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Strong Large Deviations Theorems for the Ratio of the Independent Random Variables†

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ABSTRACT

In this paper, we prove a strong large deviations theorem for the ratio of independent random variables with error rate of $O(n^{-1})$. To obtain our results we use the inversion formula for the tail probability and apply the Chaganty and Sethuraman's(1985) approach.

KEYWORDS: Large deviations, Tail probability, Inversion formula, Ratio of random variables.

1. INTRODUCTION

Let $\{T_n, n \geq 1\}$ be a sequence of random variables. One of the important problems in probability theory is to study the behavior of the limit probability for large deviations, namely , $P_r(T_n \geq t_n)$ where $\{t_n\}$ is a sequence of constants increasing to ∞. The result to give the asymptotic expressions for *log* $P_r(T_n \geq$

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t_n) is called the weak large deviations result and the result on $P_r(T_n \geq t_n)$ is called the strong large deviations result.

Since Cramér (1938) and Chernoff (1952), many generalizations have been made in large deviations. For instance,see Sievers (1969), Steinebach (1978), Bahadur and Zabel (1979), Vandemaele and Veraverbeke (1982). However, most of large deviations theorems including the results mentioned above are weak large deviations results. Bahadur and Rao (1960) obtained a notable strong large deviations result on sample mean of independent and identically distributed (i.i.d.) random variables. Recently, Chaganty and Sethuraman (1987) extended Bahadur and Rao's (1960) result by considering an arbitrary sequence of random variables which are not necessarily sums of i.i.d. random variables under some restrict conditions for the moment generating functions.

The result of Chaganty and Sethuraman (1987) is extended by Choi, Kim and Jeon (1992), where they obtained a strong large deviations result for the ratio of independent but arbitrary random variables. Chaganty and Sethuraman's (1987) strong large deviations theorem and Choi, Kim and Jeon's(1992) give the asymptotic expressions for the tail probabilities with the remainder's error rate of $o(1)$. In this paper, we prove a strong large deviations theorem for the ratio of independent random variables with an improved error rate $O(n^{-1})$. Chaganty and Sethuraman's (1987) and Choi, Kim, Kim and Jeon's (1992) results were proved by using the Esscher transformation method. However, we use directly the inversion formula for the tail probability and apply the Chaganty-Sethuraman's (1985) approach as in the proof of large deviations local limit theorems. This is the key point by which we could obtain an improved error rate of $O(n^{-1})$.

2. MAIN THEOREM

Let $\{U_{n_1}, n_1 \geq 1\}$ be a sequence of random variables with absolutely continuous distribution function F_{n_1} and $\{S_{n_2} > 0, n_2 \geq 1\}$ be a sequence of random variables with distribution function F_{n_2}. Suppose that the two sequences are independent and $n_1 \leq n_2$. Define $\phi_{n_1}(z) = E\{\exp(zU_{n_1})\}$ and $\phi_{n_2}(z) = E\{\exp(zS_{n_2})\}$ for a complex number z. Assume that

$$\phi_{n_i}(z) \quad \text{is analytic in} \quad \Omega_i = \Big\{z \in C : |\text{Real } z| < a_i, a_i > 0\Big\}, \quad i = 1, 2.$$

Define

$$\psi_{ni}(z) = \frac{1}{n_i} \log \phi_{ni}(z) \quad \text{for} \quad z \in \Omega_i,\ i = 1, 2. \tag{2.1}$$

Denote the interval $(-b_i, b_i)$ by $J_i,\ i = 1, 2$ where $0 < b_i < a_i$. And let $\{r_{n_1,n_2}, n_1 \geq 1,\ n_2 \geq 1\}$ be a double array of real numbers such that $\sup |r_{n_1,n_2}| = r_1 < \infty$ and

$$\begin{aligned} G_{n_1,n_2}(t; r, \tau) \ &= \frac{1}{n_2}\Big[n_1\Big\{\psi_{n_1}(\tau) - \psi_{n_1}(\tau + it)\Big\} \\ &\quad + n_2\Big\{\psi_{n_2}(-r\tau) - \psi_{n_2}(-r(\tau + it))\Big\}\Big], \end{aligned} \tag{2.2}$$

for $\tau \in J_1, r\tau \in J_2$

Denote r_{n_1,n_2} by r_n and τ_{n_1,n_2} by τ_n and consider the following conditions for U_{n_1} and S_{n_2};

(A) There exist $\beta_i > 0,\ i = 1, 2$ such that

$$|\psi_{n_i}(z)| < \beta_i \quad \text{for } |z| < a_i,\ n_i \geq 1,\ i = 1, 2.$$

(B) There exists $\tau_n \in J_1$ such that $r_n\tau_n \in J_2$

$$\psi'_{n_1}(\tau_n) - r_n\psi'_{n_2}(-r_n\tau_n) = 0 \quad \text{for all } n_1 \geq 1, n_2 \geq 1.$$

(C) There exists $\alpha_1 > 0$ such that $\psi''_{n_1}(\tau) \geq \alpha_1$ for $\tau \in J_1$.

(D) There exists $\eta > 0$ such that for each $0 < \delta < \eta$,

$$\inf_{|t|\geq\delta} \text{Real } G_n(t) = \min\Big\{\text{Real } G_n(\delta), \text{Real } G_n(-\delta)\Big\},$$

for $n_1 \geq 1, n_2 \geq 1$, where $G_n(t) = G_{n_1,n_2}(t; \tau_n, r_n)$.

(E) There exists $M > 0$ such that

$$\left|\frac{\psi'_{n_2}(z)}{\psi'_{n_2}(-r_n\tau_n)}\right| \le M, \quad \text{for arbitrary } z \in \Omega_2, n_2 \ge 1.$$

(F) There exist $l > 0$ and $p > 0$ such that

$$\int \left|\frac{\phi_{n_1}(\tau+it)}{\phi_{n_1}(\tau)}\right|^{l/n_2} \cdot \left|\frac{\phi_{n_2}\{-r(\tau+it)\}}{\phi_{n_2}(-r\tau)}\right|^{l/n_2}$$
$$\times \left|\frac{\psi'_{n_2}\{-r(\tau+it)\}}{\psi'_{n_2}(-r\tau)}\right| dt = O(n_2^p), \quad \text{for } \tau \in J_1 \quad \text{and } r\tau \in J_2.$$

Theorem 2.1. Assume that the conditions (A), (C) through (F) and the following condition (B′) are satisfied.
(B′) There exists $\tau_n \in J_1$ such that $r_n\tau_n \in J_2$, $\inf_n \tau_n = \tau_0 > 0$

$$\psi'_{n_1}(\tau_n) - r_n\psi'_{n_2}(-r_n\tau_n) = 0 \quad \text{for all } n_1 \ge 1, n_2 \ge 1.$$

Then as n_1 and $n_2 \to +\infty$ with $n_1/n_2 \to c > 0$, the tail probability of $R_{n_1,n_2} = U_{n_1}/S_{n_2}$ is given by

$$\bar{H}_{n_1,n_2}(r_n) = \frac{\exp\{n_1\psi_{n_1}(\tau_n) + n_2\psi_{n_2}(-r_n\tau_n)\}}{\tau_n\sqrt{2\pi\{n_1\psi''_{n1}(\tau_n) + n_2r_n^2\psi''_{n_2}(-r_n\tau_n)\}}}$$
$$\times \left\{1 + O(n_1^{-1})\right\}. \tag{2.3}$$

We will state the next Lemma 2.2 through Lemma 2.4 before proving the above theorem whose proofs were given in Cho (1991).

Lemma 2.2. Assume that the conditions (A), (B), (C) and (D) are satisfied. Let $r_0 = \inf_n |r_n|$ and choose $\alpha_2 > 0$ such that $(n_1/n_2)\alpha_1 - r_0^2\alpha_2 = \epsilon_n > 0$. Then there exists $\delta' < \eta$ such that for $0 < \delta < \delta'$,

$$\inf_{|t|\ge\delta} \text{Real } G_n(t) \ge \epsilon_n\frac{\delta^2}{4}, \quad \text{for all } n_1 \ge 1, n_2 \ge 1, \tag{2.4}$$

where $G_n(t) = G_{n_1,n_2}(t; r_n, \tau_n)$.

Lemma 2.3. Let

$$K_{1n} = \sqrt{n_2}\left(\frac{d_n}{2\pi}\right)^{\frac{1}{2}} \int_{|t|<n_2^{-\lambda}} \exp\{-n_2 G_n(t)\}dt. \tag{2.5}$$

Then $K_{1n} = 1 + O(n_1^{-1})$.

Lemma 2.4. Assume that the condition (A) is satisfied. Then the following identity

$$\int_{-i\infty}^{+i\infty} \phi_{n_1}(z)\phi'_{n_2}(-rz)dz = \int_{\tau-i\infty}^{\tau+i\infty} \phi_{n_1}(z)\phi'_{n_2}(-rz)dz \tag{2.6}$$

holds, where $\tau \in J_1$ and $r\tau \in J_2$.

Proof of Theorem 2.1. Let H_{n_1,n_2} be the distribution function of $R_{n_1,n_2} = U_{n_1}/S_{n_2}$, then

$$H_{n_1,n_2}(r) = \int_0^{+\infty} F_{n_1}(ry)dF_{n_2}(y). \tag{2.7}$$

Thus, the p.d.f. h_{n_1,n_2} of H_{n_1,n_2} is given by

$$h_{n_1,n_2}(r) = \int_0^{+\infty} y f_{n_1}(ry)dF_{n_2}(y), \tag{2.8}$$

where f_{n_1} is the p.d.f. of U_{n_1}. The ch.f. $\hat{h}_{n_1,n_2}$ of R_{n_1,n_2} is given by

$$\hat{h}_{n_1,n_2}(t) = \int_{-\infty}^{+\infty} \exp(-itr)\ h_{n_1,n_2}(r)dr = \int_0^{+\infty} \phi_{n_1}\left(\frac{it}{y}\right) dF_{n_2}(y). \tag{2.9}$$

By using Fourier inversion formula, the p.d.f. h_{n_1,n_2} is given by

$$\begin{aligned} h_{n_1,n_2}(r) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp(-itr)\hat{h}_{n_1,n_2}(t)dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-itr}\Big\{\int_0^{\infty} \phi_{n_1}\left(\frac{it}{y}\right) dF_{n_2}(y)\Big\}dt \quad (\text{put } \frac{t}{y} = s) \end{aligned}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \phi_{n_1}(is)\Big\{\int_0^{\infty} e^{-isyr}\cdot ydF_{n_2}(y)\Big\}ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \phi_{n_1}(it)\phi'_{n_2}(-irt)dt \quad (\text{put } it = z)$$

$$= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \phi_{n_1}(z)\phi'_{n_2}(-rz)dz$$

$$= \frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty} \phi_{n_1}(z)\phi'_{n_2}(-rz)dz, \quad \tau\in J_1 \quad (\text{by Lemma 2.4})$$

$$= \frac{n_2}{2\pi}\int_{-\infty}^{+\infty} \exp\Big[n_1\psi_{n_1}(\tau+it)+n_2\psi_{n_2}\Big\{-r(\tau+it)\Big\}\Big]$$

$$\times\, \psi'_{n_2}\Big\{-r(\tau+it)\Big\}dt \tag{2.10}$$

Therefore, the tail probability of R_{n_1,n_2} is given by

$$\bar{H}_{n_1,n_2}(r) = \frac{n_2}{2\pi}\int_r^{+\infty}\int_{-\infty}^{+\infty} \exp\Big[n_1\psi_{n_1}(\tau+it)+n_2\psi_{n_2}\Big\{-r(\tau+it)\Big\}\Big]$$

$$\times\, \psi'_{n_2}\Big\{-r(\tau+it)\Big\}dtdr \tag{2.11}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp\Big\{n_1\psi_{n_1}(\tau+it)\Big\}\Big[-\frac{1}{\tau+it}\exp\Big\{n_2\psi_{n_2}(-r(\tau+it))\Big\}\Big|_r^{\infty}\Big]dt.$$

Since $\exp[n_2\psi_{n_2}\{-r(\tau+it)\}] = \phi_{n_2}\{-r(\tau+it)\} = \int_0^{+\infty}\exp\{-r(\tau+it)x\}$ $\times dF_{n_2}(x)$ converges to zero as $r\to+\infty$ and $\tau>0$, the relation (2.11) becomes

$$\bar{H}_{n_1,n_2}(r) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp\Big[n_1\psi_{n_1}(\tau+it)+n_2\psi_{n_2}\Big\{-r(\tau+it)\Big\}\Big]\frac{dt}{\tau+it}. \tag{2.12}$$

Substituting r by r_n and τ by τ_n in (2.12), we have

$$\bar{H}_{n_1,n_2}(r_n) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp\Big[n_1\psi_{n_1}(\tau_n+it)+n_2\psi_{n_2}\Big\{-r_n(\tau_n+it)\Big\}\Big]\frac{dt}{\tau_n+it}$$

$$= \frac{\exp\{n_1\psi_{n_1}(\tau_n)+n_2\psi_{n_2}(-r_n\tau_n)\}}{\tau_n\sqrt{2\pi\{n_1\psi''_{n_1}(\tau_n)+n_2r_n^2\psi''_{n_2}(-r_n\tau_n)}}\} \times \tilde{I}, \tag{2.13}$$

where

$$\tilde{I} = \sqrt{n_2}\left(\frac{d_n}{2\pi}\right)^{\frac{1}{2}}\int_{-\infty}^{+\infty}\exp\Big\{-n_2G_n(t)\Big\}\frac{\tau_n}{\tau_n+it}dt \tag{2.14}$$

and $d_n = (n_1/n_2)\psi''_{n_1}(\tau_n) + r_n^2\psi''_{n_2}(-r_n\tau_n)$, $G_n(t) = G_{n_1,n_2}(t; r_n, \tau_n)$, for $\tau_n \in J_1, r_n\tau_n \in J_2$. It is sufficient to show that $\tilde{I}$ is $1 + O(n_1^{-1})$. We can write $\tilde{I}$ as follows;

$$\begin{aligned}
\tilde{I} &= \sqrt{n_2}\left(\frac{d_n}{2\pi}\right)^{\frac{1}{2}} \int_{|t|\geq n_2^{-\lambda}} \exp\Big\{-n_2 G_n(t)\Big\} \frac{\tau_n}{\tau_n + it}\, dt \\
&\quad + \sqrt{n_2}\left(\frac{d_n}{2\pi}\right)^{\frac{1}{2}} \int_{|t|\leq n_2^{-\lambda}} \exp\Big\{-n_2 G_n(t)\Big\} \frac{\tau_n}{\tau_n + it}\, dt \\
&= \tilde{I}_1 + \tilde{I}_2 \qquad \text{(say)}, \quad \text{where } \frac{1}{3} < \lambda < \frac{1}{2}.
\end{aligned} \tag{2.15}$$

First, we will show that $\tilde{I}_1$ goes to zero exponentially. And next, we will show that $\tilde{I}_2$ is $1 + O(n_1^{-1})$. By Lemma 2.2, we can choose N such that for $n_2 \geq N$,

$$\inf_{|t|\geq n_2^{-\lambda}} \text{Real } G_n(t) \geq \epsilon_n \frac{n_2^{-2\lambda}}{4}.$$

For $n_2 \geq N$,

$$\begin{aligned}
|\tilde{I}_1| &\leq \sqrt{n_2}\left(\frac{d_n}{2\pi}\right)^{\frac{1}{2}} \int_{|t|\geq n_2^{-\lambda}} \Big|\exp\{-n_2 G_n(t)\}\Big|\, dt \\
&\leq \sqrt{n_2}\left(\frac{d_n}{2\pi}\right)^{\frac{1}{2}} \sup_{|t|\geq n_2^{-\lambda}} \Big|\exp\{-(n_2 - l)G_n(t)\}\Big| \\
&\quad \times \int \left|\frac{\phi_{n_1}(\tau_n + it)}{\phi_{n_1}(\tau_n)}\right|^{l/n_2} \left|\frac{\phi_{n_2}\{-r_n(\tau_n + it)\}}{\phi_{n_2}(-r_n\tau_n)}\right|^{l/n_2} dt \\
&= O(n_2^{\frac{1}{2}+p}) \exp\Big\{-(n_2 - l) \inf_{|t|\geq n_2^{-\lambda}} \text{Real } G_n(t)\Big\} \\
&\leq O(n_2^{\frac{1}{2}+p}) \exp\Big\{-(n_2 - l)\epsilon_n \frac{n_2^{-2\lambda}}{4}\Big\} \\
&= O(n_2^{\frac{1}{2}+p}) \exp\Big\{-\epsilon_n \frac{n_2^{1-2\lambda}}{4}\Big\},
\end{aligned} \tag{2.16}$$

which goes to zero exponentially fast, since $\frac{1}{3} < \lambda < \frac{1}{2}$. Next, we can express $\tilde{I}_2$ as follows;

$$\begin{aligned}\tilde{I}_2 &= \sqrt{n_2}\left(\frac{d_n}{2\pi}\right)^{\frac{1}{2}} \int_{|t|<n_2^{-\lambda}} \exp\Big\{-n_2 G_n(t)\Big\}\frac{\tau_n}{\tau_n + it}dt \\ &= \sqrt{n_2}\left(\frac{d_n}{2\pi}\right)^{\frac{1}{2}} \int_{|t|<n_2^{-\lambda}} \exp\Big\{-n_2 G_n(t)\Big\}dt \\ &\quad + (-)\sqrt{n_2}\left(\frac{d_n}{2\pi}\right)^{\frac{1}{2}} \int_{|t|<n_2^{-\lambda}} \exp\Big\{-n_2 G_n(t)\Big\}\frac{it}{\tau_n + it}dt \\ &= \tilde{K}_{1n} + \tilde{K}_{2n} \quad \text{(say)}. \end{aligned} \tag{2.17}$$

The fact that $\tilde{K}_{1n} = K_{1n}$ is $1 + O(n_1^{-1})$ is proved in Lemma 2.3. It is left us to show that $\tilde{K}_{2n}$ is $O(n_1^{-1})$. Now,

$$\begin{aligned}|\tilde{K}_{2n}| &\le \left(\frac{d_n}{2\pi}\right)^{\frac{1}{2}} \int_{|s|<n_2^{-\lambda+\frac{1}{2}}} \exp\Big(-\frac{s^2}{2}d_n\Big)\Big\{1 + z_n + \tilde{L}_n(s)\Big\} \left|\frac{s/\sqrt{n_2}}{\tau_n + is/\sqrt{n_2}}\right| ds \\ &\le \left|\left(\frac{d_n}{2\pi}\right)^{\frac{1}{2}} \int_{|s|<n_2^{-\lambda+\frac{1}{2}}} \exp\Big(-\frac{s^2}{2}d_n\Big)\Big\{1 + z_n + \tilde{L}_n(s)\Big\}\frac{s}{\sqrt{n_2}\tau_0}ds\right| \\ &\le \left|\left(\frac{d_n}{2\pi}\right)^{\frac{1}{2}} \int_{|s|<n_2^{-\lambda+\frac{1}{2}}} \exp\Big(-\frac{s^2}{2}d_n\Big)\Big[1 + i\frac{n_1 s^3}{6n_2\sqrt{n_2}}\psi'''_{n_1}(\tau_n)\right. \\ &\quad + i\frac{r_n^3 s^3}{6\sqrt{n_2}}\psi'''_{n_2}(-r_n\tau_n) + n_1 R_1\Big(\tau_n + i\frac{s}{\sqrt{n_2}}\Big) \\ &\quad \left. + n_2 R_2\Big\{-r_n\Big(\tau_n + i\frac{s}{\sqrt{n_2}}\Big)\Big\} + \tilde{L}_n(s)\Big]\frac{s}{\sqrt{n_2}\tau_0}ds\right|, \end{aligned} \tag{2.18}$$

where

$$\tilde{L}_n(s) = \exp(z_n) - z_n - 1$$

and

$$z_n = -i\frac{n_1 s^3}{6n_2\sqrt{n_2}}\psi'''_{n_1}(\tau_n) + i\frac{r_n^3 s^3}{6\sqrt{n_2}}\psi'''_{n_1}(\tau_n)$$
$$+ n_1 R_1\Big(\tau_n + i\frac{s}{\sqrt{n_2}}\Big) + n_2 R_2\Big\{-r_n\Big(\tau_n + i\frac{s}{\sqrt{n_2}}\Big)\Big\}.$$

The r.h.s. of (2.18) can be written as the sum of six integrals. The first integral equals zero and the second and the third integrals are $O(n_2^{-1})$.
Since

$$\Big|\Big(\frac{d_n}{2\pi}\Big)^{\frac{1}{2}} \int_{|s|<n_2^{-\lambda+\frac{1}{2}}} \exp\Big(-\frac{s^2}{2}d_n\Big)\Big[n_1 R_1(\tau_n + i\frac{s}{\sqrt{n_2}})$$
$$+ nR_2\Big\{-r_n\Big(\tau_n + i\frac{s}{\sqrt{n_2}}\Big)\Big\} + \tilde{L}_n(s)\Big]ds\Big| \tag{2.19}$$

is $O(n_1^{-1})$ (Cho (1991)), we obtain

$$\Big|\Big(\frac{d_n}{2\pi}\Big)^{\frac{1}{2}} \int_{|s|<n_2^{-\lambda+\frac{1}{2}}} \exp\Big(-\frac{s^2}{2}d_n\Big)\Big[n_1 R_1\Big(\tau_n + i\frac{s}{\sqrt{n_2}}\Big)$$
$$+ n_2 R_2\Big\{-r_n\Big(\tau_n + i\frac{s}{\sqrt{n_2}}\Big)\Big\} + \tilde{L}_n(s)\Big]\frac{s}{\tau_0\sqrt{n_2}}ds\Big|$$
$$\le \frac{n_2^{-\lambda}}{\tau_0}\Big|\Big(\frac{d_n}{2\pi}\Big)^{\frac{1}{2}} \int_{|s|<n^{-\lambda+\frac{1}{2}}} \exp\Big(-\frac{s^2}{2}d_n\Big)\Big[n_1 R_1(\tau_n + i\frac{s}{\sqrt{n_2}})$$
$$+ n_2 R_2\Big\{-r_n\Big(\tau_n + i\frac{s}{\sqrt{n_2}}\Big)\Big\} + \tilde{L}_n(s)\Big]ds\Big|$$
$$\le O(n_1^{-1-\lambda}). \tag{2.20}$$

So the proof is completed.

3. SOME REMARKS

Remark 3.1. If $n_1 \ge n_2$, we can obtain a similar strong large deviation result of Theorem 3.1′.

Theorem 2.1′. Assume that the conditions (A), (B), (C), (D) and (E) in Theorem 2.1 and the following condition (F′) hold.
(F′) There exist $l > 0$ and $p > 0$ such that

$$\int \left|\frac{\phi_{n_1}(\tau + it)}{\phi_{n_1}(\tau)}\right|^{l/n_1} \cdot \left|\frac{\phi_{n_2}(-r(\tau + it))}{\phi_{n_2}(-r\tau)}\right|^{l/n_1} \times \left|\frac{\psi'_{n_2}(-r(\tau + it))}{\psi'_{n_2}(-r\tau)}\right| dt = O(n_1^p), \quad \text{for } \tau \in J_1 \quad \text{and } r\tau \in J_2.$$

Then the tail probability of $R_{n_1,n_2} = U_{n_1}/S_{n_2}$ is given by

$$\bar{H}_{n_1,n_2}(r_n) = \frac{\exp\{n_1\psi_{n_1}(\tau_n) + n_2\psi_{n_2}(-r_n\tau_n)\}}{\tau_n\sqrt{2\pi\{n_1\psi''_{n1}(\tau_n) + n_2 r_n^2\psi''_{n_2}(-r_n\tau_n)\}}} \times \left\{1 + O(n_2^{-1})\right\}. \tag{3.1}$$

Remark 3.2. In Theorem 2.1, if $n_1 = n_2 = n$, then as $n \to \infty$, the tail probability of $R_n = U_n/S_n$ is given by

$$\bar{H}_n(r_n) = \frac{\exp[n\{\psi_{n_1}(\tau_n) + \psi_{n_2}(-r_n\tau_n)\}]}{\tau_n\sqrt{2\pi n\{\psi''_{n_1}(\tau_n) + r_n^2\psi''_{n_2}(-r_n\tau_n)\}}} \times \left\{1 + O(n^{-1})\right\}. \tag{3.2}$$

Remark 3.3. In Theorem 2.1, if $n_1 = n_2 = n$ and $S_n = n$, then as $n \to \infty$, the tail probability of $R_n = U_n/n$ is given by

$$\bar{F}_n(m_n) = \frac{\exp[n\{\psi_n(\tau_n) - \tau_n m_n\}]}{\tau_n\sqrt{2\pi n\psi''_n(\tau_n)}} \times \left\{1 + O(n^{-1})\right\}. \tag{3.3}$$

Remark 3.4. In Theorem 2.1, if $n_1 = n_2 = n, S_n = n$ and $U_n = \sum_{i=1}^n X_i$, where X_i's are i.i.d. random variables with $\phi(z) = E\{\exp(zX_1)\}$, then as $n \to \infty$, the tail probability of $R_n = \bar{X}$ is given by

$$\bar{F}_n(x_n) = \frac{\exp[n\{\psi(\tau_n) - \tau_n x_n\}]}{\tau_n\sqrt{2\pi n\psi''_n(\tau_n)}} \times \left\{1 + O(n^{-1})\right\}, \tag{3.4}$$

where $\psi(z) = log\phi(z)$ and $\psi'(\tau_n) = x_n$.

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