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STABILITIES IN DIFFERENTIAL SYSTEMS

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1. Definitions and Notations

We consider the nonlinear nonautonomous differential system

$$x' = f(t,x), \qquad x(t_0) = x_0, \tag{1}$$

where $f \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$ and $\mathbb{R}^+ = [0,\infty)$. We assume that the Jacobian matrix $f_x = \partial f/\partial x$ exists and is continuous on $\mathbb{R}^+ \times \mathbb{R}^n$ and that $f(t,0) \equiv 0$. The symbol $|\cdot|$ denotes arbitrary norm in $\mathbb{R}^n$.

Let $x(t) = x(t,t_0,x_0)$ be denoted by the unique solution of (1) through (t_0,x_0) in $\mathbb{R}^+ \times \mathbb{R}^n$ such that $x(t_0,t_0,x_0) = x_0$. The trivial solution $x = 0$ of (1) is said to be

(S) *stable* if for any $\varepsilon > 0$ and $t_0 \geq 0$, there exists a $\delta = \delta(\varepsilon, t_0) > 0$ such that $|x_0| < \delta$ and $t \geq t_0$ imply $|x(t)| < \varepsilon$,

(US) *uniformly stable* if the δ in (S) is independent of t_0,

(ULS) *uniformly Lipschitz stable* if there are constants $M > 0$ and $\delta > 0$ such that $|x(t)| \leq M|x_0|$ whenever $|x_0| \leq \delta$ and $t \geq t_0 \geq 0$,

(EAS) *exponentially asymptotically stable* if there are constants $K > 0$ and $c > 0$ such that $|x(t)| \leq K|x_0|e^{-c(t-t_0)}$ for $t \geq t_0 \geq 0$ provided that $|x_0| < \infty$,

(hS) *h-stable* there exist $c \geq 1, \delta > 0$ and a positive bounded continuous function h on $\mathbb{R}^+$ such that $|x(t)| \leq c|x_0|h(t)h(t_0)^{-1}$ for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$.

In [2] the implications for the system (1) among the above types of stability notions were presented:

$$\text{hS} \Rightarrow \text{EAS} \Rightarrow \text{ULS} \Rightarrow \text{US} \Rightarrow \text{S}.$$

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The converses of the implications do not hold in general. For the counterexamples, see [8].

Consider the functional differential equation (F.D.E.) with finite retardations

$$x'(t) = f(t, x_t), \tag{2}$$

where $f \in C(\mathbb{R}^+ \times C([-r,0],\mathbb{R}^n), f(t,0) \equiv 0, x_t = x(t+\theta), \theta \in [-r,0]$, $r > 0$. For any $\varphi \in C([-r,0],\mathbb{R}^n)$, set

$$\|\varphi\| = \sup_{-r\le\theta\le 0} |\varphi(\theta)|.$$

We always assume that the solution of (2) with initial value $x_{t_0} = \varphi_0$ is existent and unique.

Let $x_t(\cdot) \in C([0,t],\mathbb{R}^n)$. If $x \in C(\mathbb{R}^+,\mathbb{R}^n)$, then for each $t \in \mathbb{R}^+$, $x_t(\cdot)$ is the restriction of $x(s)$ given by $x_t(s) = x(s)$ for $0 \le s \le t$, and the norm is defined by

$$\|x_t(\cdot)\| = \sup_{0\le s\le t} |x(s)|.$$

Let $S_\rho = \{x \in \mathbb{R}^n : |x| < \rho,\ \rho > 0\}$. Here, we consider the system of equations of the form

$$x'(t) = F(t, x(t), x_t(\cdot)), \quad t \in \mathbb{R}^+, \tag{3}$$

where $F \in C(\mathbb{R}^+ \times S_\rho \times C(\mathbb{R}^+,\mathbb{R}^n),\mathbb{R}^n)$ and $F(t,0,0) \equiv 0$. In this paper we investigate the stability notions of ULS, EAS and hS for the systems (1), (2) and (3).

2. Uniform Lipschitz Stability (ULS)

It is well-known that ULS implies US for the system (1). Also, ULS implies US for the system (2).

The solution $x = 0$ of (2) is said to be *ULS* if there exist constants $M > 0$ and $\delta > 0$ such that $t \ge t_0$ and $\|\varphi_0\| < \delta$ imply that $|x_t(t_0,\varphi_0)| < M\|\varphi_0\|$.

THEOREM 1. *If $x = 0$ of the equation (2) is ULS, then it is US.*

Proof. Since $x = 0$ of (2) is ULS, there are constants $M > 0$ and $\delta_1 > 0$ such that $|x_t(t_0, \varphi_0)| \le M\|\varphi_0\|$ whenever $\|\varphi_0\| < \delta_1$ and $t \ge t_0$. For any given $\varepsilon > 0$, we take $\delta = \min\{\delta_1, \varepsilon/M\}$. Then we have

$$|x_t(t_0, \varphi_0)| \le M\|\varphi_0\| \le M\delta < \varepsilon$$

if $\|\varphi_0\| < \delta$. This implies that $x = 0$ of (2) is US.

We consider the homogeneous linear equation

$$x'(t) = L(t, x_t), \tag{4}$$

where $L \in C(\mathbb{R} \times C([-r, 0], \mathbb{R}^n)$. We assume the following conditions [6, p.142]:

(i) $L(t, \varphi)$ is linear in φ,

(ii) $L(t, \varphi) = \int_{-r}^{0}[d_\theta \eta(t, \theta)]\varphi(\theta)$,

where $\eta(t, \theta)$ is an $n \times n$ matrix function which is continuous on $(-r, 0)$, measurable in $\mathbb{R} \times \mathbb{R}$ and has bounded variation in θ on $[-r, 0]$ for each t,

(iii) $|L(t, \varphi)| \le m(t)|\varphi|$ for some $m \in \mathfrak{L}_1^{\mathrm{loc}}(\mathbb{R}, \mathbb{R})$, the space of functions mapping $\mathbb{R} \to \mathbb{R}$ which are Lebesgue integrable on each compact subset of $\mathbb{R}$. Now, consider the perturbation of (4):

$$x'(t) = L(t, x_t) + g(t), \tag{4-P}$$

where $g \in \mathfrak{L}_1^{\mathrm{loc}}([t_0, \infty), \mathbb{R}^n)$.

THEOREM 2. *Consider the scalar differential equation*

$$u' = Mh(u), \quad M > 0, u(t_0) = u_0 > 0 \tag{5}$$

and suppose that $u = 0$ of (5) is ULS. If $x = 0$ of (4) is ULS, then $x = 0$ of (4-P) is also ULS.

Proof. Since $x = 0$ of (4) is ULS, there exist $M > 0$ and $\delta_1 > 0$ such that $\|\varphi_0\| < \delta_1$ and $t \ge t_0$ imply that $|x_t(t_0, \varphi_0)| \le M\|\varphi_0\|$, that is, $|T(t, s)| \le M$, where $T(t, s)$ is the solution operator [6, p.162] and [5, Theorem 1]. By the equivalent form of variation of constants formula [6, p.146], the solution $x_t(t_0, \varphi_0, g)$ of (4-P) through (t_0, φ_0) is given by

$$x_t(t_0, \varphi_0, g) = T(t, t_0)\varphi_0 + \int_{t_0}^{t} T(t, s) x_0 g(s)\, ds,$$

where

$$x_0(\theta) = \begin{cases} 0 & \text{if} \quad -r \le \theta < 0, \\ I & \text{if} \quad \theta = 0. \end{cases}$$

Thus we have

$$\begin{aligned} |x_t(t_0, \varphi_0, g)| &\le |T(t, t_0)| \|\varphi_0\| + \int_{t_0}^{t} |T(t, s)| |g(s)|\, ds \\ &\le M\|\varphi_0\| + M \int_{t_0}^{t} |g(s)|\, ds. \end{aligned}$$

It follows that

$$\begin{aligned} |x_t(t_0, \varphi_0, g)| - M \int_{t_0}^{t} |g(s)|\, ds &\le M|\varphi_0\| \\ &< u_0 \qquad \text{if} \quad u_0 > M\|\varphi_0\| \\ &= u(t) - M \int_{t_0}^{t} |g(s)|\, ds. \end{aligned}$$

Hence we have

$$|x_t(t_0, \varphi_0, g)| < u(t), \quad t \ge t_0.$$

Since $u = 0$ of (5) is ULS, it easily follows that $x = 0$ of (4-P) is ULS.

We consider the system (3). Rao and Srivastava [9] studied UAS (uniform asymptotic stability) for that system (3) by using Liapunov functions.

The solution $x = 0$ of (3) is said to be *ULS* if there are constants $M > 0$ and $\delta > 0$ such that $t_0 \ge 0$ and $\|x_{t_0}(\cdot)\| < \delta$ imply that $|x(t, t_0, x_{t_0}(\cdot)| \le M\|x_t(\cdot)\|$ for all $t \ge t_0 \ge 0$.

THEOREM 3. *If there exists a continuous functional* $V(t,x,x_t(\cdot)) \in C(\mathbb{R}^+ \times S_\rho \times C(\mathbb{R}^+,\mathbb{R}^+),\mathbb{R}^n)$ *which is locally Lipschitzian in* x *and satisfies the following*

(i) $a(\|x_t(\cdot)\|) \le V(t,x,x_t(\cdot)) \le b(\|x_t(\cdot)\|)$ *for some nondecreasing continuous nonzero functions* $a(s), b(s)$ *with* $a(0) = 0 = b(0)$ *and* $a(s)$ *is submultiplicative such that* $b(s) \le a(Ms), s > 0$, *where* $M > 0$ *is a constant,*

(ii)
$$D^+_{(3)}(t,x(t),x_t(\cdot)) = \limsup_{h\to 0^+} \frac{1}{h}[V(t+h,x(t+h),x_{t+h}(\cdot)) - V(t,x(t),x_t(\cdot))] \le 0, \qquad t \ge t_0 \ge 0,$$

where $x(t)$ *is a solution of* (3) *with initial value* $(t_0, x_{t_0}(\cdot))$, *then the zero solution of* (3) *is ULS.*

Proof. Let $0 < \varepsilon < \rho$ be given. Choose a $\delta = \delta(\varepsilon) > 0$ such that $\delta < \varepsilon/M$. Let $x(t) = x(t,t_0,x_{t_0}(\cdot))$ be a solution of (3) through $(t_0, x_{t_0}(\cdot))$, existing for all $t \ge t_0 \ge 0$. Suppose that $x = 0$ of (3) is not ULS. Then there exists a $t_1 > t_0$ such that

$$\|x_{t_0}(\cdot)\| < \delta \quad \text{and} \quad |x(t_1)| = |x(t_1,t_0,x_{t_0}(\cdot))| = M\|x_{t_0}(\cdot)\|.$$

From the assumption (ii), we have

$$V(t,x(t),x_t(\cdot)) \le V(t_0,x(t_0),x_{t_0}(\cdot)), \quad t \ge t_0.$$

Then we obtain from (i),

$$\begin{aligned} a(\varepsilon) = a(|x(t_1)|) &\le a(\|x_{t_1}(\cdot)\|) \le V(t_1,x(t_1),x_{t_1}(\cdot)) \\ &\le V(t_0,x(t_0),x_{t_0}(\cdot)) \le b(\|x_{t_0}(\cdot)\|) \\ &< b(\delta). \end{aligned}$$

But

$$b(\delta) \le a(M\delta) \le Ma(\delta) < Ma(\varepsilon/M) \le a(\varepsilon)$$

since $a(s)$ is submultiplicative. This implies that

$$a(\varepsilon) < b(\delta) < a(\varepsilon),$$

a contradiction. Hence $x = 0$ of (3) is ULS. This completes the proof.

Dishliev and Bainov [4] studied ULS via limiting equations. In their theorem sufficient conditions were given under which from ULS of the zero solution of the initial equation there follows ULS of the zero solution of each one of the respective limiting equations.

3. Exponential Asymptotic Stability (EAS)

We consider the functional differential equation with finite retardations

$$x'(t) = f(t, x_t), \tag{2}$$

where $f \in C(\mathbb{R}^+ \times C_H, \mathbb{R}^n)$ and $C_H = \{\varphi \in C([-r, 0], \mathbb{R}^n) : \|\varphi\| < H\}$, and its perturbation

$$x'(t) = f(t, x_t) + g(t, x_t), \tag{2-p}$$

where $g \in C(\mathbb{R}^+ \times C_H, \mathbb{R}^n)$.

In [7, Theorem 7.4.1], Lakshmikantham and Leela assumed that

$$|g(t, \varphi)| \leq \eta \|\varphi\|, \quad t \in \mathbb{R}^+, \quad \varphi \in C_H, \tag{6}$$

η being a sufficiently small positive number, and $f(t, \varphi)$ is linear in φ. They proved that $x = 0$ of (2-P) is EAS whenever $x = 0$ of (2) is EAS. We can obtain the same result (Theorem 7) by changing the condition (3) into another one. This result for F.D.E. with infinite retardations was proved by Sawano [10].

THEOREM 4 [7, THEOREM 7.2.1]. *Assume that $f(t, \varphi)$ is linear in φ and the solution $x = 0$ of (2) is generalized exponentially asymptotically stable (GEAS), i.e., there exist a $\delta > 0$, a continuous function $K(t) > 0$ and a strictly increasing continuous function $p(t)$ with $p(0) = 0$ and $p(t) \to \infty$ as $t \to \infty$ such that $\|\varphi_0\| < \delta$ and $t \geq t_0$ imply*

$$\|x_t(t_0, \varphi_0)\| \leq K(t_0)\|\varphi_0\| e^{p(t_0) - p(t)}.$$

(The particular case when $K(t) \equiv K > 0, p(t) = \alpha t, \alpha > 0$ is referred to as EAS). Suppose further that $p(t)$ is continuously differentiable on $\mathbb{R}^+$. Then there exists a $V \in C(\mathbb{R}^+ \times C_H, \mathbb{R})$ such that

(i) $\|\varphi\| \leq V(t, \varphi) \leq K(t)\|\varphi\|, \quad t \in \mathbb{R}^+, \varphi \in C_H,$

(ii) $|V(t,\varphi) - V(t,\psi)| \le K(t)\|\varphi - \psi\|, \quad t \in \mathbb{R}^+, \varphi, \psi \in C_H,$

(iii) $D^+_{(2)}(t,\varphi) = \limsup_{h\to 0+} \frac{1}{h}[V(t+h, x_{t+h}(t,\varphi)) - V(t,\varphi)]$
$\le -p(t)V(t,\varphi), \quad t \in \mathbb{R}^+, \varphi \in C_H.$

COROLLARY 5. *If $x = 0$ of (2) is EAS, then there exists a $V \in C(\mathbb{R}^+ \times C_H, \mathbb{R})$ such that*

(i) $\|\varphi\| \le V(t,\varphi) \le M\|\varphi\|$, *$M > 0$ is a constant,*
(ii) $|V(t,\varphi) - V(t,\psi)| \le M\|\varphi - \psi\|$,
(iii) $D^+_{(2)}(t,\varphi) \le -cV(t,\varphi), C > 0$ *is a constant.*

We define

$$V'(t,z_t) = \limsup_{h\to 0+} \frac{1}{h}[V(t+h, z_{t+h}) - V(t,z_t)]$$

for an $\mathbb{R}^n$-valued function $z(s)$ such that $z_s \in C_H$ for $s \ge t$. If $V(t,\varphi)$ is Lipschitzian in φ with Lipschitz constant M and $x(s), y(s)$ are differentiable $\mathbb{R}^n$-valued function of s with $s \ge t$ such that $x_t = y_t = \varphi \in C_H$, then we have

$$V'(t,y_t) = V'(t,x_t) + M|y'(t) - x'(t)| \tag{7}$$

in [1].

COROLLARY 6. *If $x = 0$ of (4) is EAS and the perturbing term $g(t)$ is bounded, i.e., $|g(t)| \le \mathbb{R}$ for all $t \in \mathbb{R}^+$, then all solutions of* (4-P) *are bounded.*

Proof. Since $x = 0$ of (4) is EAS, there are constants $M > 0$ and $\alpha > 0$ such that $\|x_t(t_0,\varphi)\| \le M\|\varphi\|e^{-\alpha(t-t_0)}$. Let x be a solution of (4-P) through $(t_0,\varphi) \in \mathbb{R}^+ \times C_H$ and y be a solution of (4) through $(t, x_t) \in \mathbb{R}^+ \times C_H$. Then, since $x_t = y_t$, we have

$$\begin{aligned} V'(t,x_t) &\le D^+_{(1)}(t,y_t) + M|x'(t) - y'(t)| \\ &\le -cV(t,x_t) + M|g(t)| \end{aligned}$$

by (7) and (iii) in Theorem 4. Thus

$$\begin{aligned}\|x(t)\| \le V(t,x_t) &\le V(t_0,\varphi)e^{-\alpha(t-t_0)} + \int_{t_0}^t e^{-\alpha(t-t_0)}M|g(s)|\,ds \\ &\le M\|\varphi\|e^{-\alpha(t-t_0)} + \frac{MR}{\alpha}[1-e^{-\alpha(t-t_0)}] \\ &= M[\|\varphi\| + \frac{R}{\alpha}]e^{-\alpha(t-t_0)} + \frac{MR}{\alpha}.\end{aligned}$$

This implies that x is bounded.

THEOREM 7. *For the system (2-P) assume that*

$$|g(t,\varphi)| \le a(t)\|\varphi\|, \quad \varphi \in C_H, \tag{8}$$

where $a \in C(\mathbb{R}^+,\mathbb{R}^+)$ is continuous with for any $\varepsilon > 0$

$$\int_{t_0}^t a(s)\,ds \le r + \varepsilon(t-t_0), \quad t \ge t_0 \tag{9}$$

for some constant $r > 0$. If the solution $x = 0$ of (2) is EAS, then $x = 0$ of (2-P) is also EAS.

Proof. We suppose that $c > M\varepsilon$, where M is the constant in Corollary 5. Let $H_0 = H/Me^{Mr}$. For any φ with $\|\varphi\| < H_0$ and any solution $x(t_0,\varphi)$ of (2-P), we have

$$\begin{aligned}V'(t,x_t) &\le -cV(t,x_t) + M|g(t,x_t)| \\ &\le -cV(t,x_t) + Ma(t)|x_t| \\ &\le [-c+Ma(t)]V(t,x_t)\end{aligned}$$

by (7), (8) and (i) of Corollary 5, provided that $\|x_t\| < H$. Therefore we obtain

$$\begin{aligned}|x_t| &\le V(t,x_t) \\ &\le V(t_0,\varphi)\exp[-c(t-t_0) + M\int_{t_0}^t a(s)\,ds] \\ &\le M\|\varphi\|\exp\{-c(t-t_0) + M[r+\varepsilon(t-t_0)]\} \\ &= M\|\varphi\|\exp[(M\varepsilon - c)(t-t_0) + Mr] \\ &= M\|\varphi\|\frac{H}{H_0M}e^{-\alpha(t-t_0)},\end{aligned}$$

where $\alpha = c - M\varepsilon > 0$, by (i) of Corollary 5 and (9). Consequently,

$$\|x_t(t_0, \varphi)\| \le K\|\varphi\| e^{-\alpha(t-t_0)},$$

where $K = H/H_0 > 0$. This proves that $x = 0$ of (2-P) is EAS.

4. h-stability (hS)

The concept of hS was introduced by Pinto [8]. For the linear system

$$x' = A(t)x, \quad x(t_0) = x_0, \tag{10}$$

where $A(t)$ is an $n \times n$ continuous matrix, he characterized hS as the following.

LEMMA 8 [8]. *The system (10) is hS if and only if there exist a constant $c \ge 1$ and a positive continuous bounded function h defined on $\mathbb{R}^+$ such that*

$$|\Phi(t, t_0)| \le ch(t)h(t_0)^{-1}, \quad t \ge t_0 \ge 0,$$

where $\Phi(t, t_0)$ is a fundamental matrix of (10).

We consider a perturbation of (10)

$$y' = A(t)y + g(t, y), \tag{10-P}$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$.

THEOREM 9. *Assume that for any $\varepsilon > 0$,*

$$|g(t, y)| \le K|y| \tag{11}$$

for $|y| < \varepsilon, K$ being a sufficiently small constant. If $x = 0$ of (10) is hS, then $y = 0$ of (10-P) is hS.

Proof. By Lemma 8, we have

$$|\Phi(t, t_0)| \le ch(t)h(t_0)^{-1}, \quad t \ge t_0 \ge 0.$$

Choose $c \geq 1$ such that $e^{Kc(t-t_0)} < 1$. By the variation of constants formula, the solution $y(t, t_0, y_0)$ of (10-P) is given by

$$y(t, t_0, y_0) = \Phi(t, t_0)y_0 + \int_{t_0}^{t} \Phi(t, s)g(s, y(s, t_0, y_0))\, ds.$$

Thus

$$\begin{aligned} |y(t, t_0, y_0)| &\leq |\Phi(t, t_0)||y_0| + \int_{t_0}^{t} |\Phi(t, s)||g(s, y(s, t_0, y_0))|\, ds \\ &\leq ch(t)h(t_0)^{-1}|y_0| + \int_{t_0}^{t} ch(t)h(s)^{-1}K|y(s, t_0, y_0)|\, ds. \end{aligned}$$

Hence

$$h(t)^{-1}|y(t, t_0, y_0)| \leq ch(t_0)^{-1}|y_0| + K \int_{t_0}^{t} ch(s)^{-1}|y(s, t_0, y_0)|\, ds.$$

By Gronwall's inequality, we have

$$\begin{aligned} h(t)^{-1}|y(t, t_0, y_0)| &\leq ch(t_0)^{-1}|y_0| \exp(Kc \int_{t_0}^{t} ds) \\ &= ch(t_0)^{-1}|y_0|e^{Kc(t-t_0)} \\ &< ch(t_0)^{-1}|y_0|. \end{aligned}$$

Therefore $|y(t, t_0, y_0)| \leq ch(t)h(t_0)^{-1}|y_0|$. This proves that $y = 0$ of (10-P) is hS.

Now we extend Theorem 9 to the perturbed system of (1):

$$\text{(1-P)} \qquad y' = f(t, y) + g(t, y), \quad y(t_0) = y_0,$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$. To do this we need the following lemma.

LEMMA 10 [7]. *Suppose that $k(t, x) \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$ is strictly increasing in x for each $t \geq t_0 \geq 0$ with the property*

$$x(t) - \int_{t_0}^{t} k(s, x(s))\, ds \leq y(t) - \int_{t_0}^{t} k(s, y(s))\, ds, \quad t \geq t_0 \geq 0,$$

for $x, y \in C([t_0, \infty), \mathbb{R}^n)$. *If* $x(t_0) < y(t_0)$, *then* $x(t) < y(t)$ *for all* $t \geq t_0 \geq 0$.

We consider the variational system

$$z' = f_x(t, x(t, t_0, x_0))z, \quad z(t_0) = z_0 \tag{1-V}$$

for (1) and then the fundamental matrix $\Phi(t, t_0)$ of (1-V) is given by

$$\Phi(t, t_0) = \frac{\partial}{\partial x_0} x(t, t_0, x_0).$$

The system (1) (or $x = 0$ of (1)) is said to be hSV (h-stable in variation) if the system (1-V) (or $z = 0$ of (1-V)) is hS. Clearly, hSV implies hS.

THEOREM 11. *For the system* (1-P), *we assume that*

$$|g(t, y)| \leq k(t, |y|) \tag{12}$$

where $k \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$ *is strictly increasing in* u *for each fixed* $t \geq t_0 \geq 0$ *with* $k(t, 0) = 0$. *Consider the scalar differential system*

$$u' = ck(t, u), \quad u(t_0) = u_0 = c|y_0|, \quad c > 1. \tag{13}$$

Suppose that $u = 0$ *of* (13) *is hS and* $h(t)$ *is nonincreasing. If* $x = 0$ *of* (1) *is hSV, then* $y = 0$ *of* (1-P) *is also hS.*

Proof. We have

$$|y(t)| = |y(t, t_0, y_0)| \leq |x(t, t_0, y_0)| + \int_{t_0}^{t} |\Phi(t, s)||g(s, y(s))| \, ds$$

where $\Phi(t, t_0)$ is the fundamental matrix of the variational system (1-V). Then, from (12),

$$\begin{aligned} |y(t)| &\leq c|y_0|h(t)h(t_0)^{-1} + c\int_{t_0}^{t} h(t)h(s)^{-1}|g(s, y(s))| \, ds \\ &\leq c|y_0| + c\int_{t_0}^{t} k(s, |y(s)|) \, ds \end{aligned}$$

since $h(t)$ is nonincreasing. Thus

$$
\begin{aligned}
|y(t)| - c\int_{t_0}^{t} k(s, |y(s)|)\, ds &\le c|y_0| \\
&= u_0 \\
&= u(t) - \int_{t_0}^{t} ck(s, |y(s)|)\, ds.
\end{aligned}
$$

By Lemma 10, we have $|y(t)| < u(t)$ for all $t \ge t_0 \ge 0$. Since $u = 0$ of (13) is hS, we obtain

$$
\begin{aligned}
|y(t)| < u(t) &< c_1 u_0 h(t) h(t_0)^{-1}, \quad c_1 \ge 1 \\
&= c_1 c |y_0| h(t) h(t_0)^{-1} \\
&= M |y_0| h(t) h(t_0)^{-1},
\end{aligned}
$$

where $M = c_1 c > 1$. This completes the proof.

In [3] we obtained some properties of the strengthened h-stability which is called k_ε-stability.

References

1. S. K. Choi, K. S. Koo and K. H. Lee, *Lipschitz stability and exponential asymptotic stability in perturbed systems*, J. Korean Math. Soc. **29** (1992), 175–190.
2. S. K. Choi and H. S. Ryu, *h-stability in differential systems*, Bull. Inst. Math. Acad. Sinica **21** (1993), 245–262.
3. ———, *k_ε-stability in differential systems*, submitted.
4. A. B. Dishliev and D. D. Bainov, *Investigation of the Lipschitz stability via limiting equations*, Bull. Inst. Math. Acad. Sinica **20** (1992), 187–193.
5. Y. L. Fu, *On Lipschitz stability for F.D.E.*, Pacific J. Math. **151** (1991), 229–235.
6. J. Hale, *Theory of Functional Differential Equations*, Springer-Verlag, New York-Heidelberg-Berlin, 1977.
7. V. Lakshmikantham and S. Leela, *Differential and Integral Inequalities, Theory and Applications, Vol II, Functional, Partial, Abstract and Complex Differential Equations*, Academic Press, New York, 1969.
8. M. Pinto, *Stability of nonlinear differential systems* (preprint).
9. M. R. M. Rao and S. K. Srivastava, *Liapunov functions on product spaces and stability of integro-differential equations*, Dynamic Systems and Appl. **1** (1992), 93–102.

10. K. Sawano, *Exponential asymptotic stability for functional differential equations with infinite retardations*, Tôhoku Math. J. **31** (1979), 363–382.

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