Comm. Korean Math. Soc. 9 (1994), No. 3, pp. 593–598

THE JUMP OF A SEMI-FREDHOLM OPERATOR

DONG HAK LEE AND WOO YOUNG LEE

In this note we give some results on the jump (due to Kato [5] and West [7]) of a semi-Fredholm operator.

Throughout this note, suppose X is a Banach space and write $\mathcal{L}(X)$ for the set of all bounded linear operators on X. A operator $T \in \mathcal{L}(X)$ is called *upper semi-Fredholm* if it has closed range with finite dimensional null space, and *lower semi-Fredholm* if it has closed range with its range of finite co-dimension. It T is either upper or lower semi-Fredholm we shall call it *semi-Fredholm* and *Fredholm* it is both. The *index* of a (semi-) Fredholm operator T is given by

$$\text{index}\,(T) = n(T) - d(T),$$

where $n(T) = \dim T^{-1}(0)$ and $d(T) = \operatorname{codim} T(X)$. The *punctured neighborhood theorem* ([1, 3, 4]) says that if $T \in \mathcal{L}(X)$ is semi-Fredholm then there is $\epsilon > 0$ for which $n(T-\lambda)$ and $d(T-\lambda)$ are both constant for $0 < |\lambda| < \epsilon$. Thus we can define the *jump*, $j(T)$, of a semi-Fredholm operator $T \in \mathcal{L}(X)$:

$$j(T) = \begin{cases} n(T) - n(T-\lambda) & \text{for } 0 < |\lambda| < \epsilon \text{ if } T \text{ is upper semi-Fredholm,} \\ d(T) - d(T-\lambda) & \text{for } 0 < |\lambda| < \epsilon \text{ if } T \text{ is lower semi-Fredholm.} \end{cases}$$

Continuity of the index ensures that the jump is unambiguously defined for Fredholm operators. When $T \in \mathcal{L}(X)$, we can introduce ([2, 3, 6])

$$T^{\infty}(X) = \bigcap_{n=1}^{\infty} T^n(X)$$

for the *hyperrange* and

$$T^{-\infty}(0) = \bigcup_{n=1}^{\infty} T^{-n}(0)$$

Received March 4, 1994.

for the *hyperkernel* of T: it is clear that both subspaces are invariant under any operator S on X which commutes with T. West ([6]) have shown that if $T \in \mathcal{L}(X)$ is semi-Fredholm then

$$j(T) = 0 \iff T^{-\infty}(0) \subseteq T(X), \text{ or equivalently, } T^{-1}(0) \subseteq T^{\infty}(X).$$

Thus if $j(T) \neq 0$ then there is the smallest integer ν such that

$$T^{-1}(0) \subseteq T^{\nu-1}(X) \quad \text{but} \quad T^{-1}(0) \not\subseteq T^{\nu}(X).$$

Now we have a revised version of Kato's decomposition theorem ([5], Theorem 4): it was very nearly stated by West ([7]).

THEOREM 1. *If $T \in \mathcal{L}(X)$ is semi-Fredholm then*

$$T = N \oplus T_0 \quad \text{with } N = \bigoplus_{i=1}^{j(T)} N_i,$$

where T_0 is semi-Fredholm with $j(T_0) = 0$ and each N_i is a cyclic nilpotent with nilpotency n, $\nu \leq n \leq k$, where k is the smallest integer such that $T^{-1}(0) \cap T^{\infty}(X) = T^{-1}(0) \cap T^{k}(X)$. Furthermore, there is equality

$$j(T) = \dim \left[T^{-1}(0) \ominus \left\{ T^{-1}(0) \cap T^{\infty}(X) \right\} \right]. \tag{1.1}$$

Proof. Suppose ν is the smallest integer such that

$$T^{-1}(0) \subseteq T^{\nu-1}(X) \quad \text{but} \quad T^{-1}(0) \not\subseteq T^{\nu}(X).$$

We write

$$M_1 = T^{-1}(0) \cap T^{\nu}(X).$$

Then the semi-Fredholmness of T implies

$$\dim \left(T^{-1}(0) \ominus M_1 \right) < \infty.$$

We can choose a basis of $T^{-1}(0)\ominus M_1$, $\{x_1,\dots,x_r\}$ in such a way that $x_i = T^{\nu-1}(e_i)$ $(i=1,\dots,r)$. Put

$$X_1 = \operatorname{span}\{e_i, T(e_i),\dots,T^{\nu-1}(e_i)\}_{i=1}^r.$$

Then Kato's decomposition theorem gives

$$T = T_1 \oplus S,$$

where T_1 is a nilpotent acting on X_1 consisting of r cyclic nilpotent blocks with each size ν such that

$$\begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ 0 & 1 & 0 & & \\ \vdots & \ddots & \ddots & \ddots & \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix}.$$

Thus $j(T_1) = r$ and $j(S) = j(T) - r$. We now observe that

$$\begin{aligned} T^{-1}(0) &= S^{-1}(0) \oplus \operatorname{span}\{T^{\nu-1}(e_i)\}_{i=1}^r, \\ T^{\nu-1}(X) &= S^{\nu-1}(X) \oplus \operatorname{span}\{T^{\nu-1}(e_i)\}_{i=1}^r, \\ T^{\nu}(X) &= S^{\nu}(X). \end{aligned}$$

We thus have $S^{-1}(0) \subseteq S^{\nu}(X)$. If n is the smallest integer such that

$$S^{-1}(0) \subseteq S^{m-1}(X) \quad \text{but} \quad S^{-1}(0) \not\subseteq S^{m}(X),$$

then evidently, we have $\nu < m$. Applying the above process to S and again continuing this process gives that

$$T = N \oplus T_0 \quad \text{with } N = \bigoplus_{i=1}^{j(T)} N_i,$$

where T_0 is semi-Fredholm with $j(T_0) = 0$ and each N_i is a cyclic nilpotent with nilpotency $\geq \nu$. Furthermore, retracing the steps in the above argument, we can determine

$$j(T) = \dim\left[T^{-1}(0) \ominus \{T^{-1}(0) \cap T^{\infty}(X)\}\right].$$

COROLLARY 2. *If $T \in \mathcal{L}(X)$ is upper semi-Fredholm then*

$$\dim (T-\lambda)^{-1}(0) = \dim \left(T^{-1}(0) \cap T^{\infty}(X)\right) \quad \text{for sufficiently small } \lambda. \tag{2.1}$$

If T is lower semi-Fredholm then

$$\operatorname{codim} (T-\lambda)(X) = \dim \left(T(X)^{\perp} \cap T^{-\infty}(0)\right) \quad \text{for sufficiently small } \lambda. \tag{2.2}$$

Proof. (2.1) follows at once from (1.1). For (2.2), apply the dual.

We are ready for:

THEOREM 3. *If $T \in \mathcal{L}(X)$ is semi-Fredholm then*

$$T^n = 0 \oplus T_0 \quad \text{for } n \geq k,$$

where k is the smallest integer such that $T^{-1}(0) \cap T^{\infty}(X) = T^{-1}(0) \cap T^k(X)$, 0 is the finite dimensional zero operator and T_0 is semi-Fredholm with $j(T_0) = 0$.

Proof. We first claim that if $k < \nu$ then

$$n(T^k) = k\, n(T) \quad \text{and} \quad d(T^k) = k\, d(T). \tag{3.1}$$

Indeed, for the second equality of (3.1) observe that if $T : X \to Y$ and $S : Y \to Z$ are semi-Fredholm between Banach spaces then there is isomorphism

$$S(Y)/ST(X) \;\simeq\; Y/\left(T(X) + S^{-1}(0)\right). \tag{3.2}$$

Then (3.2) with $S = T$ and $T = T^k$ gives

$$T(X)/T^k(X) \;\simeq\; X/\left(T^k(X) + T^{-1}(0)\right).$$

If $k < \nu$ then the inductive step gives

$$\dim X/T^k(X) \;=\; k \dim X/T(X),$$

which gives the second of (3.1). For the first, apply the dual to the second. Thus if $j(T) = 0$ then if follows from (3.1) that $n(T^k) = k\,n(T)$ for each $k \in \mathbb{N}$. Since $j(T-\lambda) = 0$ for sufficiently small λ, it follows that

$$k\,n(T) = k\,n(T-\lambda) = n\left((T-\lambda)^k\right) = n(T^k - \mu) \quad \text{for sufficiently small } \mu.$$

The last equality comes from the punctured neighborhood theorem. We thus have

(3.3)
$$j(T^k) = n(T^k) - n(T^k - \mu) = k\,n(T) - k\,n(T) = 0 \quad \text{for each } \ k \in \mathbb{N}.$$

Now the required result at once follows from Theorem 1 and (3.3).

COROLLARY 4. *If $T \in \mathcal{L}(X)$ is semi-Fredholm then*

$$j(T^n) = n\,(j(T)) \quad \textit{for } n \leq \nu.$$

Proof. Immediate from Theorem 1 and Theorem 3.

The jump of upper semi-Fredholm operators having finite ascent is only the nullity.

THEOREM 5. *If $T \in \mathcal{L}(X)$ is upper semi-Fredholm then*

$$j(T) = n(T) \ \textit{ if and only if } \ T \textit{ has finite ascent.} \tag{5.1}$$

If $T \in \mathcal{L}(X)$ is lower semi-Fredholm then

$$j(T) = d(T) \ \textit{ if and only if } \ T \textit{ has finite descent.} \tag{5.2}$$

Proof. We observe

$$T \text{ has finite ascent } k \iff T^{-1}(0) \cap T^k(X) = \{0\}. \tag{5.3}$$

Therefore, by (2.1) and (5.3), we have

$$\begin{aligned} j(T) = n(T) &\iff T^{-1}(0) \cap T^{\infty}(X) = \{0\} \\ &\iff T^{-1}(0) \cap T^k(X) = \{0\} \ \text{ for some } \ k \in \mathbb{N} \\ &\qquad \text{(because } \dim T^{-1}(0) < \infty) \\ &\iff T \text{ has finite ascent,} \end{aligned}$$

which gives (5.1). For (5.2), apply (5.1) to the dual.

References

1. S. Goldberg, *Unbounded linear operators*, McGraw-Hill, New York, 1966.
2. R. E. Harte, *Invertibility and singularity*, Dekker, New York, 1988.
3. R. E. Harte and W. Y. Lee, *The punctured neighborhood theorem for incomplete spaces,* J. Operator Theory (to appear).
4. ———, *A note on the punctured neighborhood theorem* (submitted).
5. T. Kato, *Perturbation theory for nullity, deficiency and other quantities of linear operators,* J. Analyse Math. **6** (1958), 261–322.
6. T. T. West, *A Riesz-Schauder theorem for semi-Fredholm operators,* Proc. Roy. Irish Acad. Sect. A **87** (1987), 137–146.
7. ——— *Removing the jump-Kato's decomposition,* Rocky Mountain J. Math. **20** (1990), 603–612.

Department of Mathematics
Kang Won University
Choon Chun 200-701, Korea

Department of Mathematics
Sung Kyun Kwan University
Suwon 440-746, Korea