Journal of the Korean Society for Noise and Vibration Engineering, Vol. 4, No. 1, pp. 83~89, 1994.

⟨**Original**⟩

A Strategy for Moving Mass Systems from One Point to Another without Inducing Residual Vibration

잔류진동 없이 질량계를 한 위치에서 다른 위치로 옮기기 위한 전략

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⟨Received January 13, 1994 ; Accepted February 18, 1994⟩

ABSTRACT

In many circumstances, it is desired to move a mass from one position to another without inducing any vibration in the mass being moved. Two such problems are considered here : the motion of a mass initiated by another mass, and the motion of a pendulum initiated by the specified motion of its support. In each case, it is desired that the system start at rest and come to rest in the second position. A simple strategy for the specified motion is given here. The method is motivated by engine cam-follower design. The force required to move the system in question is determined as well as the maximum value of the force required(and the times at which these forces take place).

요　　약

물체를 한 위치에서 타 위치로 잔류 진동없이 움직이려 하는 경우가 많다. 본 논문에서는 한 질량이 다른 질량과 스프링에 의해 움직이는 경우와 진자가 지지점의 운동에 의해 움직이는 경우에 관해 논하였다. 각 경우에 있어서 시스템은 정지 상태에서 출발한 후 최종위치에서 다시 멈추는 것으로 간주하였다. 본 논문에 주어진 운동방식에 대한 기본 전략은 엔진 캠-구동자 형태의 설계시 단순화된 모형으로부터 도출되었다. 이와 같은 형태로 시스템을 움직이는데 필요한 힘과 최대힘 및 그때 소요되는 시간이 계산되었다.

1. Introduction

In many circumstances, it is desired to move a mass from one position to another without inducing any vibration in the mass being moved. Examples include valve train systems in engines, recording heads on computer disk drives and robot arms.(1) In fact, the general design of a typical engine cam gives the motivation for the specified motion we attempt as follows. There are several approaches to the problem posed here.(2,3)

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2. Case Study for Dwell-Rise-Dwell Motion

Suppose we wish to move the mass m in Fig. 1 by

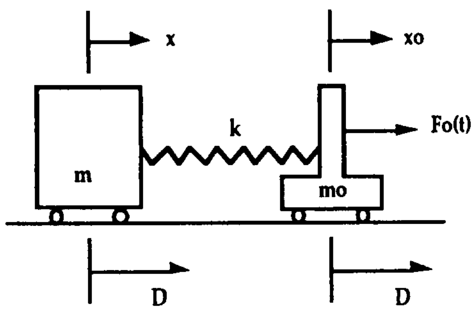

Fig. 1 The system : movement of a mass

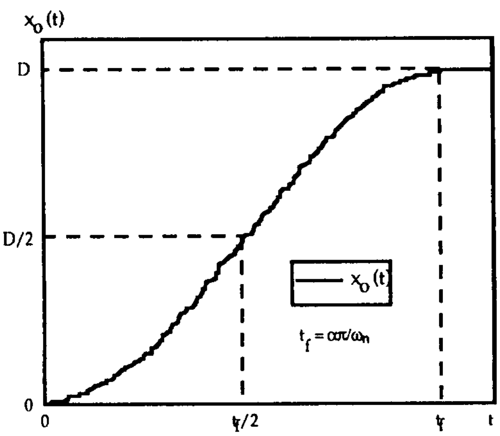

Fig. 2 Prescribed motion $x_o(t)$

moving the cart in a prescribed motion. Suppose both the mass and cart are at rest initially. And suppose that we wish to move the cart so that the mass m comes to rest again after a displacement D.

The differential equation of motion of the mass m is :

$$F_x = ma_x : k[x_o(t) - x] = m\ddot{x} \tag{1}$$

or

$$\ddot{x} + \omega_n^2 x = \omega_n^2 x_o(t) \tag{2}$$

where $\omega_n = \sqrt{\frac{k}{m}}$

Since we want m to move a distance D with no overshoot, it is important that the cart move a distance D as well. This means the spring which is unstressed initially will be unstressed in the final position as well. Thus each of the two masses will end up with a displacement D.

In order to accomplish this, let us suppose that the specified displacement has the form :

$$x_o(t) = \frac{1}{2}D\left[1 - \cos\left(\frac{\omega_n t}{\alpha}\right)\right], \quad 0 \le t \le \left(\frac{\alpha\pi}{\omega_n}\right)$$

$$= D \qquad t > \left(\frac{\alpha\pi}{\omega_n}\right) \tag{3}$$

The motivation for the $(1 - \cos\ \theta)$ in (3) comes from typical shapes used in engine valve train systems.[4] A plot of (3) is shown in Fig. 2. Notice that the shape of the curve gives hope that the strategy might work. Notice also that the speed with which we can make the move is determined by α since $\frac{\pi}{\omega_n}$ is fixed.

We will see that the values of α which will work are :

$$\alpha = 3,\ 5,\ 7,\ \cdots \tag{4}$$

That is, the frequencies of the prescribed motion $x_o(t)$ can be :

$$\frac{\omega_n}{3},\ \frac{\omega_n}{5},\ \frac{\omega_n}{7},\ \cdots,\ \frac{\omega_n}{(2k+1)} \tag{5}$$

where, $k = 1,\ 2,\ 3,\ \cdots$

We construct the solution to (2) with the specified displacement $x_o(t)$ given by (3) :

$$\ddot{x} + \omega_n^2 x = \frac{1}{2}D\omega_n^2\left[1 - \cos\left(\frac{\omega_n t}{\alpha}\right)\right] \tag{6}$$

The right hand side of (6) consists of a constant term plus a cosine term. As such, we guess a particular (forced) solution of the form :

$$x_p = P + Q\cos\left(\frac{\omega_n t}{\alpha}\right) \tag{7}$$

where P and Q are constants to be determined so that (6) is satisfied. Inserting (7) into (6) and canceling the common term (ω_n^2) gives :

$$P = \frac{1}{2}D,\ Q = -\frac{1}{2}D\left(\frac{\alpha^2}{\alpha^2 - 1}\right) \tag{8}$$

The homogeneous solution to (6) has the form :

$$x_h = A\sin(\omega_n t) + B\cos(\omega_n t) \tag{9}$$

Adding (7) and (9), we get :

$$x = \frac{1}{2}D\left[1 - \frac{\alpha^2}{\alpha^2 - 1}\cos\left(\frac{\omega_n t}{\alpha}\right)\right] + A\sin(\omega_n t) + B\cos(\omega_n t) \tag{10}$$

We determine the constants A and B from the initial conditions :

$$x(0) = \dot{x}(0) = 0 \tag{11}$$

These conditions yield

$$A=0,\ B=\frac{1}{2}D\left[\frac{1}{(\alpha^2-1)}\right] \tag{12}$$

Thus, we have the complete solution:

$$x=\frac{1}{2}D\left[1-\frac{\alpha^2}{\alpha^2-1}\cos\left(\frac{\omega_n t}{\alpha}\right)+\frac{1}{\alpha^2-1}\cos(\omega_n t)\right] \tag{13}$$

There are two conditions which we must now impose on (13). First, we want $x(t_f)=D$, where $t_f=\left(\frac{\alpha\pi}{\omega_n}\right)$. And finally, we want the velocity of m to vanish at t_f(i.e. $\dot{x}(t_f)=0$). Note that:

$$\frac{\omega_n t_f}{\alpha}=\pi \text{ (rad.)}$$
$$\text{and } \omega_n t_f=\alpha\pi \text{ (rad.)} \tag{14}$$

Thus from (13), we get:

$$x(t_f)=\frac{1}{2}D\left[1+\frac{\alpha^2}{\alpha^2-1}+\frac{1}{\alpha^2-1}\cos(\alpha\pi)\right] \tag{15}$$

The terms inside the braket of (15) must be 2 if $x(t_f)=D$. This will happen if $\cos(\alpha\pi)=-1$. Thus, this condition requires:

$$\alpha=3,\ 5,\ 6,\ \cdots,\ (2k+1) \tag{16}$$

(Note that the term $\alpha=1$ is inappropriate.)

Now we require that $\dot{x}(t_f)=0$. From (13) and (14), we get

$$\dot{x}(t_f)=-\frac{1}{2}\omega_n D\left[\frac{1}{\alpha^2-1}\sin(\alpha\pi)\right] \tag{17}$$

This expression vanishes if $\alpha=2,\ 3,\ 4,\ \cdots$.

The terms which are common to (16) and (17) are then the odd integer values of α from 3 on. Thus the solution for the motion of the mass m is:

$$\begin{aligned} x&=\frac{1}{2}D\left[1-\frac{\alpha^2}{\alpha^2-1}\cos\left(\frac{\omega_n t}{\alpha}\right)\right.\\ &\quad\left.+\frac{1}{\alpha^2-1}\cos(\omega_n t)\right],\quad 0\le t\le t_f\\ &=D\qquad\qquad\qquad,\quad t>t_f \end{aligned} \tag{18}$$

where $t_f=\left(\frac{\alpha\pi}{\omega_n}\right)$

and $\alpha=3,\ 5,\ 7,\ \cdots,\ (2n+1)$

A plot of $x_o(t)$ and $x(t)$, the specified motion of the cart and the esulting motion of m, is shown in Fig. 3. Notice that at t_f the distance between m and the cart is the same as it was initially. Thus the spring is unstressed. And since the velocity of m is zero and the prescribed motion of the cart remains at $x_o=D$, the mass m will remain at rest at the point $x=D$.

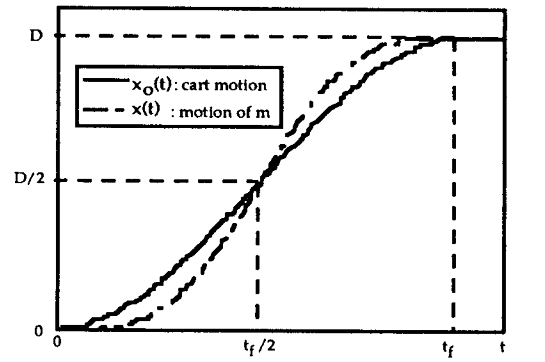

Fig. 3 The motion $x_o(t)$ and the mass motion $x(t)$

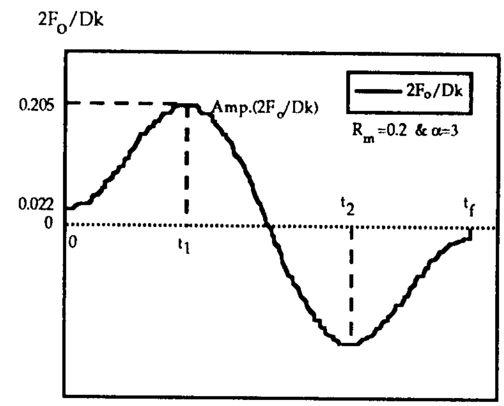

Fig. 4 The force required to initiate the motion $x_o(t)$ for the movement of a mass

Equation (18) gives the design equation for the motion shown in Fig. 3. As noted, α can take the values 3, 5, 7, $\cdots$. Thus the time required to move the mass m a distance D is:

$$t_f=\left(\frac{\alpha\pi}{\omega_n}\right) \tag{19}$$

In order to minimize the time t_f, we should select $\alpha=3$ and select a large value of the spring constant $k\left(\text{to increase the natural frequency } \omega_n=\sqrt{\frac{k}{m}}\right)$. The solution is not unique since we can take $\alpha=3,\ 5,\ 7,\ \cdots$ and adjust ω_n accordingly

The design equations for the strategy are contained in eqμations (3) and (18). One final consideration is the force $F_o(t)$ required to move the cart. Writing $F_o=m_o a_o+k(x_o-x)$ where a_o is the second (time) derivative of the motion (3), we obtain:

$$\frac{2F_o}{Dk}=\left(\frac{m_o}{m\alpha^2}+\frac{1}{\alpha^2-1}\right)\cos\left(\frac{\omega_n t}{\alpha}\right)$$

$$-\frac{1}{\alpha^2-1}\cos(\omega_n t) \qquad (20)$$

where $0\leq t\leq\frac{\alpha\pi}{\omega_n}$

The maximum amplitude of $\frac{2F_o}{Dk}$ depends on α and the mass ratio $\frac{m_o}{m}$. Suppose we take $\alpha=3$(the case for the fastest transit from A to B). Then we can determine the positions and heights of the maximum amplitudes :

The initial and final amplitude of $\frac{2F_o}{Dk}$ is also a function of the mass ratio $R_m\left(=\frac{m_o}{m}\right.$:

$$\left(\frac{2F_o}{Dk}\right)\bigg|_{t=0}=\left(\frac{m_o}{m\alpha^2}\right)=\left(\frac{R_m}{\alpha^2}\right) \qquad (21)$$

A plot of $\frac{2F_o}{Dk}$ is shown in Fig. 4. Here $\alpha=3$ and $\frac{m_o}{m}$ $=0.20$.

3. Case Study for the Pendulum Motion

Suppose now that we have a simple pendulum whose support O is to be moved a distance D. Suppose at $t=0$, the pendulum is at rest : $\theta=\dot{\theta}=0$. We then seek a strategy for moving the support o in a prescribed fashion so that when o has moved the distance D, the pendulum again comes to rest. See Fig. 5.

The situation in Fig. 5 could be a model for a crane system desgned to move material from one point to another. The amount of material moved is unimportant since the design parameter $\omega_n=\sqrt{\frac{g}{l}}$ does not depend upon the mass of the pendulum.

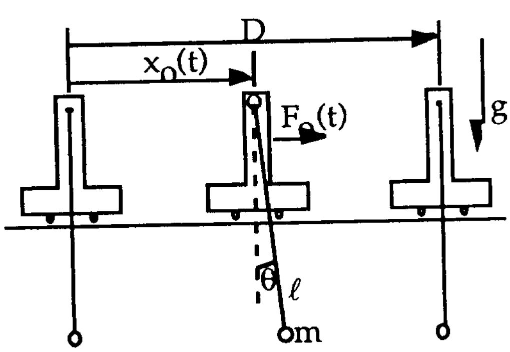

Fig. 5 The system : movement of a simple pendulum

We denote the prescribed motion of o by $x_o(t)$. Writing $F=ma$ in the direction perpendicular to the pendulum string, we have :

$$-mg\sin\theta=ml\ddot{\theta}+m\ddot{x}_o\cos\theta \qquad (22)$$

Suppose the angle θ remains small so that we can make the approximations :

$$\sin\theta\cong\theta \text{ and } \cos\theta\cong 1$$

Thus we have a differential equation for the pendulum :

$$\ddot{\theta}+\omega_n^2\theta=-\frac{\ddot{x}_o}{l} \qquad (23)$$

where $\omega_n=\sqrt{\frac{g}{l}}$

Suppose that we take the same displacement function that we chose for the motion of a mass in Fig. 2 :

$$x_o(t)=\frac{1}{2}D\left[1-\cos\left(\frac{\omega_n t}{\alpha}\right)\right],\quad 0\leq t\leq(\frac{\alpha\pi}{\omega_n})$$

$$=D \qquad t>(\frac{\alpha\pi}{\omega_n}) \qquad (23)$$

From (23), we need the second (time) derivative of x_o (t). Thus (23) becomes :

$$\ddot{\theta}+\omega_n^2\theta=-\left(\frac{D\omega_n^2}{2\alpha^2 l}\right)\cos\left(\frac{\omega_n t}{\alpha}\right) \qquad (24)$$

where $0\leq t\leq(\frac{\alpha\pi}{\omega_n})$

The initial conditions are :

$$\theta=\dot{\theta}=0 \qquad (25)$$

For a particular solution to (24), we try :

$$\theta_p=Q\cos\left(\frac{\omega_n t}{\alpha}\right) \qquad (26)$$

Inserting this in (24) and canceling the common cosine terms, we get :

$$\left(1-\frac{1}{\alpha^2}\right)Q\omega_n^2=-\frac{D\omega_n^2}{2\alpha^2 l}$$

or

$$Q=-\frac{D}{2l}\left(\frac{1}{\alpha^2-1}\right) \qquad (27)$$

Adding the particular solution to the homogeneous solution, we get :

$$\theta(t)=-\frac{D}{2l}\left(\frac{1}{\alpha^2-1}\right)\cos\left(\frac{\omega_n t}{\alpha}\right)$$
$$+A\sin(\omega_n t)+B\cos(\omega_n t) \quad (28)$$

Imposing the conditions (25), we determine :

$$A=0,\ B=\frac{D}{2l}\left(\frac{1}{\alpha^2-1}\right) \quad (29)$$

Finally, we have the motion $\theta(t)$:

$$\theta(t)=\frac{D}{2l}\left(\frac{1}{\alpha^2-1}\right)[\cos(\omega_n t)$$
$$-\cos\left(\frac{\omega_n}{\alpha}t\right)],\quad 0\le t\le t_f$$
$$=0,\quad t>t_f \quad (30)$$

At this point, we must determine the values of α which give

$$\theta(t_f)=\dot{\theta}(t_f)=0 \quad (31)$$

where $t_f=\left(\frac{\alpha\pi}{\omega_n}\right)$

Note that $\omega_n t_f=\alpha\pi$ and $\frac{\omega_n t_f}{\alpha}=\pi$. Thus from (30), we obtain

$$\theta(t_f)=\frac{D}{2l}\left(\frac{1}{\alpha^2-1}\right)[\cos(\alpha\pi)+1]$$

In order that $\theta(t_f)=0$, we set $\cos(\alpha\pi)=-1$. Thus :

$\alpha=1, 3, 5, 7, \cdots$

DIfferentiating (24) and evaluating at t_f, we obtain :

$$\dot{\theta}(t_f)=\frac{D}{2l}\left(\frac{\omega_n}{\alpha^2-1}\right)\left[\frac{1}{\alpha}\sin\left(\frac{\omega_n t_f}{\alpha}\right)-\sin(w_n t_f)\right]$$

Noting again that $\omega_n t_f=\alpha\pi$ and $\frac{\omega_n t_f}{\alpha}=\pi$, if $\alpha=1$, 2, 3, ··· the sine terms inside the brackets vanish. Thus the values of θ and $\dot{\theta}$ are zero at $t=t_f$ as desired. Once again, we note that $\alpha=1$ makes the denominator in (30) vanish. Thus acceptable values of α are 3, 5, 7, ···, $(2n+1)$.

The prescribed motion $x_o(t)=0$ and the response $\theta(t)$ are shown in Fig. 6. In order to fully understand the results shown in Fig. 6, we must compute the maximum amplitude of the response, $\theta_{\max}$.

If α is equal to 3, it can be determined that $|\theta_{\max}|$ occurs at $t=0.304\ t_f$ and $0.696\ t_f$. Plugging either value in (30), we obtain :

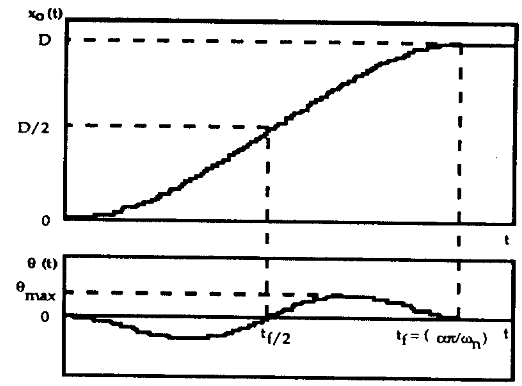

Fig. 6 The motion $x_o(t)$ and the pendulum motion $\theta(t)$

$$\theta_{\max}=0.096\frac{D}{l} \quad (32)$$

and again, the final time is :

$$t_f=\frac{3\pi}{\omega_n}\ (\alpha=3) \quad (33)$$

It is important to note that while $\alpha=3$ gives the minimum time to move the system a distance D as well as the smoothest motion, the price which is paid is that the maximum angle θ occurs at this value of α. If we select $\alpha=5, 7, \cdots, (2n+1)$ the maximum angle $\theta_{\max}$ will be reduced, but the time t_f will be extended and the motion will involve higher harmonics not seen in the case $\alpha=3$.

Suppose that we generalize the discussion by considering a compound pendulum instead of a simple pendulum. A compound pendulum is a rigid body which oscillates about a fixed horizontal axis through the body as Fig. 7. We will find that the fundamental design equations are essentially the same as above. In addition, we consider the force $F_o(t)$ required to generate the motion.

From Fig. 7 we can determine that the two equations of motion for the system(assuming small angular motions $\theta(t)$) :

$$(m_o+m)\ddot{x}_o+ml\ddot{\theta}=F_o(t) \quad (35)$$

Rewriting (34), we get :

$$\ddot{\theta}+\omega_n^2\theta=-\left(\frac{ml}{J_o}\right)\ddot{x}_o \quad (36)$$

where $\omega_n=\sqrt{\frac{mgl}{J_o}}$

J_o : mass moment of inertia of the body

l : distance from O to G(mass center)

As before, we use $x(t)$ from (24). Thus (36) becomes :

$$\ddot{\theta}+\omega_n^2\theta=-\left(\frac{D\omega_n^2 ml}{2\alpha^2 J_o}\right)\cos\left(\frac{\omega_n t}{\alpha}\right) \tag{37}$$

Comparing this to (24) shows that we can replace $\left(\frac{D}{2l}\right)$ in (24) and (27) by $\left(\frac{Dml}{2J_o}\right)$ to get the results for the present case.

$$\theta(t)=\frac{Dml}{2J_o}\left(\frac{1}{\alpha^2-1}\right)\left[\cos(\omega_n t) - \cos\left(\frac{\omega_n t}{\alpha}\right)\right],\quad 0\le t\le t_f$$
$$=0,\quad t>t_f \tag{38}$$

Similarly, we can determine θ_{max}(here for $\alpha=3$) :

$$\theta_{max}=0.096\frac{Dml}{J_o},\ \text{(rad.)} \tag{39}$$

where $\alpha=3$

and again, the final time is :

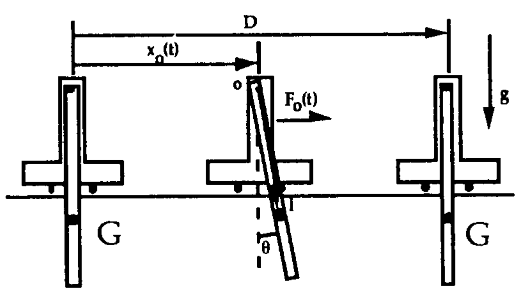

Fig. 7 The system : movement of a compound pendulum

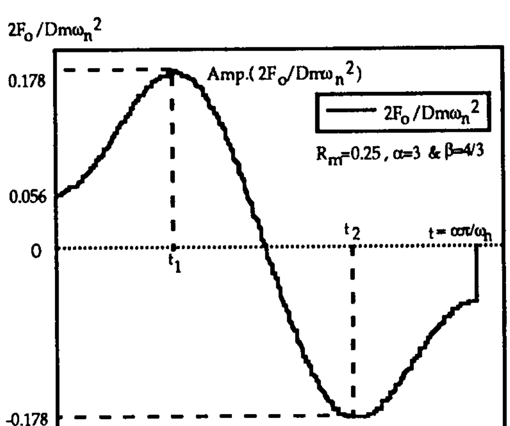

Fig. 8 The force required to initiate the motion $x_o(t)$ for the movement of a pendulum movement

$$t_f=\left(\frac{\alpha\pi}{\omega_n}\right) \tag{40}$$

where $\alpha=3, 5, 7, \cdots, (2m+1)$

To determine the force required to give the motion (38), we note the equation (35). Inserting $x(t)$ from (3) and $\theta(t)$ from (38) to (35), we get :

$$\frac{2F_o}{Dm\omega_n^2}=-\frac{1}{\beta(\alpha^2-1)}\cos(\omega_n t) +\frac{1}{\alpha^2}\cos\left(\frac{w_n t}{\alpha}\right)\left[1+R_m+\frac{1}{\beta(\alpha^2-1)}\right] \tag{41}$$

where $R_m=\frac{m_o}{m}$ and $\beta=\frac{J_o}{ml^2}$

Notice that (41) is a function of both the mass ratio R_m and the moment of inertia ratio β. In the case of the simple pendulum, $J_o=ml^2$. Thus $\beta=1$. If we take $R_m=0.2$ and $\alpha=3$, the plot of (41) is that of Fig. 8.

The minimum value of β is 1.0 (for a simple pendulum of length l). Thus $R_m\ge0$ and $\beta\ge1$. From (41), the initial and final values of $(\frac{2F_o}{Dm\omega_n^2})$ are :

Initial and Final

$$\frac{2F_o}{Dm\omega_n^2}=\frac{1}{\alpha^2}\left[-\frac{1}{\beta}+1+R_m\right] \tag{42}$$

Clearly if $\beta\ge1$ and $R_m\ge0$, this quantity is greater than or equal to zero.

4. Conclusions

Simply stated, a mass can be moved from point A to point B without inducing residual vibration if we use a $[1-\cos(\omega t)]$ specified motion where ω is one third (or one fifth $\cdots$) of the natural frequency ω_n of the system which is being moved. Clearly, the motion shape $[1-\cos(\omega t)]$ term must be applied during the time interval from $t=0$ to $t=\frac{\alpha\pi}{\omega_n}$. Taking $\alpha=3$ gives the smoothest transition from A to B in the minimum time. In the case of the pendulum, the cost of using $\alpha=3$ is that the amplitude of the pendulum is highest at that value of α.

A number of questions remain unanswered at this point. For example we have considered only undamped single degree of freedom systems here.

Future research will determine whether the ideas here can be expanded to include the systems with damping or the systems with several degrees of freedom.[5] If there is damping in a single degree of freedom system, we will not be able to bring $\dot{x}$ to zero at the end of the cycle with the open loop procedure outlined here. However, the procedure given here could be used in conjunction with a mechanical capture system or a closed loop control to achieve the desired goal.

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