed as in [1] or [15, p.249] to show that the solution $\psi(t,\vec{x})$ of the Schrödinger equation for the anharmonic oscillator:

$$i\frac{\partial}{\partial t}\Gamma(t,\vec{x}) = -\frac{1}{2}\Delta\Gamma(t,\vec{x}) + \frac{1}{2}(a\vec{x}\cdot\vec{x})\Gamma(t,\vec{x}) + V(\vec{x})\Gamma(t,\vec{x}) \qquad (4.5)$$

$$\Gamma(0,\vec{x}) = \psi(\vec{x}), \quad \vec{x} \in \mathbf{R}^{n+m}$$

can be represented by

$$\begin{aligned}&\Gamma(t,\vec{x})\\ &= \left|\det[\cos(a^{\frac{1}{2}}t)]\right|^{-\frac{1}{2}} \cdot E^{anf_{1,-1}}\left[F(\cdot,\cdot)\psi(X(\cdot,\cdot)+(\vec{\eta_1},\vec{\eta_2}))\right]\end{aligned}$$

where $F(x_1,x_2) = \int_H \exp\{i[(A^+h,x_1)\tilde{} + (A^-h,x_2)\tilde{}]d\sigma(h)$ and X is as in (3.1). Let

$$\begin{aligned}H(t,(\vec{\eta_1},\vec{\eta_2})) &= \left|\det[\cos(a^{\frac{1}{2}}t)]\right|^{-\frac{1}{2}} E^{anf_{1,-1}}\left[F|X = -(\vec{\eta_1},\vec{\eta_2})\right]\\ &\times \left(\frac{1}{2\pi i}\right)^{\frac{n}{2}}\left(\frac{-1}{2\pi i}\right)^{\frac{m}{2}} \exp\left\{\frac{i}{2}\left[|\vec{\eta_1}|^2 - |\vec{\eta_2}|^2\right]\right\}.\end{aligned}$$

Then (4.3) shows that $H(t,(\vec{\eta_1},\vec{\eta_2}))$ is the fundamental solution of the equation (4.5).

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