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COUSIN COMPLEXES AND GENERALIZED HUGHES COMPLEXES

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1. Introduction

In this paper, the ring A will mean a commutative Noetherian ring with non-zero multiplicative identity, it is understood that the ring homomorphisms respect identity elements and M will denote an A-module. Throughout this paper A and B will denote rings, $f : A \to B$ a ring homomorphism. $\mathcal{C}(A)$ (resp. $\mathcal{C}(B)$) presents the category of all A-modules (resp. B-modules) and A-homomorphisms (resp. B-homomorphisms) between them. The following ideas will be used without further explanation. B can be regarded as an A-module by means of f and $M \otimes B$ can be regarded as a B-module in the natural way. Furthermore the restriction of scalars provides a functor from $\mathcal{C}(B)$ to $\mathcal{C}(A)$.

Whenever we have a filtration $\mathcal{F}$ of Spec(A) which admits M [3, (1.1)] we can construct the Cousin complex $C(\mathcal{F}, M)$ for M with respect to $\mathcal{F}$ [3, (1.3)]. The Cousin complexes for A and B will be consistently written

$$C(A) : 0 \xrightarrow{d^{-2}} A \xrightarrow{d^{-1}} A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} \cdots \to A^n \xrightarrow{d^n} A^{n+1} \to \cdots$$

$$C(B) : 0 \xrightarrow{e^{-2}} B \xrightarrow{e^{-1}} B^0 \xrightarrow{e^0} B^1 \xrightarrow{e^1} \cdots \to B^n \xrightarrow{e^n} B^{n+1} \to \cdots$$

respectively, which have the property that, for each $n \in \mathbb{N}_0$: [1]

$$A^n = \bigoplus_{\substack{ht(p)=n \\ p \in \mathrm{Spec}(A)}} (\mathrm{Coker}\ d^{n-2})_p \quad \text{and} \quad B^n = \bigoplus_{\substack{ht(q)=n \\ q \in \mathrm{Spec}(B)}} (\mathrm{Coker}\ e^{n-2})_q.$$

Next whenever we have a family $S = (\Phi_n)_{n \in \mathbb{N}}$ of systems of ideals of A [4, (1.1)], we can construct the generalized Hughes complex $\mathcal{H}(S, M)$

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for M with respect to S [4, (1.3)]. The generalized Hughes complex for A (resp. B) with respect to $(\Phi_n)_{n\in\mathbb{N}}$ (resp. $(\Phi'_n)_{n\in\mathbb{N}}$) will be written

$$\mathcal{H}((\Phi_n)_{n\in\mathbb{N}},A)\colon 0 \longrightarrow A \xrightarrow{k^{-1}} K^0 \xrightarrow{k^0} K^1 \xrightarrow{k^1} \cdots \longrightarrow K^n \xrightarrow{k^n} K^{n+1} \longrightarrow \cdots$$

$$\mathcal{H}((\Phi'_n)_{n\in\mathbb{N}},B)\colon 0 \longrightarrow B \xrightarrow{h^{-1}} H^0 \xrightarrow{h^0} H^1 \xrightarrow{h^1} \cdots \longrightarrow H^n \xrightarrow{h^n} H^{n+1} \longrightarrow \cdots$$

where, for each $n \in \mathbb{N}$,

$$\Phi_n = \{I \mid I \text{ is an ideal of } A \text{ such that } ht_A(I) \geqslant n\}$$
$$\Phi'_n = \{J \mid J \text{ is an ideal of } B \text{ such that } ht_B(J) \geqslant n\}$$

So we have the property that, for each $n \in \mathbb{N}_0$: [5, (1.3)]

$$K^n = D_{\Phi_{n+1}}(\text{Coker } k^{n-2}) \quad \text{and} \quad H^n = D_{\Phi'_{n+1}}(\text{Coker } h^{n-2}).$$

One of the main results of R. Y. Sharp [5] is Theorem 2.3, which shows that there are A-isomorphisms $A^n \to K^n$ and B-isomorphisms $B^n \to H^n$ for every integers $n \geq -2$.

In this paper we will be interested in comparing $\mathcal{H}((\Phi_n)_{n\in\mathbb{N}},A)\otimes B$ with $\mathcal{H}((\Phi'_n)_{n\in\mathbb{N}},B)$ to prove theorem (3.1) which is the generalization of the main theorem of R. Y. Sharp [2].

2. Preliminaries

A system of ideals of A is a non-empty set Φ of ideals of A such that, whenever $I, J \in \Phi$, there exists $K \in \Phi$ such that $K \subseteq IJ$. Such a system of ideals Φ determines a Φ-torsion functor $\Gamma_\Phi : \mathcal{C}(A) \to \mathcal{C}(A)$. This is a subfunctor of the identity functor on $\mathcal{C}(A)$ for which

$$\Gamma_\Phi(M) = \{m \in M \mid Im = 0 \text{ for some } I \in \Phi\}$$

for each A-module M. A module M is called *Φ-torsion* if each element of M is annihilated by an ideal belonging to Φ. Since Φ is a direct set with respect to reverse inclusion, it gives rise to additive, left exact functors

$$\varinjlim_{I\in\Phi} \text{Hom}_A(A/I,\cdot) \quad \text{and} \quad D_\Phi := \varinjlim_{I\in\Phi} \text{Hom}_A(I,\cdot)$$

from $\mathcal{C}(A)$ to itself: the former is naturally equivalent to Γ_Φ and the latter functor D_Φ is called the Φ-*transform*.

For each $i \in \mathbb{N}_0$, let

$$H^i_\Phi := \varinjlim_{I \in \Phi} Ext^i_A(A/I, \cdot)$$

which is naturally equivalent to the i-th right derived functor of Γ_Φ and let $R^i D_\Phi$ denote the i-th right derived functor of D_Φ. Note that, for each $i \in \mathbb{N}$, the functors $R^i D_\Phi$ and H^{i+1}_Φ from $\mathcal{C}(A)$ to itself are naturally equivalent.

For each $I \in \Phi$ and $x \in M$, we define $\lambda_{I,x} : I \to M$ by $\lambda_{I,x}(a) = ax$ for all $a \in I$. Then there is a natural transformation of functors from $\mathcal{C}(A)$ to itself

$$\eta_\Phi : Id \to D_\Phi$$

such that, for each A-module M and each $x \in M$, $(\eta_\Phi(M))(x)$ is the natural image in $D_\Phi(M)$ of the homomorphism $\lambda_{I,x} \in \mathrm{Hom}_A(I, M)$ for any $I \in \Phi$. Furthermore, if M varies through the category $\mathcal{C}(B)$ then $\eta_\Phi(M)$ constitutes a morphism of functors $\eta_\Phi : Id \to D_\Phi$ from $\mathcal{C}(B)$ to itself.

LEMMA 2.1. *Suppose $U' \subseteq U$ are subsets of* Spec(A) *such that each element of $U \backslash U'$ is minimal with respect to inclusion in U. Let M be an A-module such that* $\mathrm{Supp}_A(M) \subseteq U$. *There is an A-homomorphism*

$$\zeta(M) : M \to \bigoplus_{p \in U \backslash U'} M_p$$

for which, if $m \in M$, the component of $\zeta(M)(m)$ in the direct summand M_p is $m/1$. If M is a B-module such that $\mathrm{Supp}_A(M) \subseteq U$, *$\zeta(M)$ is a B-homomorphism. Furthermore* $\mathrm{Supp}(\mathrm{Ker}\ \zeta(M)) \subseteq U'$ *and* $\mathrm{Supp}(\mathrm{Coker}\ \zeta(M)) \subseteq U'$

Proof. Confer [1, (2.2), (2.3) and (2.5)].

LEMMA 2.2. *Let $n \geqslant 0$. Then the exact sequence $A^{n-2} \xrightarrow{d^{n-2}} A^{n-1} \to \mathrm{Coker}\ d^{n-2} \to 0$ induces a natural B-isomorphism*

$$\theta^n : B \underset{A}{\otimes} (\mathrm{Coker}\ d^{n-2}) \to \mathrm{Coker}(id_B \otimes d^{n-2})$$

for which the diagram

$$\begin{array}{ccccccc} B\otimes A^{n-2} & \longrightarrow & B\otimes A^{n-1} & \longrightarrow & B\otimes \mathrm{Coker}\ d^{n-2} & \longrightarrow & 0 \\ \| & & \| & & \downarrow \theta^{n} & & \\ B\otimes A^{n-2} & \longrightarrow & B\otimes A^{n-1} & \longrightarrow & \mathrm{Coker}(id_B\otimes\ d^{n-2}) & \longrightarrow & 0 \end{array}$$

commutes.

Proof. Confer [2, (2.2)].

LEMMA 2.3. *Let Φ be a system of ideals of A and $g : M \to N$ be an A-module homomorphism such that* Ker g *and* Coker g *are Φ-torsion, then g induces an A-isomorphism*

$$D_\Phi(g) : D_\Phi(M) \to D_\Phi(N).$$

Furthermore, if $g : M \to N$ is a B-module homomorphism such that Ker g *and* Coker g *are Φ-torsion, it induces an B-isomorphism*

$$D_\Phi(g) : D_\Phi(M) \to D_\Phi(N).$$

Proof. Since D_Φ is an left exact functor, the exact sequences

$$0 \to \mathrm{Ker}\ g \to M \to \mathrm{Im}\ g \to 0,$$

$$0 \to \mathrm{Im}\ g \to N \to \mathrm{Coker}\ g \to 0$$

induces a long exact sequences

$$0 \to D_\Phi(\mathrm{Ker}\ g) \to D_\Phi(M) \to D_\Phi(\mathrm{Im}\ g) \to R^1 D_\Phi(\mathrm{Ker}\ g) \to R^1 D_\Phi(M) \to \cdots$$

and

$$0 \to D_\Phi(\mathrm{Im}\ g) \to D_\Phi(N) \to D_\Phi(\mathrm{Coker}\ g) \to R^1 D_\Phi(\mathrm{Im}\ g) \to R^1 D_\Phi(N) \to \cdots .$$

But, by [5, (1.5)], $D_\Phi(\mathrm{Ker}\ g) = 0$, $R^1 D_\Phi(\mathrm{Ker}\ g) = H^2_\Phi(\mathrm{Ker}\ g) = H^2_\Phi(D_\Phi(\mathrm{Ker}\ g)) = 0$, and $D_\Phi(\mathrm{Coker}\ g) = 0$. Hence we have an isomorphism $D_\Phi(g) : D_\Phi(M) \to D_\Phi(N)$.

LEMMA 2.4. *Let $n \in \mathbb{N}$, $\Phi_n = \{I \mid I$ is an ideal of A such that $ht(I) \geqslant n\}$, $\Phi'_n = \{J \mid J$ is an ideal of B such that $ht(J) \geqslant n\}$, and $f : A \to B$ a ring homomorphism. If $ht(q) = ht(q^c)$ for every $q \in \operatorname{Spec}(B)$, then $\Phi^f_n = \{I^e \mid I \in \Phi_n\}$ is a cofinal subset of Φ'_n.*

Proof. Let J be in Φ'_n and $J = Q_1 \cap Q_2 \cap \cdots \cap Q_n$ with $\sqrt{Q_i} = P_i$ $(i = 1, 2, \ldots, n)$ a minimal primary decomposition of J. Then, for every $\mathfrak{p} \in \operatorname{Spec}(A)$ containing J^c, $\mathfrak{p}$ must contain P_i^c for some i. Hence we have $ht_A(\mathfrak{p}) \geqslant ht(P_i^c) = ht(P_i) \geqslant n$. Therefore, for every J in Φ'_n, there is J^c in Φ_n such that $(J^c)^e \subseteq J$.

LEMMA 2.5. *Let Φ be a system of ideals of A. Then $\Phi^f = \{I^e \mid I \in \Phi\}$ is a system of ideals of B and there is a natural equivalence of functors $\varepsilon_\Phi : D_{\Phi^f}(\cdot) \to D_\Phi(\cdot)$ from $\mathcal{C}(B)$ to itself satisfying for each B-module G, the diagram*

$$\begin{array}{ccc} G & \xrightarrow{\eta_{\Phi^f}(G)} & D_{\Phi^f}(G) \\ & \searrow^{\eta_\Phi} & \downarrow{\scriptstyle \varepsilon_\Phi(G)} \\ & & D_\Phi(G) \end{array}$$

commutes.

Proof. Confer [6, (4.2.3)].

LEMMA 2.6. *Suppose that G be an A-module such that $\operatorname{Supp}_A(G) \subseteq \{P \in \operatorname{Spec}(A) \mid ht_A(P) \geqslant n\}$. There is an A-isomorphism*

$$\sigma_n(G) : D_{\Phi_{n+1}}(G) \to \bigoplus_{\substack{ht(P)=n \\ P \in \operatorname{Spec}(A)}} G_P$$

such that the diagram

$$\begin{array}{ccc} G & \xrightarrow{\eta_{\Phi_{n+1}}(G)} & D_{\Phi_{n+1}}(G) \\ & \searrow^{\varsigma(G)} & \downarrow{\scriptstyle \sigma_n(G)} \\ & & \oplus_{ht(P)=n} G_P \end{array}$$

commutes. Furthermore if G is a B-module, then $\sigma_n(G)$ is a B-isomorphism.

Proof. The diagram below is commutative

$$\begin{array}{ccc} G & \xrightarrow{\eta_{\Phi_{n+1}}(G)} & D_{\Phi_{n+1}}(G) \\ \zeta(G)\big\downarrow & & \big\downarrow D_{\Phi_{n+1}}(\zeta(G)) \\ \bigoplus\limits_{ht(P)=n} G_P & \xrightarrow{\eta_{\Phi_{n+1}}(\oplus G_P)} & D_{\Phi_{n+1}}(\bigoplus\limits_{ht(P)=n} G_P). \end{array}$$

It follows from Lemma 2.1 and Lemma 2.3 that $D_{\Phi_{n+1}}(\zeta(G))$ is an A-isomorphism and, by [4, (1.2)], $\eta_{\Phi_{n+1}}(\oplus G_P)$ is an A-isomorphism because $H^i_{\Phi_{n+1}}(\oplus G_P) = 0$ for each $i \in \mathbb{N}_0$. We define

$$\sigma_n(G) = \eta_{\Phi_{n+1}}(\oplus G_P)^{-1} \circ D_{\Phi_{n+1}}(\zeta(G)).$$

3. Main Theorem

THEOREM 3.1. *Let A and B be rings and $f : A \to B$ be a ring homomorphism. Assume that $ht_B(q) = ht_A(q^c)$ for all prime ideals q of B. Then there is a natural isomorphism of complexes of B-modules and homomorphisms*

$$\psi : B \underset{A}{\otimes} C(A) \to C(B).$$

Proof. It is enough to show that $B \otimes_A K^n$ is B-isomorphic to H^n for every $n = -2, -1$ and $n \in \mathbb{N}_0$. Define $\psi^{-2} : B \otimes K^{-2} \to H^{-2}$ to be a zero map and $\psi^{-1} : B \otimes_A A \to B$ to be the natural B-isomorphism. We shall construct a family $\{\psi^n\}_{n \geqslant 0}$ of B-isomorphism $\psi^n : B \otimes_A K^n \to H^n$ by induction on n. So suppose, inductively, that $n \geqslant 0$ and we have constructed B-isomorphisms ψ^{-2}, ψ^{-1}, $\psi^0, \ldots, \psi^{n-1}$ such that the digram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B \otimes A & \xrightarrow{id_B \otimes k^{-1}} & B \otimes K^0 & \xrightarrow{id_B \otimes k^0} & B \otimes K^1 & \longrightarrow & \cdots \\ & & \big\downarrow \psi^{-1} & & \big\downarrow \psi^0 & & \big\downarrow \psi^1 & & \\ 0 & \longrightarrow & B & \xrightarrow{h^{-1}} & H^0 & \xrightarrow{h^0} & H^1 & \longrightarrow & \cdots \end{array}$$

$$\begin{array}{ccccc} \longrightarrow & B \otimes K^{n-2} & \xrightarrow{id_B \otimes k^{n-2}} & B \otimes K^{n-1} \\ & \big\downarrow \psi^{n-2} & & \big\downarrow \psi^{n-1} \\ \longrightarrow & H^{n-2} & \xrightarrow{h^{n-2}} & H^{n-1} \end{array}$$

commutes. From our inductive assumptions, we obtain a commutative diagram

$$\begin{array}{ccc} B\otimes K^{n-1} & \longrightarrow & \operatorname{Coker}(id_B\otimes k^{n-2}) \\ \Big\downarrow{\scriptstyle \psi^{n-1}} & & \Big\downarrow{\scriptstyle \bar{\psi}^{n-1}} \\ H^{n-1} & \longrightarrow & \operatorname{Coker}(h^{n-2}) \end{array}$$

in which horizontal maps are natural and $\bar{\psi}_{n-1}$ is the induced isomorphism. Lemma 2.2 gives rise to B-isomorphism $\bar{\psi}^{n-1}\circ\theta^n : B\otimes\text{Coker } k^{n-2} \to \text{Coker } h^{n-2}$ such that

$$\begin{array}{ccc} B\otimes K^{n-1} & \xrightarrow{id_B\otimes\pi} & B\otimes\text{Coker } k^{n-2} \\ \Big\downarrow{\scriptstyle \psi^{n-1}} & & \Big\downarrow{\scriptstyle \bar{\psi}^{n-1}\circ\theta^n} \\ H^{n-1} & \xrightarrow{\pi'} & \text{Coker } h^{n-2} \end{array}$$

commutes in which π and π' are natural maps. The functor $D_{\Phi_{n+1}}$ gives rise to the commutative diagram

$$\begin{array}{ccccc} B\otimes K^{n-1} & \xrightarrow{id_B\otimes\pi} & B\otimes\text{Coker } k^{n-2} & \longrightarrow & D_{\Phi_{n+1}}(B\otimes\text{Coker } k^{n-2}) \\ \Big\downarrow{\scriptstyle \psi^{n-1}} & & \Big\downarrow{\scriptstyle \bar{\psi}^{n-1}\circ\theta^n} & & \Big\downarrow{\scriptstyle D_{\Phi_{n+1}}(\bar{\psi}^{n-1}\circ\theta^n)} \\ H^{n-1} & \xrightarrow{\pi'} & \text{Coker } h^{n-2} & \longrightarrow & D_{\Phi_{n+1}}(\text{Coker } h^{n-2}). \end{array}$$

We know that $\text{Supp}_A(\text{Coker } k^{n-2}) \subseteq \{P\in\text{Spec}(A) \mid ht(P)\geqslant n\}$. Hence it follows from Lemma 2.6 that there is a B-isomorphism

$$\alpha : B\otimes D_{\Phi_{n+1}}(\text{Coker } k^{n-2}) \to D_{\Phi_{n+1}}(B\otimes\text{Coker } k^{n-2})$$

such that the diagram

$$\begin{array}{ccc} B\otimes\text{Coker } k^{n-2} & \xrightarrow{id_B\otimes\eta_{\Phi_{n+1}}(\text{Coker } k^{n-2})} & B\otimes D_{\Phi_{n+1}}(\text{Coker } k^{n-2}) \\ & \searrow{\scriptstyle \eta_{\Phi_{n+1}}(B\otimes\text{Coker } k^{n-2})} & \Big\downarrow{\scriptstyle \alpha} \\ & & D_{\Phi_{n+1}}(B\otimes\text{Coker } k^{n-2}) \end{array}$$

commutes.

On the other hand, by Lemma 2.5 and Lemma 2.4, there is a B-isomorphism

$$\beta : D_{\Phi_{n+1}}(\text{Coker } h^{n-2}) \to D_{\Phi'_{n+1}}(\text{Coker } h^{n-2})$$

such that

$$\begin{array}{ccc} \text{Coker } h^{n-2} & \xrightarrow{\eta_{\Phi_{n+1}}(\text{Coker } h^{n-2})} & D_{\Phi_{n+1}}(\text{Coker } h^{n-2}) \\ & \searrow^{\eta_{\Phi'_{n+1}}(\text{Coker } h^{n-2})} & \downarrow \beta \\ & & D_{\Phi'_{n+1}}(\text{Coker } h^{n-2}) \end{array}$$

commutes. Therefore we obtain a B-isomorphism

$$\psi^n := \beta \circ D_{\Phi_{n+1}}(\bar{\psi}^{n-1} \circ \theta^n) \circ \alpha : D_{\Phi_{n+1}}(\text{Coker } k^{n-2}) \otimes B \\ \to D_{\Phi'_{n+1}}(\text{Coker } h^{n-2})$$

such that the diagram

$$\begin{array}{ccc} B \otimes K^{n-1} & \xrightarrow{id_B \otimes k^{n-1}} & B \otimes K^n \\ \downarrow \psi^{n-1} & & \downarrow \psi^n \\ H^{n-1} & \xrightarrow{h^{n-1}} & H^n \end{array}$$

commutes.

The corollary below follows easily, which is the one of the main theorems of [1].

Corollary 3.2. *Let S be a multiplicative closed subset of A and let $C(A)$(resp. $C(S^{-1}A)$) be a Cousin complex for A(resp. $S^{-1}A$). There is an isomorphism of $S^{-1}A$-modules and homomorphisms $\psi = \{\psi^n\}_{n \geqslant 2}$: $S^{-1}(C(A)) \to C(S^{-1}A)$ which is such that $\psi^{-1} : S^{-1}A \to S^{-1}A$ is identity.*

Example 3.3. Let A be a subring of the integral domain B, and suppose that A is integrally closed and that B is integral over A. Then $ht_B(q) = ht_A(q^c)$ for every $q \in \text{Spec}(B)$. Therefore we have a natural isomorphism of complexes of B-modules and homomorphisms

$$\psi : B \otimes_A C(A) \to C(B).$$

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