한국통계학회 논문집 제1권1호
1994년 12월 pp. 112-118

Equivalence of GLS and Difference Estimator in the Linear Regression Model under Seasonally Autocorrelated Disturbances

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Abstract

The generalized least squares estimator in the linear regression model is equivalent to the difference estimator irrespective of the particular form of the regressor matrix when the disturbances are generated by a seasonally autoregressive process and autocorrelation is closed to unity

1. Introduction

Consider the standard linear regression model

$$y = X\beta + u, \tag{1.1}$$

where y is the $T\times 1$ vector of observations on the dependent variable and X is a non-stochastic $T\times k$ matrix of rank $k\leq T$ of k regressors $x_1, x_2, \cdots, x_k$. The $k\times 1$ vector β contains the unknown regression coefficients and u is a $T\times 1$ disturbance vector with $E(u)=0$. In model (1.1), assume that the disturbances follow a stationary seasonal autoregressive process (see Thomas and Wallis (1971), Wallis (1972) and King (1984)),

$$u_t = \rho u_{t-s} + \varepsilon_t, \tag{1.2}$$

where the s denotes the ''seasons'' per year, $|\rho|<1$ and ε_t is a sequence of independent and identically distributed (i.i.d.) random variables with $E(\varepsilon_t)=0$ and $Var(\varepsilon_t)=\sigma_\varepsilon^2$, assuming σ_ε^2 is constant hereafter. If observations are taken over m years,

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then $T=ms$. The autocovariance $E(u_t u_{t-j})=\sigma_u^2 \rho^{j/s}$, if j is an integer multiple of s and $E(u_t u_{t-j})=0$, otherwise, where $\sigma_u^2=\sigma_\varepsilon^2/(1-\rho^2)$. Thus we have

$$Cov(u)=E(uu')=\sigma_u^2 V_s=\sigma_u^2(V_1 \otimes I_s)=\frac{\sigma_\varepsilon^2}{1-\rho^2}(V_1 \otimes I_s) \quad , \tag{1.3}$$

where
$$V_1 = \begin{bmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{m-1} \\ \rho & 1 & \rho & \cdots & \rho^{m-2} \\ \rho^2 & \rho & 1 & \cdots & \rho^{m-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho^{m-1} & \rho^{m-2} & \rho^{m-3} & \cdots & 1 \end{bmatrix},$$

I_s is the $s \times s$ identity matrix and $\otimes$ denotes the Kronecker product. According to a property of the Kronecker product the inverse matrix V_s^{-1} is given by

$$V_s^{-1} = V_1^{-1} \otimes I_s, \tag{1.4}$$

where
$$V_1^{-1}=\frac{1}{1-\rho^2}\begin{bmatrix} 1 & -\rho & 0 & \cdots & 0 & 0 \\ -\rho & 1+\rho^2 & -\rho & \cdots & 0 & 0 \\ 0 & -\rho & 1+\rho^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1+\rho^2 & -\rho \\ 0 & 0 & 0 & \cdots & -\rho & 1 \end{bmatrix}.$$

Since the covariance matrix of u is nonspherical (i.e., not a scalar multiple of the identity matrix), ordinary least squares (*OLS*) is not efficient, although unbiased, compared to generalized least squares by Atiken's theorem (see Fomby et al. (1984, p. 19)). In model (1.1) with known parameter ρ, the best linear unbiased estimator (*BLUE*) of β is the generalized least squares estimator (*GLSE*)

$$\tilde{\beta} = (X'V_s^{-1}X)^{-1}X'V_s^{-1}y. \tag{1.5}$$

2. GLS and Difference Estimator

The *GLS* estimator $\tilde{\beta}$ of β can be obtained by applying *OLS* to the transformed observation matrix $[Py, PX]$. That is,

$$Py = PX\beta + Pu, \tag{2.1}$$

where
$$P = P_1 \otimes I_s, \tag{2.2}$$

$$P_1 = \begin{bmatrix} (1-\rho^2)^{1/2} & 0 & 0 & \cdots & 0 & 0 \\ -\rho & 1 & 0 & \cdots & 0 & 0 \\ 0 & -\rho & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & -\rho & 1 \end{bmatrix}.$$

The $T \times T$ matrix P is chosen that $PV_sP' = c\, I_T$ with some scalar c. The transformation (2.1) is known as the Prais-Winsten transformation (see Judge et al. (1988 p. 390)) The error terms of the transformed model are i.i.d. and *OLS* applied to (2.1) produces *GLS* estimator of β,

$$\tilde{\beta} = [(PX)'(PX)]^{-1}(PX)'(Py) \tag{2.3}$$

and the covariance matrix for $\tilde{\beta}$ is given by

$$\begin{aligned} \hat{V} &= Cov(\tilde{\beta}) = \sigma_u^2(X'V_s^{-1}X)^{-1}X'V_s^{-1}V_s^{-1}V_sV_sV_s^{-1}X(X'V_s^{-1}X)^{-1} \\ &= \sigma_u^2(X'V_s^{-1}X)^{-1} = \sigma_u^2(1-\rho^2)(X'P'PX)^{-1} \\ &= \sigma_\varepsilon^2\,[(PX)'(PX)]^{-1}. \end{aligned} \tag{2.4}$$

Similarly the s-th difference estimator, *DE(s)*, of β is obtained by applying *OLS* to the difference transformed model,

$$Dy = DX\beta + Du, \tag{2.5}$$

where
$$D = D_1 \otimes I_s, \tag{2.6}$$

$$D_1 = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{bmatrix}.$$

D_1 is the $(m-1)\times m$ difference matrix. Therefore, the s-th difference estimator is

$$\beta^* = [(DX)'(DX)]^{-1}(DX)'(Dy) \tag{2.7}$$

and the covariance matrix for β^* is given by

$$V^* = Cov(\beta^*) = \sigma_u^2[(DX)'(DX)]^{-1}(DX)'DV_sD'(DX)[(DX)'(DX)]^{-1}. \tag{2.8}$$

In model (1.1)-(1.2), we are interrested in comparing *GLSE* with *DE(s)* of β, when the disturbances follow a stationary seasonal autoregressive process and autocorrelation ρ is closed to unity.

3. Main Results

Let u in (1.1) be generated by (1.2). Then the *GLS*-transformation matrix P is obtained from (2.2). This implies

$$\lim_{\rho\to 1} P = \lim_{\rho\to 1}(P_1 \otimes I_s) = [0 \ : \ D]' \quad \text{and} \quad \lim_{\rho\to 1}(PX)'(PX) = [DX]'[DX],$$

where $\mathbf{0}$ is an $(s\times T)$ matrix. Then we have the following theorem.

Theorem 3.1: (Homogeneous regression)

Both V^* and $\hat{V}$ tend to the same nonsingular and finite matrix Q as $\rho \to 1$.

Proof: From the Lemma in Appendix and constant σ_ε^2, $(DX)'(DX)$ is nonsingular.

Then

$$\lim_{\rho\to 1}\hat{V} = \lim_{\rho\to 1}\sigma_\varepsilon^2[(PX)'(PX)]^{-1} = \sigma_\varepsilon^2[(DX)'(DX)]^{-1} = Q,$$

which is some nonsingular and finite matrix. In (2.8),

$$\lim_{\rho\to 1} DV_sD' = \lim_{\rho\to 1}(D_1\otimes I_s)(V_1\otimes I_s)(D_1\otimes I_s)' = \lim_{\rho\to 1}(D_1V_1D_1'\otimes I_s).$$

where $\lim_{\rho\to 1} D_1V_1D_1'$

$$= \frac{\sigma_\varepsilon^2}{1+\rho}\begin{bmatrix} 2 & \rho-1 & \rho(\rho-1) & \cdots & \rho^{T-3}(\rho-1) \\ \rho-1 & 2 & \rho-1 & \cdots & \rho^{T-4}(\rho-1) \\ \rho(\rho-1) & \rho-1 & 2 & \cdots & \rho^{T-5}(\rho-1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho^{T-3}(\rho-1) & \rho^{T-4}(\rho-1) & \rho^{T-5}(\rho-1) & \cdots & 2 \end{bmatrix}.$$

Thus $\lim_{\rho\to 1} DV_sD' = \sigma_\varepsilon^2 I_m\otimes I_s = \sigma_\varepsilon^2 I_T$. Accordingly, $\lim_{\rho\to 1} V^* = Q$. □

When the first column of X consists of ones (inhomogeneous regression), the first column of DX becomes zero and is dropped before applying OLS to (2.5). Now, the standard linear regression model (1.1) may be written as:

$$y = [\,i : X_1]\begin{bmatrix}\alpha\\ \beta_1\end{bmatrix} + u,$$

where i is a $T\times 1$ vector of ones and α denotes the intercept.

Then we have the following theorem in the case of an inhomogeneous regression.

Theorem 3.2: (Inhomogeneous regression)

Both V^* and the lower right $(k-1)\times(k-1)$ submatrix of $\hat{V}$ tend to the same nonsingular and finite $(k-1)\times(k-1)$ matrix Q_1 as $\rho\to 1$.

Proof: $V_1^* = Cov(\beta_1^*)$ is given by (2.8) after substituting X_1 for X :

$$V_1^* = Cov(\beta_1^*) = [(DX_1)'(DX_1)]^{-1}(DX_1)'DV_sD'(DX_1)[(DX_1)'(DX_1)]^{-1},$$

whereas $\widehat{V}_1 = Cov(\beta_1^*)$ is the lower right $(k-1)\times(k-1)$ submatrix of

$$\widehat{V} = \begin{bmatrix} (P\iota)'(P\iota) & (P\iota)'(PX_1) \\ (PX_1)'(P\iota) & (PX_1)'(PX_1) \end{bmatrix}^{-1}.$$

By the well-known inversion formular of partitioned matrices (see Graybill, (1983 p.184)) we get

$$\widehat{V}_1 = \sigma_\varepsilon^2\{(PX_1)(PX_1)' - (PX_1)'(P\iota)\,[(P\iota)'(P\iota)]^{-1}(P\iota)'(PX_1)\}. \tag{3.1}$$

Further,

$$(P\iota)(P\iota)' = [(P_1\otimes I_s)\iota]'[(P_1\otimes I_s)\iota] = s(1-\rho^2)+(T-s)(1-\rho)^2,$$

which has a single zero at $\rho = 1$, and

$$(P\iota)(P\iota)' = [(P_1\iota)(P_1\iota)'\otimes I_s],$$

where
$$(P_1\iota)(P_1\iota) = \begin{bmatrix} (1-\rho)(1+\rho) & \sqrt{1-\rho^2}(1-\rho) & \cdots & \sqrt{1-\rho^2}(1-\rho) \\ \sqrt{1-\rho^2}(1-\rho) & (1-\rho)^2 & \cdots & (1-\rho)^2 \\ \vdots & \vdots & \ddots & \vdots \\ \sqrt{1-\rho^2}(1-\rho) & (1-\rho)^2 & \ddots & (1-\rho)^2 \\ \sqrt{1-\rho^2}(1-\rho) & (1-\rho)^2 & \cdots & (1-\rho)^2 \end{bmatrix}.$$

Therefore, $\lim_{\rho\to 1}(P\iota)[(P\iota)'(P\iota)]^{-1}(P\iota)'$ has zeros everywhere except in the upper left corner. From $\lim_{\rho\to 1}PX = [0 \,:\, DX]'$, the second term in (3.1) tends to zero as $\rho\to 1$. This implies $\lim_{\rho\to 1}\widehat{V}_1 = \sigma_\varepsilon^2[(DX_1)'(DX_1)]^{-1} = \lim_{\rho\to 1}V_1^*$ and proves the theorem. □

4. Remarks

The theorems 3.1 and 3.2 show that the GLSE is equivalent to the DE(s) irrespective of the particular form of the regressor matrix, if ρ is closed to unity. In particular, when $\rho=1$ (u_t follows a random walk process), the GLSE is equal to the DE(s). Specially if $s=1$, the GLSE is equivalent to the first difference estimator.

Appendix

Lemma: If $\iota_T \notin C(X)$, where ι_T is a $T\times 1$ vector of ones and $C(X)$ is a column space of X, then $Q=X'D'DX$ is nonsingular.

Proof: Suppose Q is singular. Then there exists a nonzero vector v such that $Qv=0$ which implies $D'DXv=0$. Since the null space of $D'D$ is generated by the vector ι_T, we see that $Xv=\lambda_0 \iota_T$ for some scalar λ_0. Furthermore, as $v\neq 0$ and X is column nonsingular we have $\lambda_0\neq 0$. This entails $\iota_T \in C(X)$ which by assumption is excluded.

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