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조화함수법을 이용한 주기 비선형 시스템의 Chaos 해석

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Chaos Analysis for the Periodic Nonlinear System Using Harmonic Balance Method

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ABSTRACT

주기함수의 외력을 갖는 비선형 시스템의 다양한 응답 특성을 구하기 위해 새로운 조화함수법(HBM)을 적용하였다. 새로운 조화함수법의 해는 비선형항을 선형항으로부터 따로 분리시킨 다음 같은 주파수 성분을 갖는 비선형 방정식들을 Newton-Raphson법으로 풀어서 구하였다. 다양한 천이(Bifurcation) 특성을 해석적으로 판별하기 위하여 HBM의 해를 이용하여 구한 섭동 방정식의 Floquet 지수의 고유해를 사용하였다. 새로이 개발한 HBM과 천이 판별법을 1차원 비선형항을 갖는 구조물인 ALP(Articulated Loading Platform)모델과 다차원인 비선형 회전체 모델에 적용시켜 HBM의 해의 정확성과 이들 시스템의 천이 특성의 하나인 Chaos 존재를 확인 하였다.

Key Words : Bifurcation, Chaos, HBM(Harmonic Balance Method), Poincare mapping, Poincare point, Floquet multipliers, AFT(Alternating Frequency / Time)

1. Introduction

Modern mechanical systems are being recently designed for higher performance, reliability and smooth operation within compact configuration. These requirements often cause significant and various nonlinear effects which could not be predicted by linear models. In general, the following classifications could describe relevant nonlinear characteristics in many applications in mechanical systems : i) piecewise-linear nonlinearity (e.g. dead-space in gears) ii) piecewise-smooth nonlinearity (e.g. offshore articulated loading form structure), iii) polynomial type nonlinearity (e.g Duffing type stiffness). The equation of motion describing the behavior of such systems includes abrupt variations of stiffness and damping coefficients. The equation of motion is catagorized as an ordinary, second order differential equation with periodically switching boundary conditions in a mathemat-

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ical sense, and from a physical point of view a nonconservative, nonautonomous strongly nonlinear system. Due to the strong nonlinearity of the equations describing the motion of such systems, their dynamic response is characterized by possible multiple solutions, frequency demultiplication, initial condition dependence, chaotic response. General analytical methods do not exist for obtaining steady-state solutions for such system.

A common procedure for strong nonlinear vibration problem is the direct numerical integration. The disadvantages of this approach are : i) it would be too expensive and tedious to completely characterize the dynamic behavior of the system. ii) all possible solutions could be found by changing the initial conditions if a multiple valued response exists. iii) the time increment would have to be chosen carefully to guarantee the convergence of all solutions. An analytical approach to finding the steady-state response of forced piecewise-linear oscillators was proposed by Masri[1] in which boundary values at the contact points are matched and the resulting nonlinear algebraic equations are then solved using an iterative method. Maezawa[2][3] used a Fourier expansion to represent the unknown contact forces as external forces. The forces are determined through enforcing periodicity and therefore also obtaining the solutions sought. Unfortunately, such a solution method based on assumed displacement and contact patterns often suffers restrictions.

Unpredicted types of vibration, not included in the assumed response shapes, may be excluded. Furthermore, it is almost impossible to find the exact duration of contact for some parameter ranges because of the severly nonlinear nature of the algebraic relations involving the duration of contact.

An Incremental Harmonic Balance (IHB) method which accommodate multiple harmonic components in the description of the nonlinear forces was introduced by Lau et al. in 1982.[4][5] The method was originally applied to continuous (cubic) nonlinearities by Lau[6], but pierre et al.[7] extended the method to analyze the rigid/perfectly-plastic friction damper problem in which the nonlinear force is discontinuous. The IHB method appears to handle strong nonlinearities as well, and provides accurate solutions over a much broader range than the one term harmonic balance method since it accommodates multiple harmonics. However, disadvantage of the IHB method is the amount of analytical work required. Functions in the equation should be expanded in a first Taylor series (which may be difficult to obtain for discontinuous functions) and substituted into the original equation to obtain linear "incrementalized" equation.[8]

The present paper aims at providing a versatile solution and bifurcation analysis for steady-state response of general piecewise-smooth type nonlinear systems by employing explicit Jacobian formulation associated with application of Galerkin technique. First, the paper consists of explaining the new harmonic balance method using AFT (Alternating Frequency / Time) technique and the bifurcation determination. Conclusion will be made by showing two examples. One is the offshore articulated loading platform (ALP) system in one degree of freedom with piecewise-smooth nonlinearity. Another one is the MDOF rotor model consists of disks and a nonlinear bearing.

2. Harmonic Balance Method (HBM)

Consider a real periodic differential system

$$\frac{dx}{dt} = X(x,t) \tag{1}$$

where x and $X(x, t)$ are vectors of the same dimension and $X(x, t)$ is periodic in t with period T. To obtain the periodic solution of equation (1), trigonometric polynomial can be used as

$$x_m(t) = a_o + \sum_{n=1}^{m}(a_n \cos nt + b_n \sin nt) \tag{2}$$

with undetermined coefficients $(a_o,\ldots,a_n,b_n)$ where $a_o,\ldots,a_n,b_n$ are vectors of the same dimension as x and $X(x, t)$. Next, substituting (2) into (1), the following equations can be easily obtained.

$$\begin{aligned} F_o(\alpha) &= \frac{1}{T}\int_o^T X[x_m(s), s]ds = 0 \\ F_{2n}(\alpha) &= \frac{2}{T}\int_o^T X[x_m(s), s]\cos ns\, ds - nb_n = 0 \\ F_{2n+1}(\alpha) &= \frac{2}{T}\int_o^T X[x_m(s), s]\sin ns\, ds + na_n = 0 \end{aligned} \tag{3}$$

Where $\alpha = [a_o,\ldots,a_m,b_o,\ldots,b_m]^T$. Suppose (3) has a real solution $\bar{\alpha} = [\bar{a}_o,\ldots,\bar{a}_m,\bar{b}_o,\ldots,\bar{b}_m]^T$. Then the trigonometric polynomial

$$\bar{x}_m(t) = \bar{a}_o + \sum_{n=1}^{m}(\bar{a}_n \cos nt + \bar{b}_n \sin nt) \tag{4}$$

is obtained from $x_m(t)$ replacing α by $\bar{\alpha}$ Equation (4) will be considered as an approximate solution of (1). The trigonometric polynomial (4) and equation (3) are respectively called the Galerkin approximation. The Galerkin approximation is obtained by solving the determining equation (3). However, the left member of the equation is usually nonlinear. Consequently, at first glance, it seems very complicated to solve the determining equation. For that purpose, one can consider soving equation (3) in two cases : i) if the nonlinear vector functions have simple forms (e.g. cubic nonlinearities), equation (3) can be used directly to obtain the periodic steady-state solutions; and ii) if the nonlinear vector functions have more complicated forms (e.g. piecewise-smooth nonlinearities) the nonlinear part can be decoupled from the linear part in order to apply special treatment in discrete time domain (AFT method). The determining equation (3) can usually be solved by iterative methods (i.e. Newton-Raphson) as

$$F_k(\alpha)^{(i)} = 0(k = 0,1,\ldots,2n+1) \tag{5}$$

where superscript i denotes the ith iteration.

Newton- Raphson Method

The solution procedure of determining α in equation (3) involves the integration of the nonlinear function, X, with one period. In general the nonlinear function X has complicated expressions with x (e.g. discontinuous or square forms), therefore, the explicit expressions of Fs in terms of α are not always easy. To avoid the difficulties, the solution procedure of determining the periodic solutions of equation (3) can be performed numerically with a discrete time domain. The discrete time domain approach is much more efficient than a frequency domain approach used in a conventional harmonic balance method or IHB(Incremental Harmonic Balance).[9][10] Generally the Fourier coefficients in the discrete time domain can be evaluated approximately by the expansions

$$\begin{aligned} a_o &= \frac{1}{2N}\sum_{i=1}^{2N} f(t_i) \\ a_n &= \frac{1}{N}\sum_{i=1}^{2N} f(t_i)\cos nt_i \\ b_n &= \frac{1}{N}\sum_{i=1}^{2N} f(t_i)\sin nt_i \end{aligned} \tag{6}$$

where $(n\text{-}1, 2, \ldots, N\text{-}1)$ and $t_i = \frac{2_i - 1}{2N}\pi$, $(i = 1, 2, \ldots, 2N)$. one solves the determining equations (3) using Newton-Rapson method. Using (6), the determining equation can be rewritten in discrete time domain as follows :

$$F_o(\alpha) = \frac{1}{2N}\sum_{i=1}^{2N} X[x_m(t_i),t_i] = 0$$
$$F_{2n}(\alpha) = \frac{1}{N}\sum_{i=1}^{2N} X[x_m(t_i),t_i]\cos nt_i - nb_n = 0 \quad (7)$$
$$F_{2n+1}(\alpha) = \frac{1}{N}\sum_{i=1}^{2N} X[x_m(t_i),t_i]\sin nt_i - na_n = 0$$

where $(n = 1,2,\ldots,m)$, $\alpha = [a_0,\ldots,a_n,b_n]T$ and

$$x_m(t_i) = a_o + \sum_{i=1}^{m}(a_n \cos nt_i + b_n \sin nt_i).$$

The elements of the jacobian matrix of the left member of (7) are given by

$$J_{0,0}(\alpha) = \frac{1}{2N}\sum_{i=1}^{2N}\Psi[X_m(t_i,t_i)]$$
$$J_{0,2p}(\alpha) = \frac{1}{2N}\sum_{i=1}^{2N}\Psi[X_m(t_i,t_i)]\cos pt_i$$
$$J_{0,2p+1}(\alpha) = \frac{1}{2N}\sum_{i=1}^{2N}\Psi[X_m(t_i,t_i)]\sin pt_i$$
$$J_{2n,0}(\alpha) = \frac{1}{N}\sum_{i=1}^{2N}\Psi[X_m(t_i,t_i)]\cos nt_i$$
$$J_{2n,2p}(\alpha) = \frac{1}{N}\sum_{i=1}^{2N}\Psi[X_m(t_i,t_i)]\cos nt_i \cos pt_i$$
$$J_{2n,2p+1}(\alpha) = \frac{1}{N}\sum_{i=1}^{2N}\Psi[X_m(t_i,t_i)]\cos nt_i \sin pt_i - n\delta_{np} \quad (8)$$
$$J_{2n+1,0}(\alpha) = \frac{1}{N}\sum_{i=1}^{2N}\Psi[X_m(t_i,t_i)]\sin nt_i$$
$$J_{2n+1,2p}(\alpha) = \frac{1}{N}\sum_{i=1}^{2N}\Psi[X_m(t_i,t_i)]\sin nt_i \cos pt_i + n\delta_{np}$$
$$J_{2n+1,2p+1}(\alpha) = \frac{1}{N}\sum_{i=1}^{2N}\Psi[X_m(t_i,t_i)]\sin nt_i \sin pt_i$$

where $n, p = 1,2,\ldots,m$, $\Psi(x,t)$ is the Jacobian matrix of $X(x,t)$ with respect to x and δ_{np} is unity if $n=p$, otherwise zero. Newton-Raphson method has the following form

$$\sum_{\nu=0}^{2n} J_{\mu\nu}(\alpha)h_\nu + F_\mu(\alpha) = 0, (\mu = 0,1,\ldots,2m) \quad (9)$$

The unknowns in the above equation, h_ν and the values of $F_\mu(\alpha)$, $J_{\mu\nu}(\alpha)$ $(\mu,\nu = 0,1,\ldots,2m)$ can be determined when the values of α are given.

This iterative procedure for solving the nonlinear algebraic equation can be summed up as follows :

(1) Estimate the initial value of α_0 in equation (a).

(2) Calculate the discrete data point of $t_i = \frac{2_i - 1}{N}\pi$

(3) Calculate the discrete value for the nonlinear function of $F_\mu(\alpha)$.

(4) Repeat the following steps with $k = 0,1,2,\ldots$

i) Calculate [J].

ii) Solve equation (a) to decide the correctors of $\Delta\alpha$.

iii) End the iteration if $\Delta\alpha$ is small enough; otherwise set $\alpha_{k+1} = \alpha_k + \Delta\alpha_k$ and iterate again from step i.

The AFT method

Since the nonlinear equations (8) and (9) are frequently very complicated (e.g. involving high order polynomial terms or discontinuous functions), the solution procedure can be very involved. An AFT (Alternating Frequency Time) methods is an effective approach to remedy this complexity. Generally equation (1) can be written as[11].

$$\frac{dx}{dt} = X_L(x,t) + X_N(x,t) \quad (10)$$

Where $X_L(x,t)$ and $X_N(x,t)$ represent the linear and nonlinear parts of $X(x,t)$ respectively. Then the nonlinear part $X_N(x,t)$ can be expressed

$$X_N(x,t) = c_0 + \sum_{n=1}^{m}(c_n \cos nt + d_n \sin nt) \quad (11)$$

The AFT method can be applied to obtain $\beta = [c_0, \ldots, c_m, d_m]^T$ from $\alpha = [a_0, \ldots, a_m, b_m]^T$ using DFT (Discrete Fourier Transform) and IDFT (Inverse Discrete Fourier Transform) procedures. The determining equation along with an AFT in discrete time domain can be defined as following :

$$F_0(\alpha) = \frac{1}{2M}\sum_{i=1}^{2N} X_L[x_m(t_i), t_i] + c_0 = 0 \quad (12)$$

$$F_{2n}(\alpha) = \frac{1}{M}\sum_{i=1}^{2N} X_L[x_m(t_i), t_i]\cos nt_i + c_n - nb_n = 0$$

$$F_{2n+1}(\alpha) = \frac{1}{M}\sum_{i=1}^{2N} X_L[x_m(t_i), t_i]\sin nt_i + d_n - na_n = 0$$

The coefficients of the approximated Fourier expansion of $X[x_m(t_i), t_i]$ can be written, using equation (6) as

$$c_0 = \frac{1}{2M}\sum_{i=1}^{2N} X_N[x_m(t_i), t_i]$$

$$c_n = \frac{1}{M}\sum_{i=1}^{2N} X_N[x_m(t_i), t_i]\cos nt_i \quad (13)$$

$$d_n = \frac{1}{M}\sum_{i=1}^{2N} X_N[x_m(t_i), t_i]\sin nt_i$$

Where, $t_i = \frac{2_i - 1}{2M}\pi (i = 1, \ldots, 2M)$

Equation (12) is much easier to solve than equation (7) since it decouples the nonlinear terms from the linear terms and the AFT easily yields the β's from α's. The elements of the Jacobian matrix of the left member of (12) are given by

$$J_{0,0}(\alpha) = \frac{1}{2N}\sum_{i=1}^{2N} \Phi_L[x_m(t_i), t_i] + \frac{\partial c_0}{\partial a_n}$$

$$J_{0,2p}(\alpha) = \frac{1}{2N}\sum_{i=1}^{2N} \Phi_L[x_m(t_i), t_i]\cos pt_i + \frac{\partial c_0}{\partial a_n}$$

$$J_{0,2p}(\alpha) = \frac{1}{2N}\sum_{i=1}^{2N} \Phi_L[x_m(t_i), t_i]\sin pt_i + \frac{\partial c_0}{\partial b_n}$$

$$J_{2n,0}(\alpha) = \frac{1}{N}\sum_{i=1}^{2N} \Phi_L[x_m(t_i), t_i]\cos pt_i + \frac{\partial c_n}{\partial a_0}$$

$$J_{2n,2p}(\alpha) = \frac{1}{N}\sum_{i=1}^{2N} \Phi_L[x_m(t_i), t_i]\cos nt_i \cos pt_i + \frac{\partial c_n}{\partial a_n} \quad (14)$$

$$J_{2n,2p+1}(\alpha) = \frac{1}{N}\sum_{i=1}^{2N} \Phi_L[x_m(t_i), t_i]\cos nt_i \sin pt_i + \frac{\partial c_n}{\partial b_n} - n\delta_{np}$$

$$J_{2n+1,0}(\alpha) = \frac{1}{N}\sum_{i=1}^{2N} \Phi_L[x_m(t_i), t_i]\sin nt_i + \frac{\partial d_n}{\partial a_0}$$

$$J_{2n+1,2p}(\alpha) = \frac{1}{N}\sum_{i=1}^{2N} \Phi_L[x_m(t_i), t_i]\sin nt_i \cos pt_i + \frac{\partial d_n}{\partial a_n} + n\delta_{np}$$

$$J_{2n+1,2p+1}(\alpha) = \frac{1}{N}\sum_{i=1}^{2N} \Phi_L[x_m(t_i), t_i]\sin nt_i \sin pt_i + \frac{\partial d_n}{\partial b_n}$$

where $\Phi_L(x, t)$ is the Jacobian matrix of $X_L(x, t)$ with respect to x, and δ is the Cronecker delta.

3. Stability and Bifurcation Analysis

Let γ be a periodic orbit of some flow, ϕ, in R^n arising from a nonlinear vector field $f(x)$. One first take a cross section $\Sigma \subset R^n$, of dimension n-1.

Denote the point where γ intersects Σ by p(Poincare point in general), and let $U \subseteq \Sigma$ be some neighborhood of p. Then the first return or Poincare mapping $f: U \rightarrow \Sigma$ is defined for a point $q \subseteq U$ by (see Guckenheimer[12])

$$f(q) = \phi_T(q) \quad (15)$$

where T is the time taken for the orbit ϕ (q) based at q to the first return to Σ. Generally T is the period of γ. Clearly p is a fixed point of the mapping f, and it is not difficult to see that the stability of p for f reflects the stability of γ for the flow ϕ. The examination of the stability of the fixed point, x_p, of the poincare mapping f(i.e. $x_p=f(x_p)$) is performed by disturbing x_p by a small perturbation ξ, or setting

$$x = x_p + \xi \tag{16}$$

If $\bar{x}(t)=\bar{x}(t+T)$ is a solution lying on the closed orbit γ, based at $x(0)p\in\Sigma$, then after linearizing the differential equation about γ, one obtains the system

$$\dot{\gamma} = Df(\bar{x}(t))\zeta \tag{17}$$

where $Df(\bar{x}(t))$ is an n×n, T-periodic matrix. It can be shown that any fundamental solution matrix of such a T-periodic system can be written in the form[13]

$$X(t) = Z(t)e^{tR}, \quad Z(t) = Z(t+T) \tag{18}$$

where X, Z and R are n×n matrices. In particular, one can chose $X(0)=Z(0)=I$, so that

$$X(T) = Z(T)e^{TR} = Z(0)e^{TR} = e^{TR} \tag{19}$$

It then follows that the behavior of solutions in the neighborhood of γ is determined by the eigenvalues of a constant matrix e^{TR}. These eigenvalues, $\lambda_1,\ldots,\lambda_n$ are called the characteristic(Floquet) multipliers or roots, and the eigenvalues $\mu_1,\ldots,\mu_n$ of R are the characteristic exponents of the closed orbit γ. The multipliers of the remaining n-1 determine the stability γ. If all the multipliers are inside of the unit disk, the periodic solution is asymptotically stable. If at least one multipliers is outside the unit disk, the periodic solution is unstable. If one of the multipliers is at 1, and all the other multipliers are inside the unit disk, the bifurcation equation (17) can have a possible bifurcation point. Depending on the nonlinear terms. one of three type of bifurcations (flip, fold, Hopf) to a stable periodic solution may occur[14].

4. Applications

1) SDOF system with piecewise-smooth nonlinearity

The nonlinear restoring moments applied to the ALP is expanded into a power series only retaining the first two terms and using an equivalent linear damping. The ALP system is then represented by the following equation[15] (refer to Fig. 1)

$$I_v\ddot{v} + C\dot{v} + g(v) = M_0 \cos\omega t \tag{20}$$

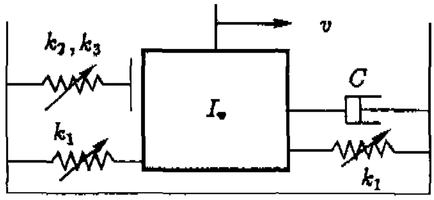

Fig. 1-a An ALP model

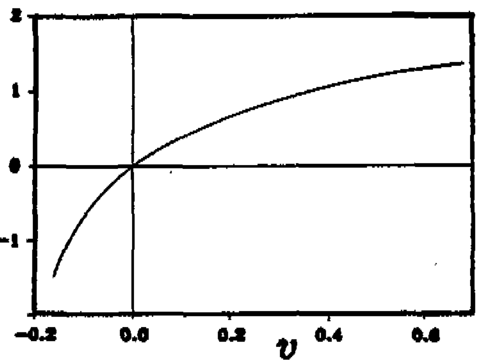

Fig. 1-b Nonlinear restoring force

where

$$g(v) = \begin{cases} k_1(v - v^3/6), & if\ v \ge 0 \\ k_2 v + k_3 v^3, & otherwise \end{cases} \tag{21}$$

ν is the angular displacement, I_v is the actual and virtual moment of inertia of the tower, C is the linearized damping coefficient, and M_0 and ω are the amplitude and frequency of the wave excitation, respectively. By introducing the following nondimensional parameters:bilinear stiffness ratio $\alpha_1 = \frac{k_2}{k_1}$; nonlinear stiffness ratio $\alpha_2 = \frac{k_3}{k_e}$; frequency ratio $\Omega = \frac{\omega}{\omega_b}$; static displacement $M_s = M_0/K_e$; and time scale ν $\tau = \omega t$:equivalent stiffness $K_e = \frac{4k_1k_2}{(\sqrt{k_1}+\sqrt{k_2})^2}$; circular frequency $\omega_b = \sqrt{\frac{K_e}{I_v}}$; equation (21) can be cast into the following nondimensional form,

$$\nu'' + 2\zeta\frac{\nu}{\Omega}\nu' g^*(\nu) = \frac{M_s\nu^2}{\Omega^2}\cos\nu\tau \qquad (22)$$

where

$$g^*(\nu) = \begin{cases} \dfrac{\nu^2(1+\sqrt{\alpha_1})^2}{4\Omega^2\alpha_1}(\nu - \nu^3/6), & if\ \nu \geq 0 \\ \dfrac{\nu^2(1+\sqrt{\alpha_1})^2}{4\Omega^2\alpha_1}\nu + \dfrac{\nu^2\alpha^2}{\Omega^2}\nu^3, & otherwise \end{cases} \qquad (23)$$

The accuracy of the periodic solutions of the SDOF system with piecewise-smooth nonlinearity as obtained using HBM (with truncat-

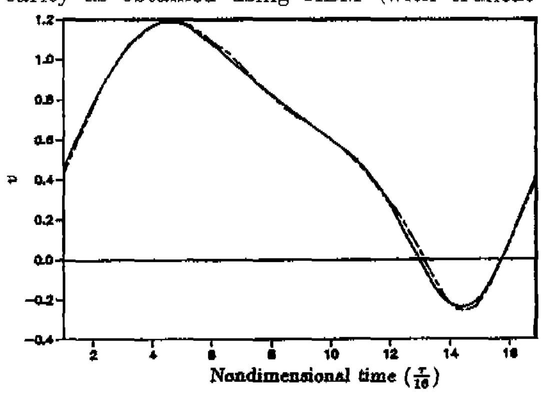

Fig. 2 Comparison between HBM and numerical integration
ζ=0.1, α_1=10, α_2=100, M_s=0.43, Ω=2
(— HBM - - numerical integration)

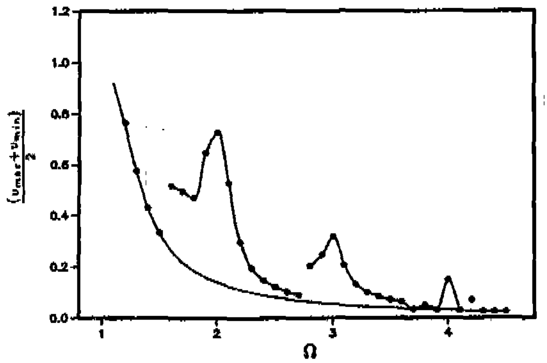

Fig. 3 Comparison between HBM and numerical integration
ζ=0.1, α_1=10, α_2=100, Ms=0.43
(— HBM ○ numerical integration)

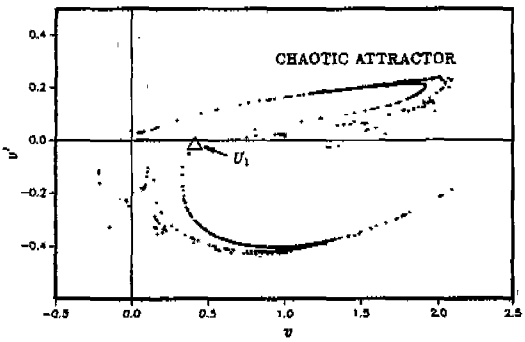

Fig. 4 Poincare map for Ω=1.8615

ing order m = 4) is checked against the solutions using numerical integration. The result is shown in Fig. 2, where good agreement between HBM and numerical integration can be seen. Shown in Fig. 3 are the results of the HBM and numerical integration for the average displacements as the function of the dimensionless frequency Ω. The figure reveals the high accuracy of HBM for broad range of Ω. Using the stability criteria explained in the paper, flip bifurcation was checked and the route to chaos was found in the model. Flip bifurcation is the process of period doubling, and it can lead to chaotic attractor. Shown in Fig. 4 is the chaotic attractor with

Table 1. Rotor configuration and physical parameters

	item	value
E	Young's modulus	2.07×10^{11} N/m^2
m_1	Mass of first disk	1.18 kg
m_2	Mass of second disk	5.76kg
O.D.	Shaft Outer Diameter	0.054m
I.D.	Shaft Inner Diameter	0.019m
L_1	1st shaft length	0.091m
L_2	2nd shaft length	0.116m
L_3	3rd shaft length	0.1m
K_B	Nonlinear spring stiffness	7.7×10^{7} N/m
J_{d1}	1st disk diametrical moment of inerta	0.056kg-m^2
J_{d2}	2nd disk diametrical moment of inertia	0.114kg-m^2

Ω = 1.8615. In the figure the dots represent Poincare points. and the marker U_1 denotes the unstable saddle point.

2) Nonlinear multi-degree-of-freedom rotor system

A typical multi-disk rotor system with nonlinear bearing is considered as shown in Fig. 5. The rotor is supported on a linear bearing at the left and by a nonlinear bearing with a gap at the middle. The detailed rotor configuration is shown in Table 1. Using a finite element formulation, the system equations of motion can be expressed in terms of the assembled mass, damping and stiffness matrices $[M]$, $[C]$ and $[K]$ as

$$[M]\ddot{q} + [C]\dot{q} + [K]q = f_u + f_n \tag{24}$$

where $q = [y, z]^T$ and y, z denote the $2L \times 1$ assembled state vectors in the X-Y and X-Z planes, where L is the total number of nodes. The vector f_u represents the vector of the disk's mass imbalances and side forces on the disk, while the f_n vector denotes the force vector at the nonlinear bearings. In particular, the nonlinear restoring forces in the y, z - directions on the bearing with clearances, at the node j, are expressed as

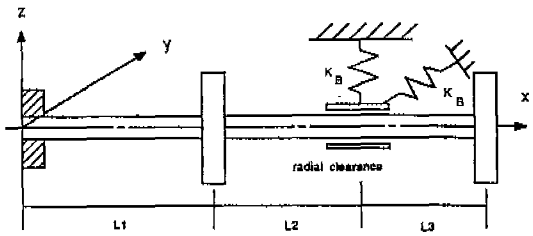

Fig. 5 MDOF rotor model with a piecewise-linear bearing clarance

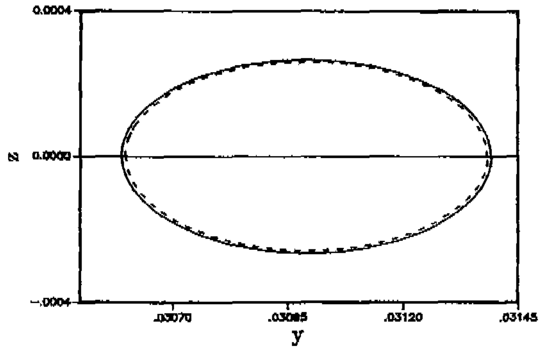

Fig. 6 Comparison between HBM and numerical integration
speed=2000 rad/s. gap=3 mm, side force=1400 N
(— HBM - - numercal integration)

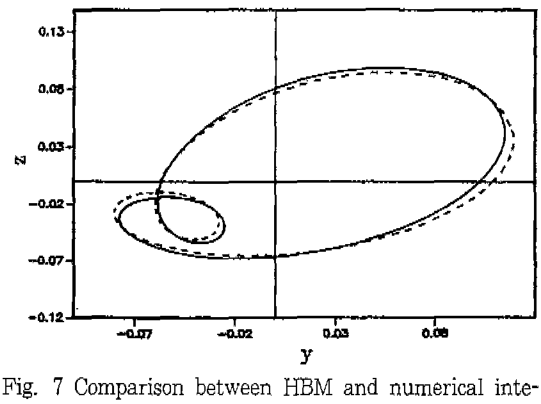

Fig. 7 Comparison between HBM and numerical integration
speed=1600 rad/s. gap=3 mm, side force=1400 N
(—HBM - - numerical integration)

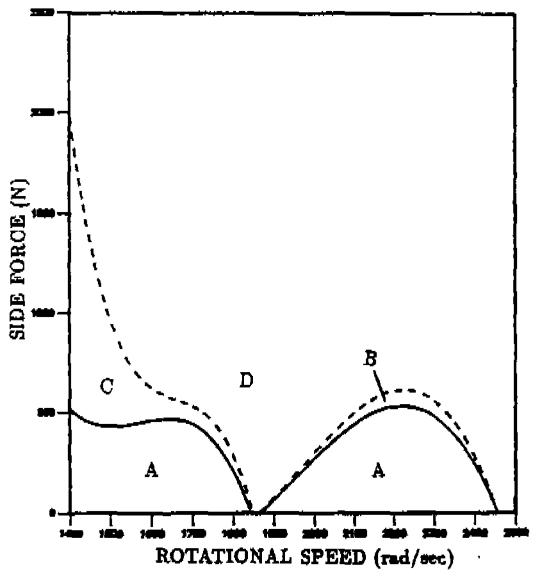

Fig. 8 Effect of side force and speed (flip bifurcation boundaries)

$$f_{yj} = k_b y_j \left(1 - \frac{\delta}{\sqrt{y_j^2 + z_j^2}}\right) \quad (25a)$$

$$f_{zj} = k_b z_j \left(1 - \frac{\delta}{\sqrt{y_j^2 + z_j^2}}\right) \quad (25b)$$

where k_b is the bearing stiffness and δ is the clearance between bearing and outer race and stator. The bearing forces f_{yj}, or f_{zj} will banish if δ is larger than the radial displacement, otherwise the bearing forces will be as given by equation (25). The whirling orbit at the nonlinear bearing is obtained for a rotor with 2000 rad/sec rotational speed, as shown in Fig. 6. The solid line in this figure represents the HBM solution (with truncation order m = 4) which is seen to be accurate. The minor discrepancies between numerical integration and the HBM solutions are due to truncation errors in assuming a finite number of harmonics for the steady-state solution and for the restoring nonlinear force. This error can be reduced by retaining larger number of harmonic terms. A more complicated subsynchronous whirling response at a rotor spinning speed of 1600 rad-sec is shown in Fig. 7. The solid line represents the response obtained by the HBM, while the dotted line stands for the solution of direct numerical integration. The results show good agreement between the HBM and the numerical integration methods. Another important characteristic of the behavior of the nonlinear rotor is the sudden change of its whirling shape with small changes of certain parameters. This is due to bifurcation since it can lead to a sudden change from synchronous to subsynchronous motion (including subharmonic or quasi-periodic motion) or vice versa. Fig. 8 shows effects of the rotational speed and side force

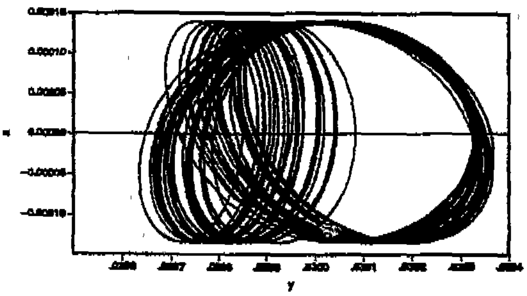

Fig. 9 Chaotic whirling response (side force=550 N)

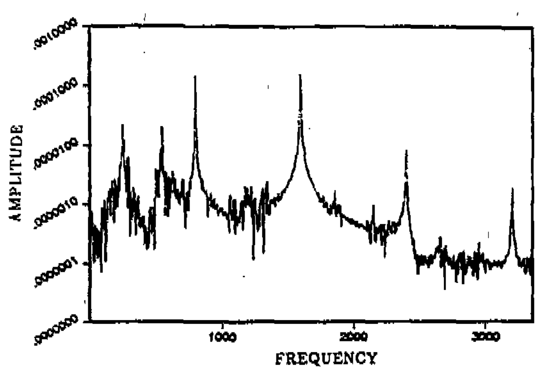

Fig. 10 Power spectrum of chaotic response (side force=550 N)

on the flip (period multiplying) bifurcation boundaries. Region A of this figure represents the stable whirling motion, region B is the subsynchronous whirling of order 1/4 which can cause violent whirling motion, region C is the subsynchronous whirling with order 1/3, and region D shows another synchronous whirling motion. In the regions B and C, one of the Floquet multiplier left unit circle through -1(which is a flip bifurcation case) and the motions can bifurcate further to reach chaotic whirling motion by increasing the side force variation. As the HBM can handle only finite number of low order subharmonic terms, numerical integration is used to study further flip bifurcation in the region C. With small increase of the side force beyond subsynchronous regions, whirling can become almost non-periodic (i.e. "chaotic") as shown in Fig. 9. Fig. 10 shows the response spectrum of that in Fig. 9, clearly indicates chaotic motion, since the spike due to synchronous whirling is 1600 rad/s (which is dominant in the figure), but other dominant spikes at other frequencies indicate chaotic whirling.

5. Conclusion

HBM with AFT method is developed as an approximated solution of periodic nonlinear systems. The method can be widely used to any types of periodic nonlinear systems. Poincare mapping is used to check the stability of the solution obtained by HBM. Floquet multipliers can tell not only the stability criteria but also the type of bifurcations (e.g. flip, fold, Hopf). Two examples, one is single degree of freedom ALP system with piecewise-smooth nonlinearity and the other one multi-degree of freedom rotor system with nonlinear bearing, are solved using HBM and investigated their bifurcation characteristics. Two examples both show chaotic responses through flip bifurcation.

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