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ON CERTAIN SUBCLASSES OF MEROMORPHICALLY MULTIVALENT FUNCTIONS

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1. Introduction

Let $\sum_p$ denote the class of functions of the form

$$f(z) = \frac{a_{-p}}{z^p} + \sum_{k=0}^{\infty} a_k z^k \quad (a_{-p} \neq 0, p \in N = \{1, 2, \dots\})$$

which are regular in the punctured disk $D = \{z : \ 0 < \ |z| \ < 1\}$. For any integer n, let the operator I^n operating on $f \in \sum_p$ be defined by

$$I^n f(z) = \frac{a_{-p}}{z^p} + \sum_{k=1}^{\infty} (p+k)^{-n} a_{k-1} z^{k-1}.$$

Obviously, we have

$$I^n(I^m f(z)) = I^{n+m} f(z)$$

for all integers m and n. For any nonpositive integer n and $p = 1$, the operators I^n are the differential operators studied by Uralegaddi and Somanatha [6,7]. Also the operators I^n are closely related to the multiplier transformations introduced by Flett [2].

For any integer n, let $\sum_{n,p}(\alpha)$ denote the class of functions $f \in \sum_p$ satisfying the condition

$$\mathrm{Re}\left\{ \frac{I^{n-1} f(z)}{I^n f(z)} - (p+1) \right\} < -\alpha \quad (0 \leq \alpha < p, z \in U = \{z : |z| < 1\}).$$

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In this paper, we prove that for the classes $\sum_{n,p}(\alpha)$ of functions in $\sum_p$, $\sum_{n,p}(\alpha) \subset \sum_{n+1,p}(\alpha)$ holds. Since $\sum_{0,p}(\alpha)$ equals $\sum_p^*(\alpha)$ (the class of meromorphically p-valent starlike functions of order α [4]), all members in $\sum_{n,p}(\alpha)$ are p-valent starlike for any nonpositive integer n. Further properties preserving integrals are considered. Our results generalize some results of Bajpai [1], Goel and Sohi [3] and Uralegaddi and Somanatha [7].

2. Main results

We begin with the statement of the following lemma due to Miller and Mocaun [5].

LEMMA. *Let $\phi(u,v)$ be a complex valued function, $\phi : D \to C$, $D \subset C^2$ (C is the complex plane), and let $u = u_1 + iu_2, v = v_1 + iv_2$. Suppose that the function $\phi(u,v)$ satisfies the following conditions:*

(i) *$\phi(u,v)$ is continuous in D;*
(ii) *$(1,0) \in D$ and $\mathrm{Re}\{\phi(1,0)\} > 0$;*
(iii) *$\mathrm{Re}\{\phi(iu_2, v_1)\} \le 0$ for all $(iu_2, v_1) \in D$ such that $v_1 \le \frac{-(1+u_2^2)}{2}$.*

Let $q(z) = 1 + q_1 z + q_2 z^2 + \cdots$ be regular in U such that $(q(z), zq'(z)) \in D$ for all $z \in U$. If

$$\mathrm{Re}\{\phi(q(z), zq'(z))\} > 0 \quad (z \in U).$$

Then $\mathrm{Re}\{q(z)\} > 0$ $(z \in U)$.

With the aid of above Lemma, we derive

THEOREM 1. *If $f \in \sum_{n,p}(\alpha)$, then $f \in \sum_{n+1,p}(\beta)$, where*

$$\beta = \frac{3 + 2(p+\alpha) - \sqrt{(2(p-\alpha)+1)^2 + 8}}{4}. \tag{2.1}$$

Proof. Define the function $q(z)$ by

$$\frac{I^n f(z)}{I^{n+1} f(z)} = \gamma + (1-\gamma) q(z), \tag{2.2}$$

where

$$\gamma = \frac{2(p-\alpha)+1+\sqrt{(2(p-\alpha)+1)^2+8}}{4} \quad (\gamma > 1). \tag{2.3}$$

We see that $q(z) = 1 + q_1 z + q_2 z^2 + \cdots$ is regular in U. Making use of the logarithmic differentiations of both sides in (2.2) and using the identity

$$z(I^n f(z))' = I^{n-1} f(z) - (p+1) I^n f(z), \tag{2.4}$$

we obtain

$$\frac{I^{n-1} f(z)}{I^n f(z)} = \gamma + (1-\gamma) q(z) + \frac{(1-\gamma) z q'(z)}{\gamma + (1-\gamma) q(z)} \tag{2.5}$$

or

$$\begin{aligned} -\operatorname{Re}&\left\{\frac{I^{n-1} f(z)}{I^n f(z)} - (p+1) + \alpha\right\} \\ &= \operatorname{Re}\left\{p + 1 - (\alpha+\gamma) - (1-\gamma) q(z) - \frac{(1-\gamma) z q'(z)}{\gamma + (1-\gamma) q(z)}\right\} > 0. \end{aligned} \tag{2.6}$$

Let us define the function $\phi(u, v)$ by

$$\phi(u, v) = p + 1 - (\alpha+\gamma) - (1-\gamma) u - \frac{(1-\gamma) v}{\gamma + (1-\gamma) u}. \tag{2.7}$$

Then $\phi(u, v)$ satisfies

(i) $\phi(u, v)$ is continuous in $D = (C - \{\frac{\gamma}{\gamma-1}\}) \times C$;

(ii) $(1, 0) \in D$ and $\operatorname{Re}\{\phi(1, 0)\} = p - \alpha > 0$;

(iii) for all $(iu_2, v_1) \in D$ such that $v_1 \leq \frac{-(1+u_2^2)}{2}$,

$$\begin{aligned} \operatorname{Re}\{\phi(iu_2, v_1)\} &= p + 1 - (\alpha+\gamma) - \frac{\gamma(1-\gamma) v_1}{\gamma^2 + (1-\gamma)^2 u_2^2} \\ &\leq p + 1 - (\alpha+\gamma) + \frac{\gamma(1-\gamma)(1+u_2^2)}{2(\gamma^2 + (1-\gamma)^2 u_2^2)} \\ &\leq 0. \end{aligned}$$

Thus the function $\phi(u,v)$ satisfies the conditions in our Lemma. This shows that $\mathrm{Re}\{q(z)\} > 0$ $(z \in U)$, hence

$$\mathrm{Re}\left\{ \frac{I^n f(z)}{I^{n+1} f(z)} \right\} < \gamma \quad (z \in U) \tag{2.8}$$

or

$$\mathrm{Re}\left\{ \frac{I^n f(z)}{I^{n+1} f(z)} - (p+1) \right\} < -\beta \quad (z \in U,\ 0 \le \beta < p), \tag{2.9}$$

where β is given by (2.1). Therefore we complete the proof of the theorem.

Since $\beta - \alpha > 0$ in Theorem 1, we have

COROLLARY 1. *$\sum_{n,p}(\alpha) \subset \sum_{n+1,p}(\alpha)$ for any integer n.*

REMARK. For nonpositive integers n and $p = 1$, Corollary 1 is a similar result obtained by Uralegaddi and Somanatha [7].

Putting $n = 0$, $p = 1$ and $\alpha = 0$ in Corollary 1, we obtain the following result of Bajpai [1].

COROLLARY 2. *If $f(z) = \frac{a_{-1}}{z} + \sum_{k=0}^{\infty} a_k z^k$ $(a_{-1} \neq 0)$ is meromorphically starlike, then so is*

$$F_1(z) = \frac{1}{z^2} \int_0^z t f(t) dt. \tag{2.10}$$

Next, we prove

THEOREM 2. *Let $f \in \sum_{n,p}(\alpha)$ and let*

$$F_c(z) = \frac{c}{z^{c+p}} \int_0^z t^{c+p-1} f(t) dt \quad (c \ge 1). \tag{2.11}$$

Then $F_c \in \sum_{n,p}(\beta)$, where

$$\beta = \frac{2(p+\alpha) + 2c + 1 - \sqrt{(2(p-\alpha) - 2c + 3)^2 + 8(2(p+1-\alpha)(c-1)+1)}}{4} \tag{2.12}$$

Proof. Let $f \in \sum_{n,p}(\alpha)$. Then we have

$$\text{(2.13)} \qquad \operatorname{Re}\left\{\frac{I^{n-1}f(z)}{I^n f(z)} - (p+1)\right\} < -\alpha.$$

From the definition of F_c, we obtain

$$\text{(2.14)} \qquad z(I^n F_c(z))' = cI^n f(z) - (c+p)I^n F_c(z)$$

and also

$$\text{(2.15)} \qquad z(I^n F_c(z))' = I^{n-1}F_c(z) - (p+1)I^n F_c(z).$$

Using (2.14) and (2.15), the condition (2.13) may be written as

$$\text{(2.16)} \qquad \operatorname{Re}\left\{\frac{\frac{I^{n-2}F_c(z)}{I^{n-1}F_c(z)} + (c-1)}{1 + (c-1)\frac{I^n F_c(z)}{I^{n-1}F_c(z)}} - (p+1)\right\} < -\alpha.$$

Define the function $q(z)$ by

$$\text{(2.17)} \qquad \frac{I^{n-1}F_c(z)}{I^n F_c(z)} = \gamma + (1-\gamma)q(z),$$

where

(2.18)

$$\gamma = \frac{2(p-\alpha) - 2c + 3 + \sqrt{(2(p-\alpha) - 2c + 3)^2 + 8(2(p+1-\alpha)(c-1) + 1)}}{4} \quad (\gamma > 1).$$

Then $q(z) = 1 + q_1 z + q_2 z^2 + \cdots$ is regular in U. Differentiating (2.17) logarithmically and simplifying, we have

$$\text{(2.19)} \qquad \frac{\frac{I^{n-2}F_c(z)}{I^{n-1}F_c(z)} + (c-1)}{1 + (c-1)\frac{I^n F_c(z)}{I^{n-1}F_c(z)}} - (p+1) = -(p+1) + \gamma + (1-\gamma)q(z) + \frac{(1-\gamma)zq'(z)}{(\gamma + c - 1) + (1-\gamma)q(z)}.$$

It follows from (2.19) that

$$(2.20)\qquad -\operatorname{Re}\left\{\frac{\frac{I^{n-2}F_c(z)}{I^{n-1}F_c(z)}+(c-1)}{1+(c-1)\frac{I^nF_c(z)}{I^{n-1}F_c(z)}}-(p+1)+\alpha\right\}$$
$$=\operatorname{Re}\left\{p+1-(\alpha+\gamma)-(1-\gamma)q(z)-\frac{(1-\gamma)zq'(z)}{(\gamma+c-1)+(1-\gamma)q(z)}\right\}$$
$$>0.$$

If we define the function $\phi(u,v)$ by

$$(2.21)\quad \phi(u,v)=p+1-(\alpha+\gamma)-(1-\gamma)u-\frac{(1-\gamma)v}{(\gamma+c-1)+(1-\gamma)u},$$

then $\phi(u,v)$ satisfies

(i) $\phi(u,v)$ is continuous in $D=(C-\{\frac{\gamma+c-1}{\gamma-1}\})\times C$;

(ii) $(1,0)\in D$ and $\operatorname{Re}\{\phi(1,0)\}=p-\alpha>0$;

(iii) for all $(iu_2,v_1)\in D$ such that $v_1\le\frac{-(1+u_2^2)}{2}$,

$$\operatorname{Re}\{\phi(iu_2,v_1)\}=p+1-(\alpha+\gamma)-\frac{(\gamma+c-1)(1-\gamma)v_1}{(\gamma+c-1)^2+(1-\gamma)^2u_2^2}$$
$$\le p+1-(\alpha+\gamma)+\frac{(\gamma+c-1)(1-\gamma)(1+u_2^2)}{2\{(\gamma+c-1)^2+(1-\gamma)^2u_2^2\}}$$
$$\le 0.$$

Since $\phi(u,v)$ satisfies the conditions in Lemma, we have that $\operatorname{Re}\{q(z)\}>0$ $(z\in U)$. This prove that

$$(2.22)\qquad \operatorname{Re}\left\{\frac{I^{n-1}F_c(z)}{I^nF_c(z)}\right\}<\gamma\quad(z\in U)$$

or

$$(2.23)\qquad \operatorname{Re}\left\{\frac{I^{n-1}F_c(z)}{I^nF_c(z)}-(p+1)\right\}<-\beta\quad(z\in U,0\le\beta<p),$$

where β is given by (2.12). That is, $F_c\in\sum_{n,p}(\beta)$.

Similarly, from Theorem 2, we have

COROLLARY 3. *If $f \in \sum_{n,p}(\alpha)$, then the integral operator F_c defined by (2.11) belongs to the class $\sum_{n,p}(\alpha)$.*

Taking $n = 0, p = 1$ and $\alpha = 0$ in Corollary 3, we obtain the following corresponding result of Goel and Sohi [3].

COROLLARY 4. *If $f(z) = \frac{a_{-1}}{z} + \sum_{k=0}^{\infty} a_k z^k$ $(a_{-1} \neq 0)$ is meromorphically starlike, then so is the integral operator F_c defined by (2.11).*

The following theorem gives us a characterization of the class $\sum_{n,p}(\alpha)$.

THEOREM 3. *$f \in \sum_{n,p}(\alpha)$ if and only if the integral operator F_1 defined by (2.10) belongs to the class $\sum_{n-1,p}(\alpha)$.*

Proof. For $c = 1$, the identities (2.14) and (2.15) reduce to $I^n f(z) = I^{n-1} F_1(z)$ and hence $I^{n-1} f(z) = I^{n-2} F_1(z)$. Therefore

$$\frac{I^{n-1} f(z)}{I^n f(z)} = \frac{I^{n-2} F_1(z)}{I^{n-1} F_1(z)} \tag{2.24}$$

and the result follows.

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