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EQUIVARIANT HARMONIC MAPS BETWEEN CERTAIN FOLIATED MANIFOLDS

DONG-PYO CHI AND DONG-SOONG HAN

1. Construction of equivariant manifold and maps

In this paper, as the generalization of Baird's result [1] we study the equivariant harmonic maps in foliated manifolds.

A smooth foliation $\mathcal{F}$ of dimension p (or codimension $q = m - p$) is a decomposition of manifold M into disjoint connected subset $\{\mathcal{L}_a\}_{a \in A}$, called the leaves of the foliation such that each point of M has a neighborhood U and a system of smooth coordinates $(x, y) : U \to \mathbb{R}^p \times \mathbb{R}^q$ such that for each leaf $\mathcal{L}_a$ the components of $U \cap \mathcal{L}_a$ are described by the equations;

$$\begin{aligned} y_1 &= \text{constant}, \\ &\vdots \\ y_q &= \text{constant}. \end{aligned}$$

Let (M, g_M) be a smooth foliated manifold with bundle-like metric and dimension m. The leaves of M are denoted by $\mathcal{L} \subset M$. Let $L \subset TM$ be a subbundle of codimension q. By the Frobenius Theorem, if L is involutive then it is the tangent bundle of a codimension q foliation $\mathcal{F}$ on M. The leaves of $\mathcal{F}$ are constructed as the maximal connected integral submanifolds of the subbundle $L \subset TM$. The normal bundle Q of a codimension q foliation $\mathcal{F}$ on M is the quotient bundle $Q = TM/L$. Equivalently, Q appears in the exact sequence of vector bundles such that

$$0 \longrightarrow L \underset{i}{\longrightarrow} TM \underset{d\pi}{\longrightarrow} Q \longrightarrow 0.$$

The metric g_M defines a splitting σ in the exact sequence,

$$0 \longrightarrow L \underset{i}{\longrightarrow} TM \overset{\sigma}{\underset{d\pi}{\rightleftarrows}} Q \longrightarrow 0$$

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with $\sigma Q = L^{\perp}$ the orthogonal complement of L. Thus g_M induces a metric g_Q on Q.

Then we can choose an orthonormal frame field (X_i, ξ_α) in M such that $X_i \in L_M$ and $\xi_\alpha \in \sigma Q_M$ for $TM = L \oplus \sigma Q$.

DEFINITION 1.1 [3]. A connection ∇ in Q is defined by

$$\nabla_X s = d\pi[X, Y_s] \text{ for } X \in \Gamma L,\ s \in \Gamma Q \text{ and } Y_s = \sigma(s) \in \Gamma\sigma Q,$$
$$\nabla_X s = d\pi(\nabla^M_X Y_s) \text{ for } X \in \Gamma\sigma Q,\ s \in \Gamma Q \text{ and } Y_s = \sigma(s) \in \Gamma\sigma Q$$

where $d\pi : TM \longrightarrow Q_M$.

The first condition say that ∇ is an adapted connection on Q, i.e., a connection extending the partial Bott connection along L.

DEFINITION 1.2. A $Q-$valued bilinear form on M is defined by

$$\alpha(X,Y) = -(\nabla d\pi)(X,Y) \equiv d\pi(\nabla^M_X Y) - \nabla_X d\pi(Y).$$

We call α the second fundamental form of the foliation.

This bilinear form α is symmetric. And restriction of α to any leaf $\mathcal{L}$ of the foliation is the second fundamental form of submanifold $\mathcal{L} \subset M$.

For the further discussion of α, we introduce the following map.

DEFINITION 1.3. For each $\xi \in \Gamma Q$, the map $A(\xi) : TM \longrightarrow TM$ uniquely defined by

$$g_M(A(\xi)X, Y) = g_Q(\alpha(X,Y), \xi) \text{ for } X, Y \in \Gamma TM.$$

Then $A(\xi)$ is selfadjoint by the symmetry of α. And we decompose $A(\xi)$ such that the matrix representation

$$A = \begin{pmatrix} A_1 & A_2 \\ A_2^* & A_3 \end{pmatrix}$$

with $A_1 : L \to L$, $A_3 : \sigma Q \to \sigma Q$ both selfadjoint and $A_2 : \sigma Q \to L$.

Suppose $A_2(\xi) = 0$ in TM, then by [3] σQ_M is involutive. Hence if a smooth map $\phi : M \longrightarrow N$ is horizontal, i.e., its differential preserve normal bundles, then we can take the induced map $\bar{\phi}$ of ϕ such that $\bar{\phi} : M/\mathcal{L}_M \longrightarrow N/\mathcal{L}_N$. We obtain the following commutative diagram.

$$\begin{array}{ccc} M & \xrightarrow{\phi} & N \\ \downarrow{\scriptstyle \pi_M} & & \downarrow{\scriptstyle \pi_N} \\ \mathcal{Q}_M & \xrightarrow{\bar{\phi}} & \mathcal{Q}_N \end{array}$$

where $\mathcal{Q}_M$ and $\mathcal{Q}_N$ denote the integral manifolds of σQ_M and σQ_N respectively.

Let $W(\xi) : L_p \longrightarrow L_p$ be the weingarten map along ξ defined by $W(\xi)(X) = -\pi^{\perp}(\nabla_X \xi)$; where $X \in L_p$, $\xi \in \sigma Q_M$, $\pi^{\perp} = id - \sigma \circ \pi$. Then $A(\xi) = W(\xi)$.

Let eigenvalues of W be principal curvatures $\lambda_1(\xi), \ldots, \lambda_s(\xi)$ along ξ, and $S_1(\xi), \ldots, S_s(\xi)$ be the eigenspaces of leaf at $p \in M$ with respect to λ_k for $1 \leq k \leq s$. Then

$$g_M(\alpha(X,Y), \xi) = g_M(W(\xi)(X), Y) = -\lambda_k(\xi) g_M(X,Y)$$

for $X \in S_k(\xi)$, $Y \in L$. Hence by linearity of $W(\xi)$, $W(\xi)(X) = \bar{\lambda}(\xi)X$ where $\bar{\lambda}$ is the linear function in σQ.

If λ_k is constant in L_p, then $d\pi(W(\xi)(X)) = \bar{\lambda}(\xi)$.

Hence we will define some desirable manifolds.

DEFINITION 1.4. A smooth foliated manifold M with constant sectional curvature is called equivariant with respect to $\mathcal{L}_M$ if

(1) $A_2(\xi) = 0$ for any $\xi \in \sigma Q_M$,
(2) Eigenvalues of W are constant in L_p,
(3) $\mathcal{L}_p$ has locally flat normal curvature in M.

REMARK. If there exists a smooth isoparametric map $f : M \longrightarrow R^k$, (1), (2) and (3) of Definition 1.4 hold as in [5]. Hence definition of equivariantly foliated manifold is some kind of generalization of isoparametric manifold.

From property of (3) we can show that leaves of M is decomposed by common eigenspaces.

LEMMA 1.5. *Let A and B be a linear operator of vector space V. If A and B are commute, then V is decomposed by a common eigenspaces.*

Proof. Let V be decomposed by A and B such that $V = \oplus E_i$ where $A|_{E_i} = c_i I$, and $V = \oplus E_i'$ where $A|_{E_i} = c_i' I$.

Then for any vector $e_i \in E_i$,

$$\begin{aligned} AB(e_i) &= BA(e_i) \\ &= c_i B(e_i). \end{aligned}$$

Hence $B(e_i) \subset E_i$. Also $B(E_i) \subset E_i$. For any i, we can decompose E_i such that $E_i = \oplus_j E_{ij}$ where $B|_{E_{ij}} = c_j' I$.

Hence we can write $V = \oplus_{i,j} E_{ij}$.

PROPOSITION 1.6. *If $\mathcal{L}_p$ has locally flat normal curvature in M with constant sectional curvature, then $T\mathcal{L}_p$ is decomposed by a common eigenspaces.*

Proof. Suppose $\mathcal{L}_p$ is flat, i.e.,

$$\Omega^Q(\xi_i, \xi_j) = [W(\xi_i), W(\xi_j)] = 0$$

where Ω^Q is normal curvature of $\mathcal{L}_p$.

Hence for any i, j,

$$W(\xi_i)W(\xi_j) = W(\xi_j)W(\xi_i).$$

Then by Lemma 1.5, there exists the decomposition of $T\mathcal{L}_p = \oplus E_i$ such that $W(\xi_k)|_{E_i} = \lambda_i(\xi_k)I$ for all k, l.

Now we will write an eigenspaces of M_p and N_q to S_i and T_j as common eigenspaces. We suppose that $d\phi_p(S_k(x)) \subset T_{j_k}(\phi_p(x))$, for all $x \in M_p$, and $\phi_{r,t} : M_r \longrightarrow N_t$ is harmonic for all r and t. Also when $\gamma_k(r,t) : M_p \to \mathbb{R}$ is defined by

$$\gamma_k(r,t)(x) = \operatorname{trace}_{S_k(x)} g_N(d\phi_{r,t}, d\phi_{r,t}),$$

for all $x \in M_p$, $k = 1, \ldots, s$, then for each k, $\gamma_k(r,t)(x)$ depends only on r and t. Write $\gamma(r,t) = \sum_{k=1}^{s} \gamma_k(r,t)$ for such ϕ.

DEFINITION 1.7. Suppose $\phi : M \longrightarrow N$ be a smooth horizontal map in the equivariantly foliated manifolds M, N such that

(1) $d\phi_p(S_k(x)) \subset T_{j_k}(\phi_p(x))$, for all $x \in M_p$,
(2) $\phi_{r,t} : M_r \longrightarrow N_t$ is harmonic for all r and t,
(3) for each k, $\gamma_k(r,t)(x)$ depends only on r and t,

then ϕ is called the equivariant map in an equivariantly foliated manifolds.

REMARK. If $\phi|_{S_k}$ has the constant energy density, then the condition (3) holds.

2. Reduction Theorem

THEOREM 2.1. *Let M, N be an equivariantly foliated manifolds, and $\phi : M \to N$ be an equivariant map. Then ϕ is harmonic if and only if the induced map $\bar{\phi} : \mathcal{Q}_M \longrightarrow \mathcal{Q}_N$ is satisfies*

$$\left\langle (\Delta^Q \bar{\phi}) + d\bar{\phi}(\bar{\tau}), \eta_\alpha \right\rangle = -\sum_{k=1}^{p} \mu_{j_k}(\eta_\alpha)\gamma_k$$

for any orthonormal frame field $\{\eta_\alpha\}$ in σQ_N where $\bar{\tau}$ is tension field of $\mathcal{L}_M$.

Proof. First we calculate $\Delta\phi$ at $p \in M$, then

$$
\begin{aligned}
(1)\qquad \Delta^M \phi &= \text{trace } \nabla^M d\phi \\
&= \sum_\alpha \nabla^M d\phi(\xi_\alpha, \xi_\alpha) + \sum_i \nabla^M d\phi(X_i, X_i) \\
&= \sum_\alpha \nabla^M d\phi(\xi_\alpha, \xi_\alpha) + \sum_i \nabla^M d\phi(di_p(X_i), di_p(X_i)) \\
&= \sum_\alpha \nabla^M d\phi(\xi_\alpha, \xi_\alpha) - d\phi(\tilde{\Delta} i_p) + \tilde{\Delta}(\phi \circ i_p),
\end{aligned}
$$

where $\tilde{\Delta}$ is a laplacian of M_p.

Step 1. $\Delta\phi \in \Gamma\sigma Q_N$.

Consider $\phi \circ i_p : M_p \longrightarrow N_{\phi(p)}$,

$$\Delta(\phi \circ i_p) = d\phi(\Delta i_p) + \text{ trace}\nabla d\phi(di_p, di_p).$$

Since i_p is isometric inclusion, $\Delta i_p \in \Gamma\sigma Q_N$. Also $\Delta(\phi \circ i_p)$ and $d\phi(\Delta i_p)$ is contained in $\Gamma\sigma Q_N$. Therefore

$$\text{trace}\,\nabla d\phi(di_p, di_p) \in \Gamma\sigma Q_N.$$

Also

$$\nabla d\phi(\xi_\alpha, \xi_\alpha) = -d\phi(\nabla^M_{\xi_\alpha}\xi_\alpha) + \nabla^N_{d\phi(\xi_\alpha)} d\phi(\xi_\alpha) \in \Gamma\sigma Q_N,$$

since the first term of right hand side is 0 and $d\phi(\xi_\alpha) \in \sigma Q_N$.

Hence

$$\begin{aligned}\Delta\phi &= \text{trace}\,\nabla d\phi \\ &= \text{trace}\,\nabla d\phi(di_p, di_p) + \sum_\alpha \nabla d\phi(\xi_\alpha, \xi_\alpha) \\ &\in \Gamma\sigma Q_N.\end{aligned}$$

Step 2.

$$d\pi_N\left(\sum_\alpha \nabla^M d\phi(\xi_\alpha, \xi_\alpha)\right) = \sum_\alpha \left(\nabla^Q d\bar{\phi}(\xi'_\alpha, \xi'_\alpha)\right)$$

where ξ'_α is an orthonormal frame field induced by $d\pi_M$.

Let $\sigma_M : Q_M \to TM$ be a map such that $\sigma_M \circ d\pi_M =$ identity of TM. By using the commutative of diagram and the definition of Bott connection, we have

$$
\begin{aligned}
&\sum_\alpha \nabla^Q d\bar{\phi}(\xi'_\alpha, \xi'_\alpha) \\
&= \sum_\alpha \left\{ -d\bar{\phi}(\nabla^Q_{\xi'_\alpha} \xi'_\alpha) + \nabla^{\tilde{Q}}_{d\bar{\phi}(\xi'_\alpha)} d\bar{\phi}(\xi'_\alpha) \right\} \\
&= d\pi_N \sum_\alpha \left\{ -d\phi(\nabla^M_{\xi_\alpha} \xi_\alpha) + \left(\nabla^N_{\sigma_N d\bar{\phi}(\xi'_\alpha)} \sigma_N d\bar{\phi}(\xi'_\alpha) \right) \right\} \\
&= d\pi_N \sum_\alpha \left\{ -d\phi(\nabla^M_{\xi_\alpha} \xi_\alpha) + \left(\nabla^N_{d\phi(\xi_\alpha)} d\phi(\xi_\alpha) \right) \right\} \\
&= d\pi_N \left(\sum_\alpha \nabla^M d\phi(\xi_\alpha, \xi_\alpha) \right).
\end{aligned}
$$

Step 3.

$$d\pi_N \circ d\phi(\tilde{\Delta} i_p) = d\bar{\phi}(\bar{\tau}),$$

where $i_p(\bar{X}_i) = X_i$.

$$
\begin{aligned}
0 = \tilde{\Delta}(\pi_M \circ i_p) &= d\pi_M(\tilde{\Delta} i_p) + \text{ trace } \nabla d\pi_M(di_p, di_p) \\
&= d\pi_M(\tilde{\Delta} i_p) + \sum_i (\nabla d\pi_M)(\bar{X}_i, \bar{X}_i).
\end{aligned}
$$

Hence

$$
\begin{aligned}
d\pi_N \circ d\phi(\tilde{\Delta} i_p) &= d\bar{\phi} \circ d\pi_M(\tilde{\Delta} i_p) \\
&= -\, d\bar{\phi} \left(\sum_i (\nabla d\pi_M)(\bar{X}_i, \bar{X}_i) \right) \\
&= d\bar{\phi} \left(\sum_i \alpha(\bar{X}_i, \bar{X}_i) \right).
\end{aligned}
$$

Step 4. Let ϕ_p be a harmonic map in each leaf and the eigenvalue of Weingarten map of N be μ_{j_k} in $T_{j_k}(\phi_p(x))$ such that $X_i \in S_k(x)$ and

$d\phi_p\left(S_k(x)\right) \subset T_{j_k}\left(\phi_p(x)\right)$ for any $x \in M_p$. Then

$$\left\langle \tilde{\Delta}(\phi \circ i_p), \eta_\alpha \right\rangle = \sum_i \mu_{j_k}(\eta_\alpha) \left\langle d\phi_p(\bar{X}_i), d\phi_p(\bar{X}_i) \right\rangle,$$

where η_α is an orthonormal frame of σQ_N.

For each i,

$$0 = < d(\phi \circ i_p)(\bar{X}_i), \eta_\alpha > .$$

Therefore, taking the covariant derivative with respect to $d\phi_p(\bar{X}_i)$,

$$\begin{aligned} (2) \qquad 0 &= \sum_i \langle \nabla^N_{d\phi_p(\bar{X}_i)} d\phi_p(\bar{X}_i), \eta_\alpha \rangle \\ &\qquad + \langle d\phi_p(\bar{X}_i), \nabla^N_{d\phi_p(\bar{X}_i)} \eta_\alpha \rangle \\ &= \sum_i \langle \nabla d\phi_p(\bar{X}_i, \bar{X}_i) + d\phi_p(\nabla_{\bar{X}_i} \bar{X}_i), \eta_\alpha \rangle \\ &\qquad + \langle d\phi_p(\bar{X}_i), \nabla^N_{d\phi_p(\bar{X}_i)} \eta_\alpha \rangle \\ &= \langle \tilde{\Delta}\phi_p, \eta_\alpha \rangle + \sum_i \langle d\phi_p(\bar{X}_i), -\mu_{j_k}(\eta_\alpha) d\phi_p(\bar{X}_i) \rangle. \end{aligned}$$

Hence

$$\langle \tilde{\Delta}(\phi \circ i_p), \eta_\alpha \rangle = \sum_i \mu_{j_k}(\eta_\alpha) \langle d\phi_p(\bar{X}_i), d\phi_p(\bar{X}_i) \rangle.$$

Step 5.

If ϕ is harmonic, then by (1)

$$\begin{aligned} 0 &= \Delta^M \phi \\ &= \sum_\alpha \nabla^M d\phi(\xi_\alpha, \xi_\alpha) - d\phi(\tilde{\Delta} i_p) + \tilde{\Delta}(\phi \circ i_p) \end{aligned}$$

Hence by Step 2,3,4 for any $\eta_\alpha \in \sigma Q_N$

$$0 = \left\langle d\pi_M \left(\sum_\alpha \nabla^M d\phi(\xi_\alpha, \xi_\alpha) - d\phi(\tilde{\Delta} i_p) + \tilde{\Delta}(\phi \circ i_p) \right), \eta_\alpha \right\rangle$$

$$= \langle (\Delta^Q \bar{\phi}) + d\bar{\phi}(\bar{\tau}), \eta_\alpha \rangle + \sum_{k=1}^{p} \mu_{j_k}(\eta_\alpha)\gamma_k.$$

Therefore

$$\langle (\Delta^Q \bar{\phi}) + d\bar{\phi}(\bar{\tau}), \eta_\alpha \rangle = - \sum_{k=1}^{p} \mu_{j_k}(\eta_\alpha)\gamma_k .$$

Also if

$$\langle (\Delta^Q \bar{\phi}) + d\bar{\phi}(\bar{\tau}), \eta_\beta \rangle = - \sum_{k=1}^{p} \mu_{j_k}(\eta_\alpha)\gamma_k$$

for any η_α, then $d\pi_M(\Delta^M \phi) = 0$. By Step 1. $\Delta^M \phi = 0$. Hence ϕ is harmonic.

DEFINITION 2.2. Let M be a smooth manifold with foliation $\mathcal{F}$. Then $\mathcal{F}$ is a minimal foliation if all leaves of the foliation are a minimal submanifolds of M. And $\mathcal{F}$ is a totally geodesic foliation if each leaf L is a totally geodesic submanifold of M.

COROLLARY 2.3. *Let M, N be a equivariant manifold and $\phi : M \longrightarrow N$ be equivariant. Suppose M is a minimal foliation and N is a totally geodesic foliation, then ϕ is harmonic if and only if $\bar{\phi}$ is harmonic.*

Proof. If M is a minimal foliation, then by Step 2 the tension field of $\mathcal{L}_M$ is zero. And If N is a totally geodesic foliation, then by (2)

$$0 = \sum_i \langle \nabla^N_{d\phi_p(\bar{X}_i)} d\phi_p(\bar{X}_i), \eta_\alpha \rangle + \langle d\phi_p(\bar{X}_i), \nabla^N_{d\phi_p(\bar{X}_i)} \eta_\alpha \rangle$$

$$= \sum_i < \langle \nabla d\phi_p(\bar{X}_i, \bar{X}_i) + d\phi_p(\nabla_{\bar{X}_i} \bar{X}_i), \eta_\alpha \rangle$$

$$- \langle \nabla^N_{d\phi_p(\bar{X}_i)} d\phi_p(\bar{X}_i), \eta_\alpha \rangle$$

$$= \langle \tilde{\Delta} \phi_p, \eta_\alpha \rangle.$$

Hence

$$\Delta^M \phi = \Delta^Q \bar{\phi}.$$

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Department of Mathematics
Seoul National University
Seoul 151-742 Korea

Department of Mathematics
JeonJu University
Jeonju Jeonbuk 560-759 Korea