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H-FUZZY SEMITOPOGENOUS PREORDERED SPACES

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1. Preliminaries

Throughout this paper we will let H denote the complete Heyting algebra $(H, \vee, \wedge, *)$ with order reversing involution $*$. 0 and 1 denote the supremum and the infimum of $\emptyset$, respectively. Given any set X, any element of H^X is called H-fuzzy set (or, simply f.set) in X and will be denoted by small Greek letters, such as μ, ν, ρ, σ. H^X inherits a structure of H with order reversing involution in natural way, by defining $\vee$, $\wedge$, $*$ pointwise (same notations of H are usual). If f is a map from a set X to a set Y and $\mu \in H^Y$, then $f^{-1}(\mu)$ is the f.set in X defined by $f^{-1}(\mu)(x) = \mu(f(x))$. Also for $\sigma \in H^X$, $f(\sigma)$ is the f.set in Y defined by $f(\sigma)(y) = \sup\{\sigma(x) : f(x) = y\}$ ([4]). A preorder R on a set X is reflexive and transitive relation on X, the pair (X, R) is called preordered set. A map f from a preordered set (X, R) to another one (Y, T) is said to be preorder preserving (inverting) if for $x, y \in X$, xRy implies $f(x)Tf(y)$ (resp. $f(y)Tf(x)$).

For the terminology and notation, we refer to [10, 11, 13] for category theory and [7] for H-fuzzy semitopogenous spaces.

2. Category PHFS and its subcategories

A H-fuzzy semitopogenous preordered space (or, simply phfs.space) is a triplet (X, S, R) consisting of a set X, a hfs.structure S ([7]) and a preorder R on X.

With PHFS we will denote the category whose objects are phfs.spaces and whose morphisms are continuous preorder preserving maps.

Let HFS denote the category of H-fuzzy semitopogenous spaces and continuous maps, and let PORD denote the category of preordered sets and preorder preserving maps.

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We note that PHFS is the meet category of HFS and PORD, i.e., PHFS=HFS $\wedge$ PORD (see [12] for the meet categories of two categories).

Since HFS and PORD are both topological ([7], [5]), by [12], one has the following:

THEOREM 2.1. *PHFS is a topological and cotopological category.*

Using the above theorem, one has the following:

COROLLARY 2.2. *PHFS is complete and cocomplete.*

NOTATION. Let ^{k} be an ordinary operation ([7]). We will let ^{k}-PHFS denote the full subcategory of PHFS consisting of those phfs.space (X, S, R) with $S = S^k$.

THEOREM 2.3. (1) *If ^{k} is coarser than ^{1}, then ^{1}-PHFS is coreflective in ^{k}-PHFS.*

(2) *Let ^{a} be a symmetrical elementary operation. The ^{sa}-PHFS is closed under the formation of initial sources in ^{a}-PHFS.*

Proof. (1) Let $(X, S, R) \in {}^k$-PHFS. It is clear that S^1 is ^{1}-hfs.structure on X and the identity map $1_X : (X, S^1, R) \to (X, S, R)$ is a continuous preorder preserving map. Take any (Y, V, T) in ^{1}-PHFS and any continuous preorder preserving map $f : (Y, V, T) \to (X, S, R)$. Then $f^{-1}(S^1) = (f^{-1}(S))^1 < V^1$. Since $V^1 = V$, $f : (Y, V, T) \to (X, S^1, R)$ is a continuous preorder preserving map. This completes the proof.

(2) Suppose that $(f_i : (X, S, R) \to (X_i, S_i, R_i)_{i\in I}$ is an initial source in ^{a}-PHFS. Since PORD is closed under initial sources, it is enough to show that $S = S^{sa}$. Since ^{sa}-HFS is coreflective in ^{a}-HFS, $S = (\cup f_i^{-1}(S_i))^{ga}$.

Let $\sqsubset = (\cup_{i\in F} f_i^{-1}(\sqsubset_i))^a$, where $\sqsubset_i \in S_i$ $(i \in F)$ and F is an nonempty subset of I.

$$\begin{aligned}\sqsubset^{sa} &= ((\cup_{i\in F} f_i^{-1}(\sqsubset_i)) \cup (\cup_{i\in F} f_i^{-1}(\sqsubset_i))^c)^a \\ &= ((\cup_{i\in F} f_i^{-1}(\sqsubset_i)) \cup (\cup_{i\in F} f_i^{-1}(\sqsubset_i^c)))^a.\end{aligned}$$

Since ^{a} is symmetrical and $(X_i, S_i, R_i) \in {}^{sa}$-PHFS, $\sqsubset^{sa} = \sqsubset$. This completes the proof.

DEFINITION 2.4. In a preordered set (X, R), a *f*.set μ in X will be called increasing (decreasing) if for $x, y \in X$, xRy implies $\mu(x) \leq \mu(y)$ (resp. $\mu(y) \leq \mu(x)$) in H.

REMARK 2.5. (1) If $\{\mu_i : i \in I\}$ is a nonempty family of increasing (decreasing) *f*.sets in a preordered set X, then $\vee\{\mu_i : i \in I\}$ and $\wedge\{\mu_i : i \in I\}$ are both increasing (resp. decreasing).

(2) For any *f*.set μ in a preordered set X, $i(\mu) = \wedge\{\rho : \mu \leq \rho$ and ρ is an increasing *f*.set in $X\}$ is the smallest increasing *f*.set containing μ, and $d(\mu) = \wedge\{\rho : \mu \leq \rho$ and ρ is a decreasing *f*.set in $X\}$ is the smallest decreasing *f*.set containing μ.

DEFINITION 2.6. A phfs.space (X, S, R) will be called increasing (decreasing) if for any $\ll \in S$, $\mu \ll \rho$ implies that there exists $\ll_1 \in S$ such that $\mu \ll_1 \sigma \ll_1 \rho$ for some suitable increasing (resp. decreasing) *f*.set σ in (X, R).

We will let $I(D)$ denote the set of all increasing (resp. decreasing) *f*.sets in a preordered set X.

LEMMA 2.7. *Let X be a preordered set. Then one has the following:*

(1) *We define a relation Δ on H^X as follows: $\mu\Delta\rho$ iff there exists $\sigma \in I$ such that $\mu \leq \sigma \leq \rho$. Then $\{\Delta\}$ is an increasing ^{b}-hfs. structure on X.*

(2) *We define a relation on ∇ on H^X as follows: $\mu\nabla\rho$ iff there exists $\sigma \in D$ such that $\mu \leq \sigma \leq \rho$. Then $\{\nabla\}$ is a decreasing ^{b}-hfs.structure on X.*

Proof. (1) Since 0 and 1 are increasing *f*.set in X, $0\Delta0$ and $1\Delta1$ and hence Δ satisfies SO1) in [7]. SO2) and SO3) in [7] are immediate from definition of Δ. It follows from 2.5.1 that Δ is a biperfect fs.order on X. It is clear that $\{\Delta\}$ satisfies S1) in [7]. Suppose $\mu\Delta\rho$. Then $\mu \leq \sigma \leq \rho$ for some $\sigma \in I$. Since $\sigma\Delta\sigma$, $\mu\Delta\sigma\Delta\rho$. Thus Δ satisfies S2) in [7]. This completes the proof.

(2) This is analogous to (1).

THEOREM 2.8. *Let S be a hfs.structure on a preordered set X. Then one has the following:*

(1) *S is increasing iff $S < \{\Delta\}$.*

(2) *S is decreasing iff $S < \{\nabla\}$.*

(3) $\Delta = \nabla^c$.

Proof. (1) ($\Rightarrow$) Take any $\ll \in S$ and let $\mu \ll \rho$. Since S is increasing, there exists $\sqsubset \in S$ such that $\mu \sqsubset \sigma \sqsubset \rho$ for some $\sigma \in I$. Since $\sigma \in I$, by SO2) in [7], $\mu\Delta\rho$. Thus $S < \{\Delta\}$.

($\Leftarrow$) Take any $\ll \in S$ and let $\mu \ll \rho$. Then there exists $\sqsubset \in S$ such that $\mu \sqsubset \sigma \sqsubset \nu \sqsubset \rho$ for some σ, $\nu \in H^X$. Since $S < \{\Delta\}$, $\mu \sqsubset \theta \sqsubset \rho$ for some $\theta \in I$. Thus S is increasing.

(2) This is analogous to (1).

(3) This follows from the fact that $\sigma \in I$ iff $\sigma^* \in D$.

PROPOSITION 2.9. (1) *A hfs.structure coarser than an increasing (decreasing) hfs.structure is also increasing (resp. decreasing).*

(2) *A hfs.structure S is increasing (decreasing) iff so is each of the structures $S^q, S^p, S^{qp}, S^b, S^t, S^{tq}, S^{tqp}$ and S^{tb}.*

(3) *A hfs.structure S is increasing iff S^c is decreasing.*

(4) *If f is a preorder preserving map of a preordered set (X, R) to a increasing (decreasing) phfs.space (Y, S, T), then $(X, f^{-1}(S), R)$ is also increasing (resp. decreasing).*

(5) *If f is a preorder inverting map of a preordered set (X, R) to a increasing (decreasing) phfs.space (Y, S, T), then $(X, f^{-1}(S), R)$ is also decreasing (resp. increasing).*

Proof. (1) Suppose S is an increasing (decreasing) hfs.structure and $S^{\sim} < S$. Since $<$ is a transitive relation, $S^{\sim}$ is increasing (decreasing).

(2) Let ^{a} be one of $^q, ^p, ^{qp}, ^b, ^{tb}, ^{tqp}$ and ^{tb}. S is increasing (decreasing) iff $S < \{\Delta\}$ ($S < \{\nabla\}$ iff $S^a < \{\Delta\}$ ($S^a < \{\nabla\}$) iff S^a is increasing (decreasing).

(3) S is increasing iff $S < \{\Delta\}$ iff $S^c < \{\Delta^c\}$ iff $S^c < \{\nabla\}$ iff S is decreasing.

(4) Suppose (Y, S, T) is increasing. Take any $\sqsubset \in S$ and suppose $\mu f^1(\sqsubset)\rho$. There exists $\ll \in S$ such that $\mu f^{-1}(\ll)f^{-1}(\sigma)f^{-1}(\ll)\rho$ for some $\sigma \in I$. Since f is preorder preserving and σ is an increasing (decreasing) *f*.set in Y, $f^{-1}(\sigma)$ is an increasing (decreasing) *f*.set in X. Thus $(X, f^{-1}(S), R)$ is increasing (decreasing).

(5) This is analogous to (4).

Notation let IPHFS (DPHFS) denote the full subcategory of PHFS determined by increasing (decreasing, resp.) preordered hfs.spaces.

THEOREM 2.10. *IPHFS is bireflective in PHFS.*

Proof. For $(X, S, R) \in PHFS$, $T(S) = \{\sqsubset : \sqsubset$ is a fs.order on X and there is a fs.order $\ll \in S$ which is finer than $\sqsubset\}$ and let $S^u = \{\ll:$

there exists a sequence $(\ll_n)$ in $T(S)$ such that $\ll = \ll_1$ and for each n, $\mu \ll_n \rho$ implies that there exists an increasing f.set σ in X such that $\mu \ll_{n+1} \sigma \ll_{n+1} \rho\}$. It is clear that S^u is an increasing hfs.structure on X and $1_X : (X, S, R) \to (X, S^u, R)$ is a continuous preorder preserving map. Let $(Y, U, T) \in IPHFS$ and suppose $f : (X, S, R) \to (Y, U, T)$ is a continuous preorder preserving map. Then $f^{-1}(U) < S$. Since $f^{-1}(U)$ is increasing, $f^{-1}(U) < S^u$. This completes the proof.

For any $(X, S, R) \in PHFS$, S^u will be called the upper hfs.structure of (X, S, R).

REMARK. Since IPHFS and DPHFS are isomorphic, by the above theorem, DPHFS is also bireflective in PHFS. We will let S^d denote the upper hfs.structure of (X, S, R^{op}), which is called the lower hfs.structure of (X, S, R).

THEOREM 2.11. *Let ^{k} be an ordinary operation such that S^k is coarser than S^{tb} for all hfs.structure S on a set X. Then for each $(X, S, R) \in PHFS$, we have the following:*
(1) $S^{uk} < S^{ku}$.
(2) $S^{dk} < S^{kd}$.
(3) *if* $S \cong S^k$, *then* $S^u \cong S^{uk}$ *and* $S^d \cong S^{dk}$.

Proof. (1) Since $S^u < S$, $S^{uk} < S^k$. By (2.9.2) S^{utb} is increasing. It is follows from (2.9.1) that S^{uk} is increasing. Thus $S^{uk} < S^{ku}$.
(2) This is similar to (1).
(3) This is immediate from (1) and (2).

THEOREM 2.12. *For any $(X, S, R) \in PHFS$, we have the following:*
(1) $S^{dc} = S^{cu}$ *and* $S^{uc} = S^{cd}$.
(2) *if S is symmetrical, then* $S^u = S^{dc}$ *and* $S^d = S^{uc}$.

Proof. (1) Let $\sqsubset \in S^{dc}$. There exists $\ll \in S^d$ such that $\sqsubset = \ll^c$. Since $\ll \in S^d$, there exists a sequence $(\ll_n)$ in S such that $\ll = \ll_1$ and for each n, $\mu \ll_n \rho$ implies that $\mu \ll_{n+1} \sigma \ll_{n+1} \rho$ for some decreasing f.set in X. Thus $\sqsubset \in S^{cu}$, and hence $S^{dc} \subseteq S^{cu}$. By the same way, the inverse inclusion holds. The second part is similar to the first part.
(2) This is immediate from (1).

Let $\ll$ be fs.order on a set X. We will let $\ll^{u(d)}$ denote $\{\ll\}^{u(d)}$. Then $\mu \ll^{u(d)} \rho$ iff there is map $\phi : E \to I(D)$ such that $r < s$ implies

$\phi(r) \ll \phi(s)$. $\phi(0) = \mu$ and $\phi(1) = \rho$, where E denotes the set of all dyadic rationals between 0 and 1, i.e., the set of all rational numbers $r = n/2^m$, $n, m \in N$, $n \leq 2^m$.

THEOREM 2.13. *For any* $(X, S, R) \in PHFS$, *we have* (1) $S^{tu} = S^{ut}$ *and* (2) $S^{dt} = S^{td}$.

Proof. (1) Let $\ll = S^t$ and $\mu \ll^u \rho$. Then there is a sequence $(\ll_n)$ such that $\ll = \ll_n$ for all $n \in N$ and for each n, $\mu \ll_n \rho$ implies that $\mu \ll_{n+1} \sigma \ll_{n+1} \rho$ for some $\sigma \in I$. Thus $\mu S^{ut} \rho$, and hence $S^{tu} < S^{ut}$. Since $S^u < S$, $S^{ut} < S^t$. The inverse inequality can be found in (2.11.1).

(2) is similar to (1).

3. ^{a}-Convex spaces

Throughout this section all spaces are assumed to be phfs.spaces, and ^{a} will denote an elementary operation.

In a preordered set (X, R), $\mu \in H^X$ will be called convex if $\mu = \sigma \wedge \rho$, where σ, ρ are increasing, decreasing *f*.sets in (X, R), respectively.

REMARK. (1) If $\{\mu_i : i \in I\}$ is a family of convex *f*.sets in a preordered set (X, R), then $\wedge\{\mu_i : i \in I\}$ is also a convex.

(2) Let μ be a *f*.set in preordered set (X, R). Then $c(\mu) = i(\mu) \wedge d(\mu)$ is the smallest convex *f*.set containing μ.

(3) Let μ be a convex *f*.set in a preordered set (X, R). If $xRzRy$ and $\mu(x) \wedge \mu(y) > 0$, then $\mu(z) > 0$.

LEMMA 3.1. *For any order family* A *on a set* X *and any elementary operation* ^{a}, *one has the following:*

(1) $A^{ga} = A^{aga} = A^{gaga}$.

(2) $A^{ta} = A^{tat} = A^{ata} = A^{tata}$.

(3) $A^{st} = A^{ts}$.

Proof. (1) Since ^{a} and ^{ga} are ordinary operations, $A^{gaga} = A^{ga} < A^{aga}$. Since ^{g} and ^{a} are ordinary operations, $A^{aga} < A^{gaga}$. Thus $A^{ga} = A^{aga} = A^{gaga}$.

(2) Since ^{a} is an elementary operation, ^{ta} is an ordinary operation and hence $A^{ta} = A^{tat} = A^{ata} = A^{tata}$.

(3) It follows from the fact that for any order family A on a set X, $\cup\{<^c : < \in A\} = (\cup\{< : < \in A\})^c$.

LEMMA 3.2. *For a nonempty family $\{A_i : i \in I\}$ of order families on a set X, one has the following:*

(1) $(\cup\{A_i : i \in I\})^k = (\cup\{A_i^k : i \in I\})^k$, *where* ^{k} *is an ordinary operation.*

(2) $(\cup\{A_i : i \in I\})^{ga} = (\cup\{A_i^a : i \in I\})^{ga}$.

(3) *if* $I = \{1,2\}$, *then* $(A_1 \cup A_2)^{gt} \cong (A_1^t \cup A_2^t)^g$.

Proof. (1) Let $A = (\cup\{A_i : i \in I\})$. Then $A_i \subseteq A$ implies $A_i < A$, so that we get from 03) in [7], $A_i^k < A^k$ and $\cup\{A_i^k : i \in I\} < A^k$, $(\cup\{A_i^k : i \in I\})^k = A^{kk} = A^k$ from 02) in [7]. On the other hand, we get from 01) in [7] $A_i < A_i^k$, $A < \cup\{A_i^k : i \in I\}$, $A^k < (\cup\{A_i^k : i \in I\})^k$.

(2) This follows from (3.1.1) and (1).

(3) It is clear that $(A_1 \cup A_2)^{gt} = (A_1 \cup A_2)^{tg}$. Since ^{t} is an ordinary operation, by (1), $(A_1^t \cup A_2^t) = (A_1^t \cup A_2^t)^{tg} = (A_1 \cup A_2)^{tg}$. Since $(A_1^t \cup A_2^t)^g = \{A_1^t, A_2^t, A_1^t \cup A_2^t)$, $(A_1 \cup A_2)^{gt} \cong (A_1^t \cup A_2^t)^g$.

DEFINITION 3.3. A space (X, S, R) will be called:

(1) almost ^{a}-convex if $S < (S^u \cup S^d)^{ga}$.

(2) ^{a}-convex if $S \cong (S^u \cup S^d)^{ga}$.

PROPOSITION 3.4. (1) *A space (X, S, R) is almost ^{a}-convex iff $(S_1 \cup S_2)^g < S < (S_1 \cup S_2)^{ga}$, where $S_1(S_2)$ is an increasing (decreasing) hfs.structure on (X, R).*

(2) *A space (X, S, R) is ^{a}-convex iff $S \cong (S_1 \cup S_2)^{ga}$, where $S_1(S_2)$ is an increasing (decreasing) hfs.structure on (X, R).*

(3) *If a space (X, S, R) is almost ^{a}-convex then (X, S^a, R) is ^{a}-convex.*

(4) *If ^{a} is an elementary operation such that $S^{au} \cong S^{ua}$ and $S^{ad} \cong S^{da}$ for any hfs.structure S on a set X, then the converse of (3) holds.*

(5) *Let (X, S, R) be a space with $S \cong S^a$. $(X, S, \leq)$ is almost ^{a}-convex iff it is ^{a}-convex.*

(6) *Let ^{e} be an elementary operation such that S^a is coarser than S^e for any hfs.structure S on a set X. If a space (X, S, R) is almost ^{a}-convex, then it is almost ^{e}-convex.*

(7) *If a space (X, S, R) is (almost) ^{a}-convex and ^{e} is an elementary operation such that ^{ae} is also elementary operation, then (X, S^e, R) is (almost) ^{ae}-convex.*

Proof. (1) The necessity is obvious. Conversely, if this condition is fulfilled by S, then $S_1(S_2) < S^u(S^d)$, therefore $(S_1 \cup S_2)^{ga} < (S^u \cup S^d)^{ga}$, so that S is almost ^{a}-convex.

(2) The necessity is obvious. Conversely, if this condition is fulfilled by S, then S_1 $(S_2) < S^u(S^d)$, therefore $S \cong (S_1 \cup S_2)^{ga} < (S^u \cup S^d)^{ga}$. But $S^a \cong (S_1 \cup S_2)^{gaa} = (S_1 \cup S_2)^{ga} \cong S$, so that S is ^{a}-convex.

(3) If (X, S, R) is almost ^{a}-convex, then, by (1), $(S_1 \cup S_2)^g < S < (S_1 \cup S_2)^{ga}$, where $S_1(S_2)$ is an increasing (decreasing) hfs.structure on (X, R). Since ^{a} is an elementary operation, $(S_1 \cup S_2)^{ga} < S^a < (S_1 \cup S_2)^{gaa} = (S_1 \cup S_2)^{ga}$ and hence $(S_1, \cup S_2)^{ga} \cong S^a$. By (2), S^a is ^{a}-convex.

(4) If (X, S^a, R) is ^{a}-convex, then $S^a \cong (S^{au} \cup S^{ad})^{ga}$. Since $S^{au} \cong S^{ua}$ and $S^{ad} \cong S^{da}$, $S^a \cong (S^{ua} \cup S^{da})^{ga}$. By Lemma 3.1.1 and 3.2.1, $S^a \cong (S^u \cup S^d)^{ga}$. Since $S < S^a$, $S < (S^u \cup S^d)^{ga}$. Thus (X, S, R) is almost ^{a}-convex.

(5) It is immediate from the definition 3.3.

(6) If (X, S, R) is almost ^{a}-convex, then $S < (S^u \cup S^d)^{ga}$. By the assumption, $S < (S^u \cup S^d)^{ga} < (S^u \cup S^d)^{ge}$. Thus (X, S, R) is almost ^{e}-convex.

(7) If (X, S, R) is almost ^{a}-convex, then $S < (S^u \cup S^d)^{ga}$, therefore $S^e < (S^u \cup S^d)^{gae}$. Since $(S^u \cup S^d)^g < S$, $(S^u \cup S^d)^g < S^e$. By (1), $(X, S^e, \leq)$ is almost ^{a}-convex. If (X, S, R) is ^{a}-convex, then $S \cong (S^u \cup S^d)^{ga}$, and hence $S^e \cong (S^u \cup S^d)^{gae}$. By (2), (X, S^e, R) is ^{ae}-convex.

THEOREM 3.5. *Let (X, S, R) be a almost ^{a}-convex space. Then (X, S^t, R) is almost ^{a}-convex, and hence (X, S^{ta}, R) is ^{a}-convex.*

Proof. If (X, S, R) is almost ^{a}-convex, then $S < (S^u \cup S^d)^{ga}$, and hence $S^{ta} < (S^u \cup S^d)^{gata} \cong (S^{ut} \cup S^{dt})^{ga}$ ((3.1.2) and (3.2.3)). By (2.13), $S < (S^{tu} S^{td})^{ga}$. Thus (X, S^t, R) is almost ^{a}-convex. The second part is immediate from (3.4.3).

NOTATION. ^{a}-APHFS (^{a}-CPHF) denotes the full subcategory of PHFS determined by all almost ^{a}-convex (^{a}-convex, resp.) spaces.

THEOREM 3.6. (1) *^{a}-APHFS is bireflective in PHFS.*

Proof. Suppose $(f_i : (X, S, R) \to (X_i, S_i, R_i))_{i \in I}$ is an initial source in PHFS and each $(X_i, S_i, R_i) \in {}^a$-APHFS. Then $S \cong (*) = (\cup\{f^{-1}(S_i) : i \in I\})^g$ and xRy iff $f_i(x) R_i f_i(y)$ for all $i \in I$. Since for each $i \in I$, $(X_i, S_i, R_i) \in {}^a$-APHFS, $(S_{i1} \cup S_{i2})^g < S_i < (S_{i1} \cup S_{i2})^{ga}$, for some increasing (decreasing) hfs.structure S_{i1} (S_{i2}, resp.) on (X, R). For each $i \in I$, $f_i^{-1}((S_{i1} \cup S_{i2})^g) = (f_i^{-1}(S_{i1} \cup S_{i2}))^g = (f_i^{-1}(S_{i1}) \cup f_i^{-1}(S_{i2}))^g < f_i^{-1}(S_i)$ from 06) in [7].

$$\begin{aligned}(**) &= \cup\{(f_i^{-1}(S_{i1} \cup f_i^{-1}(S_{i2}))^g : i \in I\} \\ &< (\cup\{f_i^{-1}(S_{i1}) \cup f_i^{-1}(S_{i2}))^g : i \in I\})^g \\ &= ((\cup\{f_i^{-1}(S_{i1}) : i \in I\}) \cup (\cup\{f_i^{-1}(S_{i2}) : i \in I\})^g\end{aligned}$$

from (3.2.1). Thus $(**) < (*)$. Since $f_i^{-1}(S_i) < f_i^{-1}((S_{i1} \cup S_{i2})^{ga})$ $(i \in I)$, $(*) < (\cup\{f_i^{-1}((S_{i1} \cup S_{i2})^{ga}) : i \in I\})^g$. From 01) and 06) in [7] and (3.2.1),

$$\begin{aligned}(*) &< (\cup\{(f_i^{-1}(S_{i1}) \cup f_i^{-1}(S_{i2}))^{ga} : i \in I\})^{ga} \\ &= ((\cup\{f_i^{-1}(S_{i1}) : i \in I\}) \cup (\cup\{f_i^{-1}(S_{i2}) : i \in I\}))^{ga}.\end{aligned}$$

By (2.10) and (3.2.1), (X, S, R) is almost ^{a}-convex. This completes the proof.

The following lemma is the result in [6].

LEMMA 3.7. *Let $G : C \to D$ be a functor and A (B, resp.) a subcategory of C (D, resp.) such that G has a restriction $E : A \to B$, i.e., $G \circ H = F \circ E$, where H, F are embedding functors. Suppose*

(1) B is a coreflective (reflective, resp.) subcategory of D.

(2) $E : A \to B$ is full and $G : C \to D$ is faithful and full; and for each $C \in C$, there is $A \in A$ such that $E(A)$ is isomorphic to B-coreflection (B-reflection, resp.) of $G(C)$. Then A is a coreflective (reflective, resp.) subcategory of C.

The following is now immediate from the above Lemma, Theorem 2.3.1, Theorem 3.6.

COROLLARY 3.8. (1) *^{a}-CPHFS is bireflective in ^{a}-PHFS.*

(2) *^{a}-CPHFS is coreflective in ^{a}-APHFS.*

THEOREM 3.9. *Let ^{a} be a symmetrical elementary operation. Then one has the following:*

(1) A space (X, S, R) is (almost) ^{a}-convex iff (X, S^c, R) is (almost) ^{a}-convex.

(2) *If* (X, S, R) *is almost* ^{a}*-convex, then* (X, S^s, R) *is almost* ^{a}*-convex, and hence* (X, S^{sa}, R) *is* ^{a}*-convex.*

Proof. (1) If (X, S, R) is almost ^{a}-convex, then $S < (S^u \cup S^d)^{ga}$. Then $S^c < (S^u \cup S^d)^{cga} = (S^{uc} \cup S^{dc})^{ga}$. By (2.12.1), (X, S^c, R) is almost ^{a}-convex. The converse is similar to the necessity. If (X, S, R) is ^{a}-convex, $S^c \cong (S^u \cup S^d)^{gac} = (S^{uc} \cup S^{dc})^{ga}$. By (2.12.1), (X, S^c, R) is ^{a}-convex.

(2) This is immediate from (1), (3.6) and (3.2.3).

THEOREM 3.10. *Let* (X, S, R) *be a symmetrical space. Then the following statements are equivalent:*

(1) (X, S, R) *is almost* ^{a}*-convex.*

(2) *there is an increasing structure* S_1 *on* X *such that* $S_1 < S < S_1^{sa}$.

(3) *there is an decreasing structure* S_1 *on* X *such that* $S_1 < S < S_1^{sa}$.

Proof. Suppose (X, S, R) is almost ^{a}-convex, then by (2.12.2), one has $S < (S^u \cup S^d)^{ga} = (S^u \cup S^{uc})^{ga} = S^{usa}$. Clearly $S^{us} < S$. Hence S^u will do the job for S_1. The other implications are obvious.

THEOREM 3.11. *Let* (X, S, R) *be a symmetrical space. Then the following statements are equivalent:*

(1) (X, S, R) *is* ^{a}*-convex.*

(2) $S \cong S_1^{sa}$, *where* S_1 *is increasing on* (X, R).

(3) $S \cong S_1^{sa}$, *where* S_1 *is decreasing on* (X, R).

Proof. It is analogous to (3.10).

THEOREM 3.12. *Let* (X, S, R) *be a space such that* $S \cong S^k$, *where* $^{k=a}$ *or* $^{k=ta}$ *for an elementary operation* ^{a}, *which fulfills* $\ll^a$ *is coarser than* $\ll^b$ *for any semitopogenous order* $\ll$. *Then one has the following:*

(1) (X, S, R) *is* ^{a}*-convex iff there exist hfs.structures* $S_1 = S_1^k$ *and* $S_2 = S_2^k$ *such that* $S_1(S_2)$ *is increasing (decreasing) on* (X, R), *and* $S \cong (S_1 \cup S_2)^{ga}$.

(2) *If* S *is symmetrical, then* (X, S, R) *is* ^{a}*-convex iff* $S \cong S_1^{sk}$ *for an increasing or decreasing hfs.structure* $S_1 = S_1^k$ *on* (X, R).

Proof. (1) The sufficiency of the condition is clear even in the case of $^{k=ta}$ (see (3.4.2) and (3.2.3)). Conversely, if (X, S, R) is ^{a}-convex, then $S \cong S^k \cong (S^u \cup S^d)^{gk} = (S^{uk} \cup S^{dk})^{gk}$ ((3.1.2), (3.2.2) and (3.2.3)). Put $S_1 = S^{uk}$ and $S_2 = S^{dk}$. ^{k} is an ordinary operation, for which S^k is coarser than S^{tb}, thus from (2.11.1) and (2.11.2), $S_1 \cong S^u$ and $S_2 \cong S^d$, so that $S_1(S_2)$ is increasing (decreasing, resp.). Finally from (02) or [6], $S_1 = S_1^k$ and $S_2 = S_2^k$.

(2) Suppose that the condition is satisfied by S. For ${}^{k=a}$ S is ^{a}-convex by (3.11). For ${}^{k=ta}$ we have $S \cong S_1^{sta} = S_1^{tsa}$ ((3.1.3)), thus we can refer to (3.11) and (2.9.2). Conversely, let (X, S, R) be ^{a}-convex and $S \cong S_1^{sa}$ for an increasing S_1 ((3.11.2)). Then $S \cong S_1^{sa}$ is coarser than $S^{sa} = S^a \cong S$, therefore $S \cong S^k \cong S^{usak}$. From (3) in [7] and (2.11), $S^u \cong S^{uk}$, thus $S_1 = S^{uk}$ is increasing, $S_1 = S_1^k$ and $S_1 \cong S^{usk} \cong S^{uksk} = S_1^{sk}$.

NOTATION. Let μ be a *f*.set in a set X. Then $\mu_x(y) = \mu(x)$ if $y = x$ and $\mu_x(y) = 0$ if $y \neq x$, and $\mu^x(y) = \mu(x)$ if $y = x$ and $\mu^x(y) = 1$ if $y \neq x$.

LEMMA 3.13. *Let be X a set. Then one has the following:*

(1) *Let $\{\ll_i : i = 1, \ldots, n\}$ be a finite sequence of fs.orders on X. Then $(\cup\{\ll_i : i = 1, \ldots, n\})^{qp}$ is coarser than $(\cup\{\ll_i^{pc} : i = 1, \ldots, n\})^{qpc}$.*

(2) *Let $\{\ll_i : i = 1, \ldots, n\}$ be a finite sequence of biperfect fs.orders on X. Then $(\cup\{\ll_i : i = 1, \ldots, n\})^{qp} = (\cup\{\ll_i : i = 1, \ldots, n\})^b$.*

Proof. (1) Put $\ll = \cup\{\ll_i : i = 1, \ldots, n\}$, $\ll_1 = \cup\{\ll_i^{pc} : i = 1, \ldots, n\}$. Suppose $\mu \ll^{qp} \rho$. Then, for each $x \in X$, $\mu_x \ll^q \rho$, thus for each $y \in X$, $\mu_x \ll^q \rho^y$. Hence $\mu_x \ll \rho^y$ for all $x, y \in X$. Thus, for each $y \in X$, $\vee\{\mu_x : \mu_x \ll_i \rho^y\} \ll_i^p \rho^y$ and hence $(\rho^y)^* \ll_i^{pc} (\vee\{\mu_x : \mu_x \ll_i \rho^y\})^*$. Then $(\rho^y)^* \ll_1^q \wedge\{(\vee\{\mu_x : \mu_x \ll_i \rho^y\})^* : i = 1, \ldots, n\} = (\vee\{\vee\{\mu_x : \mu_x \ll_i \rho^y\} : i = 1, \ldots, n\})^* = \mu^*$ and hence $\vee\{(\rho^y)^* : y \in X\} \ll_1^\infty \mu^*$. Since $\rho^* = \vee\{(\rho^y)^* : y \in X\}$, $\mu \ll_1^{qpc} \rho$. Thus $\ll^{qp}$ is coarser than $\ll_1^{qpc}$.

(2) Put $\ll = \cup\{\ll_i : i = 1, \ldots, n\}$. By (1), $\ll^{qpc} = \ll^{qcp}$ and hence $\ll^{qp}$ is a biperfect fs.order on X finer than $\ll$. By (1.1.3) in [7], $\ll^b$ is coarser than $\ll^{qp}$. On the other hand, $\ll$ is coarser than $\ll^b$ ((1.1.3) in [7]). Thus $\ll^{qp}$ is coarser than $\ll^{bqp} = \ll^{bp} = \ll^b$. Thus $\ll^{qp} = \ll^b$.

THEOREM 3.14. *Any ^{b}-convex space is ^{qp}-convex.*

Proof. If (X, S, R) is ^{b}-convex, then $S \cong S^b$. Because of (3.13.2) and (3.12.1) $S \cong (S_1 \cup S_2)^b = (S_1 \cup S_2)^{gqp}$, where $S_1(S_2)$ is increasing (decreasing) biperfect hfs.structure on (X, R).

NOTATION. We will let C_σ denote the family $\{\ll_{\mu,\sigma} : \mu \leq \sigma \leq 1,\ \sigma$ is convex$\}$.

THEOREM 3.15. *A space (X, S, R) is almost ^{a}-convex iff for any $\ll \in S$, there is a family $\mathcal{C} \subseteq \mathcal{C}_\sigma$ such that $\{\ll\} < \mathcal{C}^{ta} < (S_1 \cup S_2)^{ga}$, where $S_1(S_2)$ is increasing (decreasing) hfs.structure on (X, R) such that $(S_1 \cup S_2)^g < S$.*

Proof. Since (X, S, R) is almost ^{a}-convex, $(S_1 \cup S_2)^g < S < (S_1 \cup S_2)^{ga}$ for $S_1(S_2)$ is an increasing (decreasing) hfs.structure on (X, R). If $\ll \in S_1(S_2)$ and $\ll_1 \in S_1(S_2)$ are such that $\ll$ is coarser than $\ll_1^2$, then $\mu \ll \rho$ implies $\mu \ll_1 \nu \ll_1 \rho$, and we can find an increasing (decreasing) *f*.set σ in (X, R) for which $\nu \leq \sigma \leq \rho$. Then $\mu \ll_1 \sigma$, hence $\mathcal{C}_\ll = \{\ll_{\alpha,\sigma} \colon \alpha \ll_1 \sigma,\ \sigma \text{ is increasing (decreasing)}\}$ is a family of elementary fs.orders such that $\mathcal{C} \subseteq \mathcal{C}_\sigma$ (increasing or decreasing *f*.sets are convex), and $\{\ll\} < \mathcal{C}^{ta} < \{\ll_1^a\}$. By the same way, if $\ll_0 \in S_1(S_2)$ and $\ll_{01} \in S_1(S_2)$ are such that $\ll_0$ is coarser than $^{20}_{01}$, then there exists $\mathcal{C}^{ta}_{\ll 0}$ such that $\{\ll 0\} < \mathcal{C}^{ta}_{\ll 0} < \{\ll^a_{01}\}$. Put $\mathcal{C} = \mathcal{C}_\ll \cup \mathcal{C}_{\ll 0}$. Then $\mathcal{C} \subseteq \mathcal{C}_\sigma$, further $\mathcal{C}^{ta} = (\mathcal{C}_\ll \cup \mathcal{C}_{\ll 0})^{ta} = (\mathcal{C}^{ta}_\ll \cup \mathcal{C}^{ta}_{\ll 0})^{ta} < (S_1 \cup S_2)^{ga}$. Then converse is obvious.

COROLLARY 3.16. *If ^{q}-space (X, S, R) is almost ^{i}-convex then for every $\ll \in S$ there exists $\ll_1 \in S$ such that $\mu \ll \rho$ implies $\mu \leq \vee_{i=1}^m \alpha_i$, $\vee_{i=1}^m \sigma_i \leq \rho$, where m is a suitable natural number, $\alpha_i \ll_1 \sigma_i$, and σ_i is convex in (X, R) for each $1 \leq i \leq m$.*

Proof. Suppose that (X, S, R) is almost ^{i}-convex, and put $\ll \in S$. There exists $\mathcal{C} \subseteq \mathcal{C}_\sigma$ and $\ll_1 \in S$ such that $\{\ll\} < \mathcal{C}^t < \{\ll_1\}$. Disregarding the trivial cases of $\mu = 0$ or $\rho = 1$, from $\mu \ll \rho$,

$$\mu \leq \cup_{i=1}^m \cap_{j=1}^{ni} \alpha_{ij}, \quad \cup_{i=1}^m \cap_{j=i}^{ni} \sigma_{ij} \leq \rho,$$

where σ_{ij} is convex and $\alpha_{ij} \ll_1 \sigma_{ij}$ $(1 \leq i \leq m,\ 1 \leq j \leq n_i)$ by (2.1) in [8]. Then $\sigma_i = \wedge_{j=i}^{ni} \sigma_{ij}$ is convex, and $\alpha_i = \wedge_{j=1}^{ni} \alpha_{ij}$ we have $\mu \leq \vee \alpha_i$, $\wedge_i^m = {}_i\sigma_i \leq \rho$, further $\alpha_i \ll_1 \sigma_i$, because $\ll_1$ is topogeneous fs.order.

In the rest of this section we suppose that all spaces are ^{q}-phfs.spaces.

DEFINITION 3.17. A space (X, S, R) will be called symmetrizable if there exists a symmetrical ^{i}-convex S_1 on (X, R) such that $S_1 < S < S_1^p$.

The following is immediate from the above definition, (3.4.1), (3.4.3) and (3.10).

REMARK. (1) If (X,S,R) is a symmetrizable space, then it is almost ^{p}-convex, and hence (X,S^p,R) is ^{p}-convex.

(2) If (X,S,R) is ^{i}-or almost ^{p}-convex symmetrical space, then it is symmetrizable.

NOTATION. Let SyPHFS denote the full subcategory of ^{q}-PHFS determined by symmetrizable spaces.

THEOREM 3.18. *SyPHFS is bireflective in ^{q}-PHFS.*

Proof. Suppose $((f_i:(X,S,R)\to(X_i,S_i,R_i))_{i\in I}$ is initial source in ^{q}-PHFS and each $(X_i,S_i,R_i)\in SyPHFS$. Then $S\cong(\cup f_i^{-1}(S_i))^{gq}$. For each $i\in I$, there exists symmetrical ^{i}-convex structure S_{i1} on (X_i,R_i) such that $S_{i1}<S_i<S_{i1}^p$. Then $(\cup f_i^{-1}(S_{i1}))^{gq}<S<(\cup(f_i^{-1}(S_{i1})^p)^{gqp}$. Since $(\cup(f_i^{-1}(S_{i1})^p)^{gqp}<(\cup(f_i^{-1}(S_{i1}))^{gqp}$, $S<(\cup(f_i^{-1}(S_{i1}))^{gqp}$. From Theorem 2.12 in [7] and (3.6.2), $(\cup f_i^{-1}(S_i))^{gq}$ is symmetrical ^{i}-convex structure on (X,R), and hence (X,S,R) is symmetrizable.

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