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A Note on Admissibility and Finite Admissibility in Estimation

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Abstract

Consider the problem of estimating the parameter of the model in which an observable random variable is represented by a unknown scalar parameter plus another random variable and the parameter, sample, and decision spaces consist of all integers. We first characterize the class of all admissible estimators and then characterize the class of all finitely admissible estimators. Finally, we show that two classes are identical.

Keywords : Admissibility, Finite admissibility, Estimation, Loss function, Risk function

1. Introduction

The notion of finite admissibility was first introduced by Meeden and Ghosh (1982). The basic idea of this notion is to have admissibility on certain finite subset of the parameter space. The precise definition is as follows :

Definition 1.1 An estimator δ is said to be finitely admissible if for any parameter point $w_0 \in \Theta$, the parameter space, there exists a finite subset Θ_0 of Θ containing w_0 such that when Θ_0 is taken as a restricted parameter space, δ is admissible.

Using this idea together with the stepwise Bayes technique, Meeden and Ghosh (1982, 1983), Ghosh and Meeden (1983), and Vardeman and Meeden (1984) gave various admissibility results in finite population sampling. For the notion of stepwise Bayes procedure see Hsuan (1979) and for interesting applications of this technique see Meeden (1989) and Meeden, Ghosh, Srinivasan, and Vardeman (1989). Also, Kim (1987) studied the relationship between finite admissibility and stepwise Bayes procedure in some estimation problem.

This paper treats the problem of estimating a unknown scalar parameter $w \in \Theta =$

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$\{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$ using the loss function $L(w,d) = \ell(|w-d|)$ for $d \in \mathbb{D}$, the decision space, in a model

$$X = w + E \tag{1.1}$$

where the random variable E takes $+1$ and -1 with probability $\frac{1}{2}$, respectively, and $\ell(\cdot)$ is strictly increasing with $\ell(0) = 0$. Note that the random variable X takes values in the sample space $\mathfrak{X} = \Theta$. Assume that $\mathbb{D} = \Theta$ and $\ell(2) = 1$. Without loss of generality we can assume $\ell(2) = 1$ since when any loss function is multiplied by a constant, the admissibility or inadmissibility of an estimator is unchanged.

The purpose of this note is to illustrate the notions of admissibility and finite admissibility of estimators in the model (1.1).

In Section 2 we develop some necessary notation. In Section 3 we first characterize the class of all admissible estimators and the class of all finitely admissible estimators, and then show that two classes are identical.

2. Some Notation

Consider the model (1.1). Then the risk function of an estimator δ is given by

$$R(w,\delta) = \frac{1}{2}\{\ell(|w-\delta(w-1)|) + \ell(|w-\delta(w+1)|)\}. \tag{2.1}$$

For odd values of w the risk function depends on δ only through $\delta(x)$ for x even. Similarly, for even w, the risk function depends on δ only through $\delta(x)$ for x odd. So we restrict ourselves to even w's. We also just consider estimators of the form $\delta(X) = X-1$ or $X+1$. Note that $R(w,\delta)$ in (2.1) can be either 0, $\frac{1}{2}$, or 1 since for a fixed w $|w-\delta(x)| = 0$ or 2 for $x = w \pm 1$.

For each odd x, let's denote, for notational convenience, $x-1$ and $x+1$ by D and U respectively. Now, we can represent a typical estimator as follows :

$$
\begin{array}{ccccccccccccccc}
\cdots & & U & & U & & D & & U & & D & & D & & \cdots \\
\cdots & -6 & -5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\
\cdots & & & \frac{1}{2} & & 0 & & 1 & & 0 & & \frac{1}{2} & & & \cdots
\end{array}
$$

or

$$
\begin{array}{ccccccccccccc}
\cdots & U & & U & & D & & U & & D & & D & \cdots \\
\cdots & & \frac{1}{2} & & 0 & & 1 & & 0 & & \frac{1}{2} & & \cdots
\end{array}
$$

with the understanding that the risk at a particular even value of w is given below. Some examples of inadmissible estimators are given as follows :

$$
\begin{array}{ccccccccccccccccc}
\cdots & U & & U & & U & & U & & U & & U & & U & & U & \cdots \\
\cdots & & \frac{1}{2} & & \frac{1}{2} & & \frac{1}{2} & & \frac{1}{2} & & \frac{1}{2} & & \frac{1}{2} & & \frac{1}{2} & & \cdots
\end{array}
\tag{2.2}
$$

is dominated by

$$
\begin{array}{ccccccccccccccccc}
\cdots & U & & U & & U & & U & & D & & D & & D & & D & \cdots \\
\cdots & & \frac{1}{2} & & \frac{1}{2} & & \frac{1}{2} & & 0 & & \frac{1}{2} & & \frac{1}{2} & & \frac{1}{2} & & \cdots,
\end{array}
$$

$$
\begin{array}{ccccccccccccccccc}
\cdots & D & & D & & D & & D & & D & & D & & D & & D & \cdots \\
\cdots & & \frac{1}{2} & & \frac{1}{2} & & \frac{1}{2} & & \frac{1}{2} & & \frac{1}{2} & & \frac{1}{2} & & \frac{1}{2} & & \cdots
\end{array}
\tag{2.3}
$$

is dominated by

$$
\begin{array}{ccccccccccccccccc}
\cdots & U & & U & & U & & U & & D & & D & & D & & D & \cdots \\
\cdots & & \frac{1}{2} & & \frac{1}{2} & & \frac{1}{2} & & 0 & & \frac{1}{2} & & \frac{1}{2} & & \frac{1}{2} & & \cdots,
\end{array}
$$

$$
\begin{array}{ccccccccccccccccc}
\cdots & D & & D & & D & & D & & D & & * & & * & & * & \cdots \\
\cdots & & \frac{1}{2} & & \frac{1}{2} & & \frac{1}{2} & & \frac{1}{2} & & & & & & & & \cdots
\end{array}
\tag{2.4}
$$

is dominated by

$$\begin{array}{cccccccccc} \cdots\cdots & U & U & D & D & D & * & * & * & \cdots\cdots \\ \cdots\cdots & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & & & & & \cdots\cdots , \end{array}$$

$$\begin{array}{ccccccccc} * & * & * & U & U & U & U & U & \cdots\cdots \\ & & & & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots\cdots \end{array} \qquad (2.5)$$

is dominated by

$$\begin{array}{ccccccccc} * & * & * & U & U & D & D & D & \cdots\cdots \\ & & & & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & \cdots\cdots . \end{array}$$

3. Characterizing Admissible and Finitely Admissible Estimators

In the model (1.1), the following theorem provides a characterization of the class of all admissible estimators among all estimators of the form $\delta(X) = X - 1$ or $X + 1$.

Theorem 3.1 An estimator is inadmissible if and only if it is one of the following two types :

i) all U 's from some point on

or

ii) all D 's up to some point.

Proof The proof of sufficiency was already done in (2.2) ~ (2.5). It remains to prove necessity. Suppose that a given estimator δ is inadmissible. Then, there exists another estimator δ^* which dominates it, i.e.,

$$R(w, \delta^*) \le R(w, \delta) \quad \text{for all } w \text{ even}$$

and

$$R(w_0, \delta^*) < R(w_0, \delta) \quad \text{for some } w_0 \text{ even} .$$

Now looking at w_o one of the following must happen :

$$U \underset{\frac{1}{2}}{\quad\quad} U \text{ is dominated by } U^* \underset{0}{\quad\quad} D^*, \tag{3.1}$$

or

$$D \underset{\frac{1}{2}}{\quad\quad} D \text{ is dominated by } U^* \underset{0}{\quad\quad} D^* \tag{3.2}$$

or

$$D \underset{1}{\quad\quad} U \text{ is dominated by } U^* \underset{0}{\quad\quad} D^* \text{ or } D^* \underset{\frac{1}{2}}{\quad\quad} D^* \text{ or } U^* \underset{\frac{1}{2}}{\quad\quad} U^*. \tag{3.3}$$

In each of (3.1) ~ (3.3) one of the two following things must happen :

(a) $\delta(w_0+1)=U$ and $\delta^*(w_0+1)=D^*$

or

(b) $\delta(w_0-1)=D$ and $\delta^*(w_0+1)=U^*$.

Suppose that we have $\delta(w_0+1)=U$ and $\delta^*(w_0+1)=D^*$ and we look to the right. Since $U \underset{\frac{1}{2}}{\quad} U$ and $U \underset{0}{\quad} D$, the risk function of δ as we move up from w_0 will remain $\frac{1}{2}$ until it possibly changes, and then it will go to zero. Since $D^* \underset{\frac{1}{2}}{\quad} D^*$ and $D^* \underset{1}{\quad} U^*$ as we move up from w_0, the risk function remains $\frac{1}{2}$ until it possibly changes, and then it will go to 1. Since $R(w,\delta^*) \leq R(w,\delta)$ for all w even, we see that δ must always be U after w_0+1.

For the case $\delta(w_0-1)=D$ and $\delta^*(w_0-1)=U^*$ we look to the left. Since $D \underset{\frac{1}{2}}{\quad} D$ and $U \underset{0}{\quad} D$ the risk function of δ as we move down to the left of w_0 will remain $\frac{1}{2}$ until it possibly changes, and then it will go to 0. Since $U^* \underset{\frac{1}{2}}{} U^*$ and $D^* \underset{1}{} U^*$, the risk function of δ^* as we move down to the left of w_0 will remain $\frac{1}{2}$ until it possibly changes, and then it will go to 1. Since $R(w,\delta^*) \leq R(w,\delta)$ for all w even, we see that δ must always be D before $w_0 - 1$. This completes the proof of the necessity part of the theorem.

Remark 3.1 In a similar manner, we can deal with w odd and $\delta(x) = x-1$ or $x+1$. Theorem 3.1 also holds for all w odd, and hence, the theorem holds for all $w \in \Theta$.

Now, we need the following result given in Meeden and Ghosh (1982) :

Lemma 3.1 For any estimation problem, every finitely admissible estimator is admissible.

The following theorem gives a characterization of all finitely admissible estimators in the model (1.1).

Theorem 3.2 An estimator δ is finitely admissible if and only if for any $w_0 \in \Theta$ we can find a finite subset Θ_0 of Θ containing w_0 such that $\Theta_0 = \{w_0 - 2m', w_0 - 2m' + 2, \ldots, w_0 - 2, w_0, w_0 + 2, \ldots, w_0 + 2m - 2, w_0 + 2m\}$ for some $m, m' = 0, 1, 2, \ldots$ for which $\delta(w_0 - 2m' - 1) = U$, $\delta(w_0 + 2m' + 1) = D$, and $\delta(x)$ = arbitrary for $x = w_0 - 2m' + 1, w_0 - 2m' + 3, \ldots, w_0 - 1, w_0 + 1, \ldots, w_0 + 2m - 3, w_0 + 2m - 1$.

Proof Sufficiency is clear from Theorem 3.1 that the estimator in the sufficient condition of the theorem is admissible when Θ_0 is taken as a restricted parameter space, and hence the given estimator is finitely admissible. To prove necessity suppose not, i.e., for some $w_1 \in \Theta$ we cannot find any finite subset Θ_1 of Θ containing w_1 such that $\Theta_1 = \{w_1 - 2m', w_1 - 2m' + 2, \ldots, w_1 - 2, w_1, w_1 + 2, \ldots, w_1 + 2m - 2, w_1 + 2m\}$ for some $m, m' = 0, 1, 2, \ldots$ for which $\delta(w_1 - 2m' - 1) = U$, $\delta(w_1 + 2m + 1) = D$, and $\delta(x)$=arbitrary for $x = w_1 - 2m' + 1, w_1 - 2m' + 3, \ldots, w_1 - 1, w_1 + 1, \ldots, w_1 + 2m - 3, w_1 + 2m - 1$. Then, the given estimator is one of the following two types :

i) all D 's before w_1 and arbitrary after w_1

or

ii) all U 's after w_1 and arbitrary before w_1.

Hence, the given estimator is not admissible by Theorem 3.1, and so it is not finitely admissible by Lemma 3.1 which contradicts the finite admissibility of the given estimator. This completes the proof of the theorem.

Furthermore, we have the final interesting result.

Theorem 3.3 Every admissible estimator is finitely admissible.

Proof Suppose not, then for some $w_1 \in \Theta$, one of the following two things must happen :

i) we have all D 's before w_1 and arbitrary after w_1

or

ii) we have all U 's after w_1 and arbitrary before w_1.

Hence, from Theorem 3.1, the given estimator is inadmissible which contradicts to the admissibility of the given estimator.

Finally, in the model (1.1), Theorem 3.3 together with Lemma 3.1 gives the following result :

Theorem 3.4 An estimator is admissible if and only if it is finitely admissible.

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