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ALGEBRAIC SPECTRAL SUBSPACES OF GENERALIZED SCALAR OPERATORS

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Algebraic spectral subspaces and admissible operators were introduced by K. B. Laursen and M. M. Neumann in 1988 [L88], [N]. These concepts are useful in automatic continuity problems of intertwining linear operators on Banach spaces. In this paper we characterize the algebraic spectral subspaces of generalized scalar operators. From this characterization we show that generalized scalar operators are admissible. Also we show that doubly power bounded operators are generalized scalar. And using the spectral capacity we show that a generalized scalar operator is decomposable. Then we give an example of an operator which is not admissible but decomposable.

1. Algebraic spectral subspaces

Let X be a vector space over the complex plane $\mathbb{C}$, and let $S : X \to X$ be a linear map on X.

DEFINITION 1. Let F be a subset of the complex plane $\mathbb{C}$. Consider the class of all linear subspaces Y of X which satisfy $(S - \lambda)Y = Y$ for all $\lambda \in \mathbb{C} \setminus F$ and let $E_S(F)$ denote the span of all such subspaces Y of X. $E_S(F)$ is called an *algebraic spectral subspace* of S.

It is clear that $(S - \lambda)E_S(F) = E_S(F)$ for all $\lambda \in \mathbb{C} \setminus F$ as well so that the set is the largest linear subspace with this property.

LEMMA 2. *The space $E_S(F)$ is the union of all sets $M \subseteq X$ such that $M \subseteq (S - \lambda)M$ for all $\lambda \in \mathbb{C} \setminus F$.*

Proof. Denote by Z the union of all sets M with the given property. Clearly Z is a linear subspace of X with the property that

$$Z \subseteq (S - \lambda)Z \text{ for all } \lambda \in \mathbb{C} \setminus F.$$

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On the other hand, applying the operator $S-\lambda$ to both sides of the above inclusion we get

$$(S-\lambda)Z \subseteq (S-\lambda)[(S-\lambda)Z] \text{ for all } \lambda \in \mathbb{C}\setminus F.$$

Hence, the set $(S-\lambda)Z$ has the given property, and we have

$$(S-\lambda)Z \subseteq Z \text{ for all } \lambda \in \mathbb{C}\setminus F.$$

by the definition of Z. Thus we have shown that $(S-\lambda)Z = Z$ for all $\lambda \in \mathbb{C}\setminus F$. Since $E_S(F)$ is the largest linear subspace of X with this property, we have $Z \subseteq E_S(F)$, but the inclusion $E_S(F) \subseteq Z$ is obvious. Therefore, $E_S(F) = Z$.

REMARK 3. It is clear from the definition that

$$E_S(F) \subseteq \bigcap_{\lambda\notin F, n\in\mathbb{N}} (S-\lambda)^n X.$$

For a one to one linear map S we will show that the above inclusion becomes in fact an equality.

PROPOSITION 4. *If S is a one to one linear map on a vector space X then*

$$E_S(F) = \bigcap_{\lambda\notin F, n\in\mathbb{N}} (S-\lambda)^n X$$

for any subset F of $\mathbb{C}$.

Proof. Suppose that $\lambda \notin F$. Let $x \in \bigcap_{n\in\mathbb{N}} (S-\lambda)^n X$. Then $x = (S-\lambda)^n x_n$ for some x_n in X and for all $n \in \mathbb{N}$. Since S is 1-1, we have

$$x_1 = (S-\lambda)x_2 = (S-\lambda)^2 x_3 = \cdots.$$

Thus $x_1 \in \bigcap_{n\in\mathbb{N}} (S-\lambda)^n X$ but $x = (S-\lambda)x_1$ and we get

$$\bigcap_{n\in\mathbb{N}} (S-\lambda)^n X \subseteq (S-\lambda)[\bigcap_{n\in\mathbb{N}} (S-\lambda)^n X].$$

By the lemma 2, $\bigcap_{n\in\mathbb{N}}(S-\lambda)^n X \subseteq E_S(\mathbb{C}\setminus\{\lambda\})$ for all $\lambda \notin F$. Since $E_S(\cdot)$ preserves an arbitrary intersection [L88], we have

$$\bigcap_{\lambda\notin F, n\in\mathbb{N}} (S-\lambda)^n X \subseteq \bigcap_{\lambda\notin F} E_S(\mathbb{C}\setminus\{\lambda\}) = E_S(\bigcap_{\lambda\notin F} \mathbb{C}\setminus\{\lambda\}) = E_S(F).$$

From this inclusion and the remark 3, we have the proposition.

A linear subspace Z of X is called a *divisible subspace* of S, if

$$(S-\lambda)Z = Z \quad \text{for all } \lambda \in \mathbb{C}.$$

It is immediate that $E_S(\emptyset)$ is precisely the largest divisible subspace for the operator S. The divisible subspaces are useful in automatic continuity theory.

2. Spectral capacities

One of the most important tools in the classical spectral theory of self adjoint operators in Hilbert spaces is the set of orthonormal projections which is extended to the resolution of the identities in Dunford's theory of spectral operators. There is an analogue of these concepts for more general operators which possess some kind of spectral theory, namely the spectral capacity.

Let $\mathcal{L}(X)$ be the Banach algebra of all bounded linear operators on X, where X is a Banach space over the complex plane $\mathbb{C}$. We denote by $\mathcal{F}(\mathbb{C})$ the family of all closed subsets of $\mathbb{C}$ and by $\mathcal{S}(X)$ the family of all closed linear subspaces of X.

DEFINITION 5. A map $\mathcal{E}(\cdot) : \mathcal{F}(\mathbb{C}) \to \mathcal{S}(X)$ is called a *spectral capacity* if it satisfies the following conditions:

(i) $\mathcal{E}(\emptyset) = \{0\}$, $\mathcal{E}(\mathbb{C}) = X$.

(ii) $\mathcal{E}(\bigcap_{n=1}^{\infty} F_n) = \bigcap_{n=1}^{\infty} \mathcal{E}(F_n)$ for any sequence $\{F_n\}$ in $\mathcal{F}(\mathbb{C})$.

(iii) $X = \sum_j \mathcal{E}(\overline{G_j})$ for every finite open cover $\{G_j\}$ of $\mathbb{C}$.

DEFINITION 6. A linear operator $T \in \mathcal{L}(X)$ is said to possess a spectral capacity $\mathcal{E}(\cdot)$ if there exists a spectral capacity $\mathcal{E}(\cdot)$ which satisfies the following conditions:

(iv) $\mathcal{E}(F) \in \mathrm{Lat}(T)$ for all $F \in \mathcal{F}(\mathbb{C})$.

(v) $\sigma(T|\mathcal{E}(F)) \subseteq F$ for each $F \in \mathcal{F}(\mathbb{C})$.

Here $\mathrm{Lat}(T)$ and $\sigma(T|\mathcal{E}(F))$ denote the collection of all closed T-invariant linear subspaces of X and the spectrum of restriction of T on $\mathcal{E}(F)$, respectively.

3. Generalized scalar operators

We denote by $C^\infty(\mathbb{C})$ the Fréchet algebra of all infinitely differentiable complex valued functions $\varphi(z)$, $z = x_1 + ix_2$, $x_1, x_2 \in \mathbb{R}$, defined on the complex plane $\mathbb{C}$ with the topology of uniform convergence of every derivative on each compact subset of $\mathbb{C}$. i.e., with the topology generated by a family of pseudo-norm $|\varphi|_{K,m} = \max_{|p|\le m} \sup_{z\in K} |D^p\varphi(z)|$, where K is an arbitrary compact subset of $\mathbb{C}$, m a non-negative integer, $p = (p_1, p_2)$, $p_1, p_2 \in \mathbb{N}$, $|p| = p_1 + p_2$ and

$$D^p\varphi = \frac{\partial^{|p|}\varphi}{\partial {x_1}^{p_1} \partial {x_2}^{p_2}}, \quad (z = x_1 + ix_2).$$

An operator $T \in \mathcal{L}(X)$ is called a *generalized scalar operator* if there exists a continuous algebra homomorphism $\Phi : C^\infty(\mathbb{C}) \to \mathcal{L}(X)$ satisfying $\Phi(1) = I$, the identity operator on X, and $\Phi(z) = T$ where z denotes the identity function on $\mathbb{C}$. Such a continuous function Φ is in fact an operator valued distribution and it is called a *spectral distribution* for T. The class of generalized scalar operators was introduced by C. Foiaş [C F]. Every linear operator on a finite dimensional space as well as every spectral operator of finite type are generalized scalar operators. For more examples and properties of generalized scalar operators one may refer to [C F], [V].

An invertible operator $T \in \mathcal{L}(X)$ is called *doubly power bounded* [K T] if

$$\sup\{\|T^k\| : k \in \mathbb{Z}\} < \infty .$$

For $a \in \mathbb{R} \setminus \{0\}$ and a function $f : \mathbb{R} \to \mathbb{C}$, the *shift operator* S_a is defined as usual by $(S_af)(t) := f(t-a)$. If $p \in [1, \infty)$ then $S_a : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ is an invertible isometry. So it is a doubly power bounded operator.

EXAMPLE 7. Let T be a doubly power bounded operator on Banach space X. For any $f \in C^\infty(\mathbb{C})$ and $n \in \mathbb{Z}$ let $\hat{f}(n)$ denote the n-th Fourier coefficient of the restriction of f to the unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. Since $\hat{f}(n) = o(n^{-k})$, as $|n| \to \infty$ for any $k \in \mathbb{N}$ the map

$$\Phi(f) := \sum_{n=-\infty}^{\infty} \hat{f}(n)T^n \quad \text{for all } f \in C^\infty(\mathbb{C})$$

is a continuous algebra homomorphism from $C^\infty(\mathbb{C})$ into the Banach algebra $\mathcal{L}(X)$, for which $\Phi(1) = I$ on X, and $\Phi(z) = T$. i.e., Φ is a spectral distribution. Hence T is a generalized scalar operator.

Let T be a generalized scalar operator with spectral distribution Φ. For a given closed set $F \subseteq \mathbb{C}$ we define

$$\mathcal{E}_T(F) := \{x \in X : \Phi(u)x = 0 \text{ for every } u \in C^\infty(\mathbb{C}),\ \operatorname{supp}(u) \cap F = \emptyset\},$$

where $\operatorname{supp}(u)$ is the support of the function u. And for a given open set $G \subseteq \mathbb{C}$ we define

$$X^G := \operatorname{span}\{\Phi(u)x : x \in X,\ u \in C^\infty(\mathbb{C}) \text{ with } \operatorname{supp}(u) \subseteq G\}.$$

For a generalized scalar operator, we show that $\mathcal{E}_T(F)$ can be expressed by X^G as follows.

PROPOSITION 8. *Let T be a generalized scalar operator. For a closed set $F \subseteq \mathbb{C}$,*

$$\mathcal{E}_T(F) = \bigcap \{X^G : F \subseteq G,\ G \text{ is open}\}.$$

Proof. Let $x \in \mathcal{E}_T(F)$ and G an open set for which $F \subseteq G$. Set $G_0 = \mathbb{C} \setminus F$, then $\{G, G_0\}$ is an open cover of $\mathbb{C}$. By the normality of $C^\infty(\mathbb{C})$ there exist $f, f_0 \in C^\infty(\mathbb{C})$ for which

$$f + f_0 = 1, \quad \operatorname{supp}(f) \subseteq G, \quad \operatorname{supp}(f_0) \subseteq G_0.$$

Since $\operatorname{supp}(f_0) \cap F = \emptyset$, we have

$$0 = \Phi(f_0)x = \Phi(1 - f)x = x - \Phi(f)x.$$

So $x = \Phi(f)x$, hence $x \in X^G$. Since G is an arbitrary open set containing F, $x \in \bigcap_{F \subseteq G} X^G$.

Conversely, let $x \in \bigcap_{F \subseteq G} X^G$. Let u be a function in $C^\infty(\mathbb{C})$ with $\mathrm{supp}(u) \cap F = \emptyset$. We want to show that $\Phi(u)x = 0$. We may choose an open set G_u such that $F \subseteq G_u$ and $G_u \cap \mathrm{supp}(u) = \emptyset$. Then $x \in X^{G_u}$. Hence x is of the form

$$x = \sum_{k=1}^{n} \alpha_k \Phi(f_k) x_k, \ \text{ for some } x_k \in X, \ \alpha_k \in \mathbb{C},$$

$$f_k \in C^\infty(\mathbb{C}) \ \text{ and } \ \mathrm{supp}(f_k) \subseteq G_u \ \text{ for } \ k = 1, 2, \ldots, n.$$

Then $\Phi(u)x = \sum_{k=1}^{n} \alpha_k \Phi(u f_k) x_k = 0$, since $\mathrm{supp}(u) \cap \mathrm{supp}(f_k) = \emptyset$ for $k = 1, 2, \ldots, n$. Since u is an arbitrary function in $C^\infty(\mathbb{C})$ such that $\mathrm{supp}(u) \cap F = \emptyset$, $x \in \mathcal{E}_T(F)$.

PROPOSITION 9. *For a generalized scalar operator T the map $\mathcal{E}_T(\cdot)$ is a spectral capacity.*

Proof. For a closed set $F \in \mathcal{F}(\mathbb{C})$, clearly $\mathcal{E}_T(F)$ is a closed linear subspace of X. And $\mathcal{E}_T(\emptyset) = \{0\}$, $\mathcal{E}_T(\mathbb{C}) = X$ are obvious.

Let $\{F_n\}$ be a sequence in $\mathcal{F}(\mathbb{C})$. Then

$$\bigcap_{n \in \mathbb{N}} \mathcal{E}_T(F_n) = \bigcap_{n \in \mathbb{N}} \bigcap_{F_n \subseteq G} X^G = \bigcap_{\cap F_n \subseteq G} X^G = \mathcal{E}_T\Big(\bigcap_{n \in \mathbb{N}} F_n\Big).$$

Let $\{G_i\}_{i=1}^n$ be a finite open cover of $\mathbb{C}$. By the normality of $C^\infty(\mathbb{C})$, there exist functions $\{f_i\}_{i=1}^n$ in $C^\infty(\mathbb{C})$ for which

$$\sum_{i=1}^{n} f_i = 1\,, \quad \mathrm{supp}(f_i) \subseteq G_i\,, \quad \text{for } i = 1, 2, \ldots, n.$$

Let $x \in X$ then

$$x = \Phi(f_1)x + \Phi(f_2)x + \cdots + \Phi(f_n)x.$$

Since $\mathrm{supp}(f_i) \subseteq G_i$, $\Phi(f_i)x \in \mathcal{E}_T(\overline{G_i})$, so $X = \sum_{i=1}^{n} \mathcal{E}_T(\overline{G_i})$.

For $T \in \mathcal{L}(X)$ having the single valued extension property and for $x \in X$ the *local resolvent* $\rho_T(x)$ of x is defined as the union of all open subsets of $\mathbb{C}$ on which the equation

$$(T - \lambda)f(\lambda) = x$$

has an analytic solution $f(\lambda) : \mathbb{C} \to X$. The *local spectrum* $\sigma_T(x)$ of x is then

$$\sigma_T(x) := \mathbb{C} \setminus \rho_T(x).$$

It is easy to see that for $F \subseteq \mathbb{C}$

$$X_T(F) := \{x \in X : \sigma_T(x) \subseteq F\}.$$

is a T-invariant linear subspace of X. This space is said to be an *analytic spectral subspace.* For the properties of analytic spectral subspaces and local spectrum refer to [E L], [C F] and [V].

Given $T \in \mathcal{L}(X)$, T is called *decomposable*, if for every open covering $\{U, V\}$ of complex plane $\mathbb{C}$, there exist $Y, Z \in \mathrm{Lat}(T)$ such that

$$Y + Z = X, \ \ \sigma(T|Y) \subseteq U, \ \ \sigma(T|Z) \subseteq V.$$

This definition of operator decomposability is equivalent to the slightly more complicated original version introduced by C. Foiaş in 1963. The class of decomposable operators is rich; it contains, for instance, all spectral operators, all normal operators on Hilbert spaces, and all operators with totally disconnected spectrums.

It is well known that if T is decomposable then T possesses a spectral capacity, in this case the spectral capacity is unique and the following theorem holds.

THEOREM 10 [E L]. *$T \in \mathcal{L}(X)$ is decomposable if and only if it possesses a spectral capacity. In this case the spectral capacity of a closed subset F of $\mathbb{C}$ is analytic spectral subspace $X_T(F)$.*

It is known that generalized scalar operators are decomposable. Thus a generalized scalar operator possesses a spectral capacity. However, we will show directly that a generalized scalar operator T possesses a spectral capacity.

THEOREM 11. *Every generalized scalar operator T possesses a spectral capacity. Moreover the spectral capacity of a closed set F of $\mathbb{C}$ is the analytic spectral subspace $X_T(F)$.*

Proof. By the theorem 10, it is enough to show that T possesses the spectral capacity $\mathcal{E}_T(\cdot)$.

Let $x \in \mathcal{E}_T(F)$ and Φ the spectral distribution for T. Then for each $u \in C^\infty(\mathbb{C})$ with $\operatorname{supp}(u) \cap F = \emptyset$

$$\Phi(u)Tx = \Phi(u)\Phi(z)x = \Phi(uz)x = 0,$$

since $\operatorname{supp}(uz) \subseteq \operatorname{supp}(u)$. Hence $\mathcal{E}_T(F)$ is a T-invariant subspace.

Let $\lambda \notin F$. And choose an open set $G \subseteq \mathbb{C}$ for which $F \subseteq G$ but $\lambda \notin G$. Set $G_0 = \mathbb{C} \setminus F$, then $\{G, G_0\}$ is an open cover of $\mathbb{C}$. By the normality of $C^\infty(\mathbb{C})$ there exist $f, f_0 \in C^\infty(\mathbb{C})$ with

$$f + f_0 = 1, \quad \operatorname{supp}(f) \subseteq G, \quad \operatorname{supp}(f_0) \subseteq G_0.$$

Define a function $f_\lambda(z)$ on $\mathbb{C}$ by

$$f_\lambda(z) = \begin{cases} \frac{f(z)}{z-\lambda} & \text{for } z \neq \lambda \\ 0 & \text{for } z = \lambda . \end{cases}$$

Since $\lambda \notin \operatorname{supp}(f)$, $f_\lambda \in C^\infty(\mathbb{C})$. And $(z-\lambda)f_\lambda = f$.

For $x \in \mathcal{E}_T(F)$ we have

$$\begin{aligned} x &= \Phi(f)x + \Phi(f_0)x \\ &= \Phi(f)x \\ &= \Phi((z-\lambda)f_\lambda)x \\ &= (T-\lambda)\Phi(f_\lambda)x. \end{aligned}$$

Hence $\Phi(f_\lambda)|\mathcal{E}_T(F)$ is the inverse of $(T-\lambda)|\mathcal{E}_T(F)$ on $\mathcal{E}_T(F)$. Hence $\sigma(T|\mathcal{E}_T(F)) \subseteq F$.

PROPOSITION 12. *Let T be a generalized scalar operator. Then $X_T(F) = E_T(F)$ for a closed subset F of $\mathbb{C}$.*

Proof. From $\sigma(T|\mathcal{E}_T(F)) \subseteq F$ and $X_T(F) = \mathcal{E}_T(F)$ we have $X_T(F) \subseteq E_T(F)$. For a generalized scalar operator T, P. Vrvoá [Vr73], proved the existence of some $p \in \mathbb{N}$ such that

$$X_T(F) = \bigcap_{\lambda \notin F} (T-\lambda)^p X.$$

From this equality and the remark 3 we have,

$$E_T(F) \subseteq \bigcap_{\lambda \notin F, n \in \mathbb{N}} (T-\lambda)^n X \subseteq \bigcap_{\lambda \notin F} (T-\lambda)^p X = X_T(F).$$

REMARK 13. For a generalized scalar operator T, from remark 3, proposition 8 and proposition 12 we have,

$$E_T(F) = X_T(F) = \mathcal{E}_T(F) = \bigcap_{F \subseteq G} X^G = \bigcap_{\lambda \notin F} (T-\lambda)^p X.$$

For a linear operator T on a Banach space X is called ***admissible*** if $E_T(F)$ is closed whenever $F \subseteq \mathbb{C}$ is closed. Such operators have been extensively studied in recent papers of K. B. Laursen and M. M. Neumann. It should be noted immediately that, as shown in [L88], if $E_T(F)$ is closed then $E_T(F) = X_T(F)$, and that admissible operator can not have non trivial divisible subspaces(which simply means that $E_T(\emptyset) = \{0\}$). Thus the admissible operators allow us to combine the analytic tools associated with the space $X_T(F)$ with the algebraic tools associated with the space $E_T(F)$.

By the proposition 12 a generalized scalar operator is admissible. In particular it has no non trivial divisible subspace.

REMARK 14. There exists a compact and quasi-nilpotent operator $T \in \mathcal{L}(X)$ on a Banach space X such that T has a non trivial divisible subspace.

If $X = C[0,1]$ and $T \in \mathcal{L}(X)$ denotes the Volterra operator given by

$$(Tf)(s) := \int_0^s f(t)dt \quad \text{for all } f \in C[0,1] \text{ and } s \in [0,1].$$

Then T is both compact and quasi-nilpotent and has the following non trivial divisible subspace

$$Y := \{f \in C^\infty[0,1] : f^{(k)}(0) = 0 \text{ for all } k = 0,1,2,\dots\}.$$

Hence the Volterra operator on $C[0,1]$ is not an admissible operator. But the Volterra operator has the totally disconnected spectrum and hence it is decomposable. Thus we have an example of an operator that is decomposable but not admissible.

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