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HYPERSURFACES IN THE UNIT SPHERE WITH SOME CURVATURE CONDITIONS

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Let M be a minimally immersed closed hypersurface in $\mathbb{S}^{n+1}$, $I\!I$ the second fundamental form and $S = \|I\!I\|^2$. It is well known that if $0 \leq S \leq n$, then $S \equiv 0$ or $S \equiv n$ and totally geodesic hyperspheres and Clifford tori are the only possible minimal hypersurfaces with $S \equiv 0$ or $S \equiv n$ ([6], [2]). From these results, Chern suggested some questions on the study of compact minimal hypersurfaces on the sphere with S =constant: what are the next possible values of S to n, and does the value S determine the minimal hypersurface up to a rigid motion in the ambient sphere? By the way, S is defined extrinsically but, in fact, it is an intrinsic invariant for the minimal hypersurfaces, i.e., $S = n(n-1) - R$, where R is the scalar curvature of M. Some partial answers have been obtained for $\dim M = 3$: Assuming $M^3 \subset \mathbb{S}^4$ is closed and minimal with S =constant, de Almeida and Brito [1] proved that if $R \geq 0$ (or equivalently $S \leq 6$), then $S = 0, 3$ or 6, Peng and Terng ([5]) proved that if M has 3 distinct principal curvatures, then $S = 6$, and in [3] Chang showed that if there exists a point which has two distinct principal curvatures, then $S = 3$. Hence the problem for $\dim M = 3$ is completely done. For higher dimensional cases, not much has been known and these problems seem to be very hard without imposing some more conditions on M.

Nice examples for this problem are isoparametric hypersurfaces in $\mathbb{S}^{n+1}$. A hypersurface is called *isoparametric* if all the principal curvatures are constants. It is well known that given an isoparametric hypersurface M in $\mathbb{S}^{n+1}$, there exists a minimal isoparametric hypersurface parallel to M. In [4], Peng and Terng showed that if M is a minimal isoparametric hypersurface in $\mathbb{S}^{n+1}$ with p distinct principal curvatures, then $S = (p-1)n$ and hence $S = 0, n, 2n, 3n$ or $5n$.

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In the following, we will put a condition on the principal curvatures that their means are constant up to certain order to find possible values of S.

THEOREM. *Suppose M^4 is a closed hypersurface in $\mathbb{S}^5$ with 4 distinct principal curvatures. Let A be the shape operator of M and let R be the scalar curvature. If $\operatorname{tr} A^i = c_i$ are constants ($i \leq 3$) and $R \geq 0$, then $\operatorname{tr} A^4$ is constant and $R = 0$. Therefore, M is isoparametric in $\mathbb{S}^5$. Moreover, if M is minimal, then $S = 12$.*

We will prove this theorem through several lemmas. Let $f = \frac{1}{4} \operatorname{tr} A^4$ and let dv be the volume form on M. Choose an orthonormal frame field e_i, ($i \leq 4$) and its coframe field ω_i such that

$$\begin{cases} Ae_i = \lambda_i e_i, \\ \omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \omega_4 = dv \end{cases} \tag{1}$$

where $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4$ are the principal curvatures of M.

The curvature form ω_{ij} corresponding to the Levi-Civita connection ∇ and the Christoffel symbol γ_{ijk} are defined as follows:

$$\nabla e_i = \sum_j \omega_{ij} \otimes e_j,$$

$$\omega_{ij} = \sum_k \gamma_{ijk} \omega_k.$$

Define a 3-form ψ by

$$\psi = \sum_{i<j} (-1)^{i+j} \omega_{ij} \wedge \theta_{ij}$$

where $\theta_{ij} = \omega_1 \wedge \cdots \widehat{\omega_i} \wedge \cdots \widehat{\omega_j} \cdots \wedge \omega_4$. This form is well-defined globally if we keep the rule (1). For, suppose ω_i' satisfies (1) and $\omega_{ij}', \theta_{ij}', \psi'$ are defined by ω_i'. From $dv = \omega_1' \wedge \omega_2' \wedge \omega_3' \wedge \omega_4'$, it suffices to prove

$$\omega_{ij}' \wedge \theta_{ij}' = \omega_{ij} \wedge \theta_{ij}$$

$$\text{for} \quad \omega_i' = -\omega_i, \ i = 1, 2, \qquad \omega_j' = \omega_j, \ j > 2.$$

It is easy to show that

$$\omega'_{ij} = \begin{cases} -\omega_{ij} & \text{if } i = 1,2 \text{ and } j > 2, \\ \omega_{ij} & \text{otherwise,} \end{cases}$$
$$\theta'_{ij} = \begin{cases} -\theta_{ij} & \text{if } i = 1,2 \text{ and } j > 2, \\ \theta_{ij} & \text{otherwise.} \end{cases}$$

Hence $\psi' = \psi$.

LEMMA 1.

$$d\psi = \frac{1}{2} R\, dv + \sum_k \sum_{i<j} (-\gamma_{kii}\gamma_{kjj} + \gamma_{kij}\gamma_{kji}) dv.$$

Proof.

$$\begin{aligned} d\psi &= \sum_{i<j} (-1)^{i+j} d\omega_{ij} \wedge \theta_{ij} + \sum_{i+j} (-1)^{i+j+1} \omega_{ij} \wedge d\theta_{ij} \\ &= \sigma_1 + \sigma_2\ . \end{aligned}$$

For σ_1, use the curvature equations

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \Omega_{ij},$$

where $\Omega_{ij} = \frac{1}{2} \sum_{k \neq l} R_{ijkl} \omega_k \wedge \omega_l$ is the curvature form. Then

$$\begin{aligned} \sigma_1 &= \sum_k \sum_{i<j} (-1)^{i+j} \omega_{ik} \wedge \omega_{kj} \wedge \theta_{ij} + \sum_{i<j} (-1)^{i+j+1} R_{ijij} \omega_i \wedge \omega_j \wedge \theta_{ij} \\ &= \sum_k \sum_{i<j} (-1)^{i+j} \omega_{ik} \wedge \omega_{kj} \wedge \theta_{ij} + \frac{1}{2} R\, dv \end{aligned}$$

since $R = \sum_{i \neq j} R_{ijij}$ and $\omega_i \wedge \omega_j \wedge \theta_{ij} = (-1)^{i+j+1} dv$.

For σ_2, use the structure equations $d\omega_i = \sum_j \omega_{ij} \wedge \omega_j$ to calculate $d\theta_{ij}$. Then it is easy to obtain

$$\sigma_2 = 2\sum_k \sum_{i<j} (-1)^{i+j+1} \omega_{ik} \wedge \omega_{kj} \wedge \theta_{ij}.$$

Hence

$$\begin{aligned} d\psi &= \frac{1}{2} R\, dv - \sum_k \sum_{i<j} (-1)^{i+j} \omega_{ik} \wedge \omega_{kj} \wedge \theta_{ij} \\ &= \frac{1}{2} R\, dv - \sum_k \sum_{i<j} (-1)^{i+j} \sum_{l,m} \gamma_{ikl} \gamma_{kjm} \omega_l \wedge \omega_m \wedge \theta_{ij} \\ &= \frac{1}{2} R\, dv - \sum_k \sum_{i<j} (-1)^{i+j} (\gamma_{iki} \gamma_{kjj} \omega_i \wedge \omega_j + \gamma_{ikj} \gamma_{kji} \omega_j \wedge \omega_i) \wedge \theta_{ij} \\ &= \frac{1}{2} R\, dv + \sum_k \sum_{i<j} (-\gamma_{kii} \gamma_{kjj} + \gamma_{kij} \gamma_{kji}) dv \end{aligned}$$

since $\gamma_{ijk} = -\gamma_{jik}$ and $(-1)^{i+j+1} \omega_i \wedge \omega_j \wedge \theta_{ij} = dv$.

Now, define h_{ij} and h_{ijk} by

$$\begin{aligned} h_{ij} &= \lambda_i \delta_{ij}, \\ \nabla A &= \sum_{i,j,k} h_{ijk} \omega_k \otimes \omega_i \otimes e_j. \end{aligned}$$

It is well known that h_{ijk} is symmetric in i, j, k and

$$\sum_k h_{ijk} \omega_k = dh_{ij} + \sum_m (h_{mj} \omega_{mi} + h_{im} \omega_{mj}). \tag{2}$$

Let $d\lambda_i = \sum_k \lambda_{ik} \omega_k$. Then by (2), we have

$$\begin{cases} \lambda_{ik} = h_{iik}, \\ \gamma_{ijk} = \frac{h_{ijk}}{\lambda_i - \lambda_j} \quad \text{if } i \neq j. \end{cases} \tag{3}$$

LEMMA 2.

$$\alpha = \sum_k \sum_{i<j} \gamma_{kij}\gamma_{kji} = 0.$$

Proof. By (3),

$$\begin{aligned}
2\alpha &= \sum_{k\neq i\neq j} \gamma_{kij}\gamma_{kji} = \sum_{k\neq i\neq j} \frac{h_{ijk}^2}{(\lambda_k - \lambda_i)(\lambda_k - \lambda_j)} \\
&= \sum_{k\neq i\neq j} \frac{(\lambda_j - \lambda_i)h_{ijk}^2}{(\lambda_i - \lambda_j)(\lambda_j - \lambda_k)(\lambda_k - \lambda_i)} \\
&= \sum_{i<j<k} \frac{h_{ijk}^2}{(\lambda_i - \lambda_j)(\lambda_j - \lambda_k)(\lambda_k - \lambda_i)} \\
&\quad \{2(\lambda_j - \lambda_i) + 2(\lambda_k - \lambda_j) + 2(\lambda_i - \lambda_k)\} \\
&= 0.
\end{aligned}$$

From $\operatorname{tr} A^i = c_i$, $i \leq 3$ and $f = \frac{1}{4}\operatorname{tr} A^4$, we have

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 \\ \lambda_1^3 & \lambda_2^3 & \lambda_3^3 & \lambda_4^3 \end{pmatrix} \begin{pmatrix} d\lambda_1 \\ d\lambda_2 \\ d\lambda_3 \\ d\lambda_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ df \end{pmatrix}. \tag{4}$$

Let

$$df = \sum_i f_i\omega_i, \quad \gamma = \prod_{i<j}(\lambda_j - \lambda_i), \quad \gamma_k = \prod_{\substack{i<j \\ (i,j\neq k)}} (\lambda_j - \lambda_i).$$

LEMMA 3.

$$\sum_k \sum_{i<j} \gamma_{kii}\gamma_{kjj} = \sum_k \sum_{i<j} \frac{(-1)^{i+j}\gamma_i\gamma_j}{(\lambda_k - \lambda_i)(\lambda_k - \lambda_j)} \frac{f_k^2}{\gamma^2}.$$

Proof. By (3),

$$\gamma_{kii} = \frac{h_{kii}}{\lambda_k - \lambda_i} = \frac{h_{iik}}{\lambda_k - \lambda_i} = \frac{\lambda_{ik}}{\lambda_k - \lambda_i}.$$

If we solve (4) for λ_{ik}, then Lemma follows from

$$\lambda_{ik} = (-1)^i \frac{\gamma_i f_k}{\gamma}.$$

As a consequence of Lemma 3, we see that

$$d\psi = \frac{1}{2} R\, dv + \sum_k \sum_{\substack{i<j \\ (i,j\neq k)}} \frac{(-1)^{i+j+1}\gamma_i\gamma_j}{(\lambda_k-\lambda_i)(\lambda_k-\lambda_j)} \frac{f_k^2}{\gamma^2}\, dv. \tag{5}$$

LEMMA 4. *For each $k \leq 4$,*

$$\beta_k = \sum_{\substack{i<j \\ (i,j\neq k)}} \frac{(-1)^{i+j+1}\gamma_i\gamma_j}{(\lambda_k-\lambda_i)(\lambda_k-\lambda_j)} > 0.$$

Proof.

$$\begin{aligned}
\beta_1 &= \frac{\gamma_2\gamma_3}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)} - \frac{\gamma_2\gamma_4}{(\lambda_2-\lambda_1)(\lambda_4-\lambda_1)} + \frac{\gamma_3\gamma_4}{(\lambda_3-\lambda_1)(\lambda_4-\lambda_1)} \\
&> \frac{\gamma_2}{\lambda_2-\lambda_1} \frac{(\lambda_4-\lambda_1)\gamma_3 - (\lambda_3-\lambda_1)\gamma_4}{(\lambda_3-\lambda_1)(\lambda_4-\lambda_1)} \\
&= \gamma_2 \frac{(\lambda_4-\lambda_1)^2(\lambda_4-\lambda_2) - (\lambda_3-\lambda_1)^2(\lambda_3-\lambda_2)}{(\lambda_3-\lambda_1)(\lambda_4-\lambda_1)} \\
&> 0.
\end{aligned}$$

$$\begin{aligned}
\beta_2 &= -\frac{\gamma_1\gamma_3}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)} + \frac{\gamma_1\gamma_4}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_4)} + \frac{\gamma_3\gamma_4}{(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)} \\
&> \frac{\gamma_1}{\lambda_2-\lambda_1} \frac{(\lambda_4-\lambda_2)\gamma_3 - (\lambda_3-\lambda_2)\gamma_4}{(\lambda_3-\lambda_2)(\lambda_4-\lambda_2)} \\
&= \gamma_1 \frac{(\lambda_4-\lambda_2)^2(\lambda_4-\lambda_1) - (\lambda_3-\lambda_2)^2(\lambda_3-\lambda_1)}{(\lambda_3-\lambda_2)(\lambda_4-\lambda_2)} \\
&> 0.
\end{aligned}$$

$$\begin{aligned}
\beta_3 &= \frac{\gamma_1\gamma_2}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)} + \frac{\gamma_1\gamma_4}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_4)} - \frac{\gamma_2\gamma_4}{(\lambda_3-\lambda_2)(\lambda_3-\lambda_4)} \\
&> \frac{\gamma_4}{\lambda_4-\lambda_3}\frac{(\lambda_3-\lambda_1)\gamma_2-(\lambda_3-\lambda_2)\gamma_1}{(\lambda_3-\lambda_2)(\lambda_3-\lambda_1)} \\
&= \gamma_4\frac{(\lambda_3-\lambda_1)^2(\lambda_4-\lambda_1)-(\lambda_3-\lambda_2)^2(\lambda_4-\lambda_2)}{(\lambda_3-\lambda_2)(\lambda_3-\lambda_1)} \\
&> 0.
\end{aligned}$$

$$\begin{aligned}
\beta_4 &= \frac{\gamma_1\gamma_2}{(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)} - \frac{\gamma_1\gamma_3}{(\lambda_4-\lambda_1)(\lambda_4-\lambda_3)} + \frac{\gamma_2\gamma_3}{(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)} \\
&> \frac{\gamma_3}{\lambda_4-\lambda_3}\frac{(\lambda_4-\lambda_1)\gamma_2-(\lambda_4-\lambda_2)\gamma_1}{(\lambda_4-\lambda_2)(\lambda_4-\lambda_1)} \\
&= \gamma_3\frac{(\lambda_4-\lambda_1)^2(\lambda_3-\lambda_1)-(\lambda_4-\lambda_2)^2(\lambda_3-\lambda_2)}{(\lambda_4-\lambda_2)(\lambda_4-\lambda_1)} \\
&> 0.
\end{aligned}$$

Proof of Theorem. Integrate (5) on M:

$$0 = \int_M d\psi = \frac{1}{2}\int_M R\,dv + \int_M \sum_k \beta_k \frac{f_k^2}{\gamma^2}\,dv.$$

Since $R \geq 0$ and $\beta_k > 0$, we have $R = 0$ and $f_k = 0\ \forall k$, i.e., $df = 0$ and hence $f = \frac{1}{4}\operatorname{tr} A^4$ is a constant.

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