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A Maximal Inequality for Partial Sums of Negatively Associated Sequences

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Abstract

For an $r>2$ and a finite B, $E|\max_{1\le k\le n}\sum_{j=m+1}^{m+k}X_j|^r \le Bn^{\frac{r}{2}}$ (all $n\ge 1$) is obtained for a negatively associated sequence $\{X_j : j\in N\}$. We also derive the maximal inequality for a negatively associated sequence. Stationarity is not required.

1. Introduction

A sequence $\{X_j : j\in N\}$ of random variables is said to be associated if for any finite collection $\{X_{j(1)},\cdots,X_{j(n)}\}$ and for any real coordinatewise nondecreasing functions f, g on

$$R^n \ \mathrm{Cov}(f(X_{j(1)},\cdots,X_{j(n)}),\ g(X_{j(1)},\cdots,X_{j(n)}))\ge 0,$$

whenever the covariance is defined.

This concept of (positively) associated random variables was introduced into the statistical literature by Esary, Proschan, and Walkup(1967). Since then a great many papers have been written on the subject and its extensions and numerous multivariate inequalities have been obtained.

A sequence $\{X_j : j\in N\}$ of random variables is said to be negatively associated(NA) if for every finite subfamily $\{X_{j(1)},\cdots,X_{j(n)}\}$ and for every pair of disjoint subsets A_1, A_2 of $\{j(1),\cdots j(n)\}$,

$$\mathrm{Cov}(f(X_{j(k)}:j(k)\in A_1),g(X_{j(l)}:j(l)\in A_2))\le 0,$$

whenever f and g are nondecreasing. Joag-Dev and Proschan(1983) have introduced the notion of negatively associated(NA) random variables, and developed applications in multivariate statistical analysis. The theory and application of NA are not simply the duals of the theory and application of positive association, but differ in important results.

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Newman(1984) showed that the independence structure of an NA sequence is highly determined by its covariance structure, i.e., by the behaviour of the coefficient

$$u(n) = \sup_{k\in N} \sum_{j:|j-k|\ge n} |\mathrm{Cov}(X_j, X_k)|, \quad n\in N\cup\{0\}.$$

Birkel(1988) showed that following moment bounds for associated processes which are a weak form of stationarity and depending on the rate of decrease of $u(n)$.

Theorem A (Birkel). Let $\{X_j : j\in N\}$ be a sequence of associated random variables with $EX_j = 0$. Assume

$$\sup_{j\in N} E|X_j|^{r+\delta} < \infty \text{ for some } r>2 \text{ and } \delta>0, \tag{1.1}$$

$$u(n) = O(n^{-(r-2)(r+\delta)/2\delta}). \tag{1.2}$$

Then there exists a constant B not depending on n such that for all $n\in N$,

$$\sup_{m\in N\cup\{0\}} E|S_{n+m} - S_m|^r \le Bn^{\frac{r}{2}}. \tag{1.3}$$

Corollary B(Birkel). Let $\{X_j : j\in N\}$ be a sequence of associated random variables with $EX_j = 0$. Assume

$$\sup_{j\in N} E|X_j|^s < \infty \text{ for some } s>2, \tag{1.4}$$

$$u(n) = O(n^{-\rho}) \text{ for some } \rho>0. \tag{1.5}$$

Then there exists $r>2$ such that (1.3) holds.

In this note, we carry out that Theorem A and Corollary B for associated sequences given in Birkel[1], still hold for NA sequences and derive the maximal inequality for NA sequences applying this moment bound and Theorem 3.7.5 of Stout[6]. No stationarity is required.

In the following paper we prove the invariance principle for NA sequence using this maximal inequality.

All results are stated in Section 2. The proofs of our theorems as well as lemmas are given in Section 3.

2. Results

Here and in the following we will use $S_n = \sum_{j=1}^{n} X_j$ and denote R as a real number and R as the set of real number, unless otherwise mentioned.

The proof of Theorem 2.2 will be basically based on the following lemma :

Lemma 2.1. Let $X_1 \geq 0$, X_2 be NA random variables and let $\rho > 0$.

(i) If $X_1 \leq R < \infty$, then

$$|\operatorname{Cov}(X_1^{1+\rho}, X_2)| \leq (1+\rho) R^{\rho} |\operatorname{Cov}(X_1, X_2)|.$$

(ii) If $|X_2| \leq R < \infty$ and $p > 1+\rho$, then

$$|\operatorname{Cov}(X_1^{1+\rho}, X_2)| \leq (1+\rho+2R)(E|X_1|^p)^{\rho/(p-1)} |\operatorname{Cov}(X_1, X_2)|^{(p-1-\rho)/(p-1)}.$$

(iii) If $\gamma > 0$ and $p, q > 1$ with $1/p + 1/q = 1$, then

$$|\operatorname{Cov}(X_1^{1+\rho}, X_2)| \leq (3+\rho)(E|X_1|^{p(1+\rho+\gamma)})^{\rho/p(\rho+\gamma)} \times (E|X_2|^q)^{\rho/q(\rho+\gamma)} |\operatorname{Cov}(X_1, X_2)|^{\gamma/(\rho+\gamma)}.$$

(iv) If $\delta > 0$, then

$$|\operatorname{Cov}(X_1^{1+\rho}, X_2)| \leq (3+\rho)(E|X_1|^{2+\rho})^{\rho(1+\rho+\delta)/(\delta+\rho(2+\rho+\delta))} \times (E|X_2|^{2+\rho+\delta})^{\rho/(\delta+\rho(2+\rho+\delta))} |\operatorname{Cov}(X_1, X_2)|^{\delta/(\delta+\rho(2+\rho+\delta))}.$$

Theorem 2.2. Let $\{X_j : j \in N\}$ be an NA sequence with $EX_j = 0$. If we assume (1.1) and (1.2) then (1.3) holds.

From Theorem 2.2 we get the following corollary :

Corollary 2.3. Let $\{X_j : j \in N\}$ be a sequence of NA random variables with $EX_j = 0$. Assume that (1.4) and (1.5) hold. Then there exists $r > 2$ such that (1.3) holds.

Proof. Put $r = 2s(1+\rho)/(s+2\rho)$, $\delta = s - r$. Then we have $r > 2$, $\delta > 0$ and $\rho = (r-2)(r+2)/2\delta$. Hence (1.4), (1.5) and Theorem 2.2 imply our assertion.

The following lemma suggests that the bound on $E\max_{1\leq k\leq n}|S_{k+m} - S_m|^r$ is of the same order of magnitude as bound on $\sup E|S_{n+m} - S_m|^r$.

Lemma 2.4(Stout, 1974). Let g be a nondecreasing function with

$$2g(n) \leq g(2n) \tag{2.1}$$

for all $n \geq 1$ and $g(n)/g(n+1) \to 1$ as $n \to \infty$. Suppose

$$E|S_{m+n} - S_m|^r \leq g^{r/2}(n) \tag{2.2}$$

for all $m \geq 0$, $n \geq 1$ and some $r > 2$. Then

$$E\max_{1\leq k\leq n}|S_{k+m} - S_m|^r \leq K_r g^{r/2}(n) \tag{2.3}$$

for all $m \geq 0$, $n \geq 1$, and some $K_r < \infty$.

Proof. See Theorem 3.7.5 of Stout(1974).

Theorem 2.5. Assume that (1.4) and (1.5) hold. Then

$$P[\max_{1\le k\le n}|S_{m+k}-S_m|\ge \varepsilon]\le Bn^{r/2}/\varepsilon^r,\ \ m\ge 0,\ \varepsilon>0. \tag{2.4}$$

Theorem 2.6. Suppose that the assumptions of Theorem 2.2 or Corollary 2.3 hold. Then, as $n\to\infty$

$$(S_{m+n}-S_m)/[n^{1/2}(\log n)^{1/r}(\log\log n)^{2/r}]\to 0 \quad \text{a.s.} \tag{2.5}$$

3. Proofs

Proof of Lemma 2.1. (i). For $t\in \mathrm{R}$ put

$$f(t) = t^{1+\rho}1_{\{0\le t\le R\}}+R^{1+\rho}1_{\{t>R\}}$$

$$g(t) = (1+\rho)R^{\rho}t1_{\{0\le t\le R\}}+(1+\rho)R^{1+\rho}1_{\{t>R\}}.$$

Then $g-f$ is nondecreasing. As X_1 and X_2 are NA, we obtain $\mathrm{Cov}(f(X_1),X_2)\ge \mathrm{Cov}(g(X_1),X_2)$, according to Property P_6 of [4]. This proves (i).

(ii). Let $N>0$ be fixed and for $t\in\mathrm{R}$ put $h(t)=t1_{\{0\le t\le N\}}+N1_{\{t>N\}}$. Then $h(X_1)^{1+\rho}$ and $X_1^{1+\rho}-h(X_1)^{1+\rho}$ are nondecreasing.

As X_1 and X_2 are NA, according to Property P_6 of [4], $h(X_1)^{1+\rho}$ and X_2 are NA and $X_1{}^{1+\rho}-h(X_1)^{1+\rho}$ and X_2 are NA and thus

$$\mathrm{Cov}(h(X_1)^{1+\rho},X_2)\le 0,\ \ \mathrm{Cov}(X_1^{1+\rho}-h(X_1)^{1+\rho},X_2)\le 0. \tag{3.1}$$

Therefore, we have

$$|\mathrm{Cov}(X_1^{1+\rho},X_2)|=|\mathrm{Cov}(h(X_1)^{1+\rho},X_2)|+|\mathrm{Cov}(X_1^{1+\rho}-h(X_1)^{1+\rho},X_2)| \tag{3.2}$$

Since X_1 and X_2 are NA, $h(X_1)$ and X_2 are NA according to Property P_6 of [4]. Using (i) of Lemma 2.1 and (3.1), we therefore obtain

$$\begin{aligned} 0\ge \mathrm{Cov}(h(X_1)^{1+\rho},X_2) &\ge (1+\rho)N^{\rho}\,\mathrm{Cov}(h(X_1),X_2) \\ &\ge (1+\rho)N^{\rho}\,\mathrm{Cov}(X_1,X_2). \end{aligned} \tag{3.3}$$

The second term on the right-hand side of (3.2) is bounded by

$$\begin{aligned} &|\operatorname{Cov}\ (X_1^{1+\rho}-h(X_1)^{1+\rho}, X_2)| \\ &\leq 2RE|X_1^{1+\rho}1_{\{X_1>N\}}| \leq 2RN^{-p+1+\rho}E|X_1|^{p.} \end{aligned} \tag{3.4}$$

Since uncorrelated NA random variables are independent(Corollary 3 of [5]), we may assume without loss of generality that $|\operatorname{Cov}(X_1, X_2)| > 0$.

Choosing $N = (E|X_1|^p / |\operatorname{Cov}(X_1, X_2)|)^{1/(p-1)}$, (3.2) (3.3) and (3.4) imply (ii).

(iii). Let $N>0$ be fixed. We proceed as in (ii). By Holder's inequality the second term on the right-hand side of (3.2) is bounded by

$$\begin{aligned} |\operatorname{Cov}(X_1^{1+\rho}-f(X_1)^{1+\rho}, X_2)| &\leq 2(E|X_1|^{p(1+\rho)}1_{\{X_1>N\}})^{1/p}(E|X_2|^q)^{1/q} \\ &\leq 2N^{-\gamma}(E|X_1|^{p(1+\rho+\gamma)})^{1/p}(E|X_2|^q)^{1/q}. \end{aligned} \tag{3.5}$$

Again we assume $|\operatorname{Cov}(X_1, X_2)| > 0$. Then, choosing $N = [(E|X_1|^{p(1+\rho+\gamma)})^{1/p} (E|X_2|^q)^{1/q} / |\operatorname{Cov}(X_1, X_2)|]^{1/(\rho+\gamma)}$, (3.2), (3.3) and (3.5) imply (iii).

(iv). (iv) follows from (iii), putting $\gamma = \delta/(2+\rho+\delta)$ and $p = (2+\rho)/(1+\rho+\gamma)$.

Proof of Theorem 2.2. Let $r = l+\rho$, where $l \in N$, $l \geq 2$ and $\rho \in (0,1]$. We proceed by induction on l. For simplicity we introduce the following notations:

$$S_{m,n} = S_{n+m} - S_m,\ a_n = \sup_{m \in N \cup \{0\}} E|S_{m,n}|^r,$$

$$S_{m,n}^+ = \max\{S_{m,n}, 0\},\ S_{m,n}^- = \max\{-S_{m,n}, 0\},$$

$$S_{m+n,n}^+ = \max\{S_{m+n,n}, 0\},\ S_{m+n,n}^- = \max\{-S_{m+n,n}, 0\}$$

We shall show that there exist $c_1 < \infty$ and $\varepsilon \in (0,1)$, such that for all $m \in N \cup \{0\}$, $n \in N$

$$E|S_{m,2n}|^r \leq 2a_n + c_1 a_n^{1-\rho} n^{\rho r/2} + c_1 a_n^{1-\varepsilon} n^{\varepsilon r/2}. \tag{3.6}$$

From (3.6) we obtain

$$a_{2n} \leq 2a_n + c_1 a_n^{1-\rho} n^{\rho r/2} + c_1 a_n^{1-\varepsilon} n^{\varepsilon r/2} \text{ for } n \in N,$$

and by induction there exists $c < \infty$ such that $a_n \leq c n^{r/2}$ for all $n \in \{2^\nu : \nu \in N \cup \{0\}\}$. Then (1.3) follows from the proof of Lemma 7.4 of Doob[2].

To prove (3.6), we show the following inequalities for $m \in N \cup \{0\}$, $n \in N$:

$$E|S_{m,2n}|^r \leq 2a_n + 2^{l+1}(E|S_{m,n}|^\rho |S_{m+n,n}|^l + E|S_{m,n}|^l |S_{m+n,n}|^\rho); \tag{3.7}$$

$$\begin{aligned} E|S_{m,n}|^\rho |S_{m+n,n}|^l &\leq a_n^{1-\rho}(E|S_{m,n}||S_{m+n,n}|^{l-1+\rho})^\rho, \\ E|S_{m,n}|^l |S_{m+n,n}|^\rho &\leq a_n^{1-\rho}(E|S_{m,n}|^{l-1+\rho}|S_{m+n,n}|)^\rho; \end{aligned} \tag{3.8}$$

$$E(|S_{m,n}||S_{m+n,n}|^{l-1+\rho}) \le |\mathrm{Cov}(|S_{m,n}|, |S_{m+n,n}|^{l-1+\rho})| + c_2 n^{r/2},$$
$$E(|S_{m,n}|^{l-1+\rho}|S_{m+n,n}|) \le |\mathrm{Cov}(|S_{m,n}|^{l-1+\rho}, |S_{m+n,n}|)| + c_2 n^{r/2}; \quad (3.9)$$

$$|\mathrm{Cov}(|S_{m,n}|, |S_{m+n,n}|^{l-1+\rho})| \le c_3 a_n^{1-\gamma} n^{\gamma r/2},$$
$$|\mathrm{Cov}(|S_{m,n}|^{l-1+\rho}, |S_{m+n,n}|)| \le c_3 a_n^{1-\gamma} n^{\gamma r/2}; \quad (3.10)$$

where $\gamma = (r-2+\delta)/s$, $s = \delta + (r-2)(r+\delta)$. Then (3.6) follows from (3.7)-(3.10), putting $\varepsilon = \rho\gamma$.

Since the proofs of (3.7), (3.8) and (3.9) are generally the same as those of (3.7), (3.8) and (3.9) of Birkel[1], it remains to prove (3.10).

To prove (3.10) : We have

$$\begin{aligned} &|\mathrm{Cov}(|S_{m,n}|, |S_{m+n,n}|^{(l-1+\rho)})| \\ &= |\mathrm{Cov}(|S_{m,n}|, |S_{m+n,n}|^{r-1})| \\ &\le |\mathrm{Cov}(S^+_{m,n}, ((S_{m+n,n})^+)^{r-1})| + |\mathrm{Cov}(S^+_{m,n}, ((S_{m+n,n})^-)^{r-1})| \\ &\quad + |\mathrm{Cov}(S^-_{m,n}, ((S_{m+n,n})^+)^{r-1})| + |\mathrm{Cov}(S^-_{m,n}, ((S_{m+n,n})^-)^{r-1})|. \end{aligned} \quad (3.11)$$

Since $S^+_{m,n}$ and $((S_{m+n,n})^+)^{r-1}$ are nondecreasing and $S^-_{m,n}$ and $(S^-_{m+n,n})^{r-1}$ are nonincreasing functions of $X_{m+1}, \cdots, X_{m+n}$ and $X_{m+n+1}, \cdots, X_{m+2n}$ respectively and since $X_{m+1}, X_{m+2}, \cdots, X_{m+2n}$ are NA, we obtain

$$\mathrm{Cov}(S^+_{m,n}, (S^+_{m+n,n})^{r-1}) \le 0, \quad \mathrm{Cov}(S^+_{m,n}, (S^-_{m+n,n})^{r-1}) \ge 0,$$
$$\mathrm{Cov}(S^-_{m,n}, (S^+_{m+n,n})^{r-1}) \ge 0, \quad \mathrm{Cov}(S^-_{m,n}, (S^-_{m+n,n})^{r-1}) \le 0.$$

Hence by (3.11),

$$\begin{aligned} &|\mathrm{Cov}(|S_{m,n}|, |S_{m+n,n}|^{l-1+\rho})| \\ &\le -\mathrm{Cov}(S^+_{m,n}, (S^+_{m+n,n})^{r-1}) + \mathrm{Cov}(S^+_{m,n}, (S^-_{m+n,n})^{r-1}) \\ &\quad + \mathrm{Cov}(S^-_{m,n}, (S^+_{m+n,n})^{r-1}) - \mathrm{Cov}(S^-_{m,n}, (S^-_{m+n,n})^{r-1}) \\ &= -\sum_{j=m+1}^{m+n} \mathrm{Cov}(X_j, (S^+_{m+n,n})^{r-1}) - \sum_{j=m+1}^{m+n} \mathrm{Cov}(-X_j, (S^-_{m+n,n})^{r-1}) \\ &= \sum_{j=m+1}^{m+n} |\mathrm{Cov}(X_j, (S^+_{m+n,n})^{r-1}| - \sum_{j=m+1}^{m+n} |\mathrm{Cov}(-X_j, (S^-_{m+n,n})^{r-1})|. \end{aligned} \quad (3.12)$$

Since nondecreasing(nonincreasing) functions defined on disjoint subsets of a set of NA random variables are NA(see Property P_6 of [4]) X_j and $S_{m+n,n}$, X_j and $S^+_{m+n,n}$, $-X_j$ and $S^-_{m+n,n}$ are NA random variables for every $j \in \{m+1, \cdots, m+n\}$.

Applying Lemma 2.1 (iv) (with $r-2$ instead of ρ and $X_1 = S^+_{m+n,n}$, resp. $X_1 = S^-_{m+n,n}$, $X_2 = X_j$ resp. $X_2 = -X_j$), Lemma 2.1(i) and our assumptions (1.1) and (1.2), we obtain from (3.12)

$$
\begin{aligned}
&|\operatorname{Cov}(|S_{m,n}|, |S_{m+n,n}|^{l-1+\rho})| \\
&\le 2(r+1)a_n^{1-\gamma} \sup_{i \in N}(E|X_i|^{r+\delta})^{(r-2)/s} \sum_{j=m+1}^{m+n}(|\operatorname{Cov}(X_j, S_{m+n,n})|)^{\delta/s} \quad \text{by (1.1)} \\
&\le d_1 a_n^{1-\gamma} \sum_{j=1}^{n} u(j)^{\delta/s} j^{(r-2)(r+\delta)/2s} j^{-(r-2)(r+\delta)/2s} \\
&\le d_2 a_n^{1-\gamma} \sum_{j=1}^{n} j^{-(r-2)(r+\delta)/2s} \quad \text{by (1.2)} \\
&\le d_3 a_n^{1-\gamma} n^{\gamma r/2}.
\end{aligned}
$$

This proves the first inequality in (3.10). The second inequality follows similarly and thus the proof of Theorem 2.2 is complete.

Proof of Theorem 2.5. From Corollary 2.3 here and Lemma 2.4 of Stout[6] that there exists a constant B such that

$$E(\max_{1 \le k \le n}|S_{m+k} - S_m|^r) \le Bn^{r/2}, \tag{3.13}$$

for some $r > 2$ and $m \in N \cup \{0\}$.

Thus using Chebyshev's inequality and (3.13) we obtain (2.4), that is,

$P\{\max_{1 \le k \le n}|S_{m+k} - S_m| \ge \varepsilon\} \le Bn^{r/2}/\varepsilon^r$, $m \ge 0$, $\varepsilon > 0$.

Proof of Theorem 2.6. It follows from Theorem 2.2 or Corollary 2.3 that (1.3) holds. Thus (1.3) implies (2.2) according to Lemma 2.4.

Remark 1. The results in Section 2 are still hold for linearly negative quadrant dependent(LNQD) sequences.

2. We may apply this moment bound to the invariance principle for NA sequences and to some multivariate variables (example, multivariate normal) in the following paper.

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