Journal of the Korean
Statistical Society
Vol. 23, No. 2, 1994

Effects of Temporal Aggregation on Hannan-Rissanen Procedure

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ABSTRACT

Effects of temporal aggregation on estimation for ARMA models are studied by investigating the Hannan & Rissanen(1982)'s procedure. The temporal aggregation of autoregressive process has a representation of an autoregressive moving average. The characteristic polynomials associated with autoregressive part and moving average part tend to have roots close to zero or almost identical. This causes a numerical problem in the Hannan & Rissanen procedure for identifying and estimating the temporally aggregated autoregressive model. A Monte-Carlo simulation is conducted to show the effects of temporal aggregation in predicting one period ahead realization.

KEYWORDS: Temporal aggregation, Hannan & Rissanen estimation, Unidentifiability, Prediction mean squares error.

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1. INTRODUCTION

Time series are often obtained in temporally aggregated form. For example, when economic theory suggests a daily model for an economic variable, the economic data may be available only in weekly totals. Also, while an engineering model for a variable needs to be on second basis, the engineering variable may be observed only on minute basis because of cost consideration. In these cases, instead of the original time series, say x_t, temporally aggregated series

$$y_1 = x_1 + \ldots + x_m, y_2 = x_{m+1} + \ldots + x_{2m}, \ldots, y_n = x_{nm-m+1} + \ldots + x_{nm} \quad (1.1)$$

are available for some positive integer m, called order of aggregation.

Among the important contributions to the identification of temporally aggregated autoregressive(AR) process are Telser (1967), Ameniya & Wu (1972), Brewer (1973), Abraham (1982), Tiao (1987), Stram & Wei (1986), Nijman & Palm (1990), Wei & Stram (1990), Shin & Kim (1993), Shin & Pantula (1993), and references in Wei & Stram (1990). In almost all the above works, they do not handle estimation for temporally aggregated series. Telser (1967) and Nijman & Palm (1990) report difficulties in identifying and estimating temporally aggregated autoregressive moving average(ARMA) process. Shin & Kim (1993) study estimation for unit root in temporally aggregated first order autoregressive process.

Hannan & Rissanen (1982) propose a three-stage procedure for identifying and estimating ARMA process. For analysis of temporal aggregation, we select the Hannan & Rissanen as an estimation procedure because each stage of the Hannan & Rissanen's procedure reveals many aspects of estimating ARMA process. Many authors have been interested in the Hannan and Rissanen (1982), among others Poskitt (1987), Hannan and Kavalieris (1984). We study the effect of temporal aggregation on Hannan & Rissanen procedure in estimating and identifying temporally aggregated model from data y_t when the original process x_t is a stationary $AR(p)$ process. It is well known that y_t has an ARMA(p, q) representation with $q = [(p+1)(m-1)/m]$, the largest integer not greater than $(p+1)(m-1)/m$. However, if we estimate ARMA(p, q)

model for y_t, we confront a severe numerical instability problem because some roots of the characteristic equation associated with autoregressive part and those with moving average part are both close to zero and / or are nearly identical. In fact, Stram and Wei (1986) show that there are cases in which the ARMA(p, q) representation can be reduced to lower order ARMA model. We study an estimation procedure which avoids the instability problem. The procedure adopts the Hannan & Rissanen(1982) strategy which is an order selection rule for ARMA process. If a temporally aggregated AR series has both autoregressive roots and moving average roots close to zero, or if roots corresponding to autoregressive part and moving average part are nearly identical, the order selection rule automatically drops some variables associated with those roots.

In a Monte-Carlo experiment, we compare the prediction mean squares errors of two predictors for y_{n+1}. One predictor is computed from the true ARMA(p, q) model with the true parameter values. The other predictor is computed from the model selected by Hannan & Rissanen procedure in which the parameters are estimated. We show that the procedure yields predictor comparable to that obtained from the true model.

In section 2, a model for temporally aggregated autoregressive process is presented. In section 3, effects of aggregation on estimation and prediction are discussed. In section 4, a procedure for estimation of temporally aggregated autoregressive process is proposed. In section 5, a result of Monte-Carlo simulation is reported.

2. MODEL FOR TEMPORAL AGGREGATION

Consider

$$x_t + \phi_1 x_{t-1} + \phi_2 x_{t-2} + \cdots + \phi_p x_{t-p} = \epsilon_t, \tag{2.1}$$

where $\{\epsilon_t\}$ is an iid $(0, \sigma_\epsilon^2)$ sequence. We assume that the roots $\lambda_1, \ldots, \lambda_p$ of the characteristic equation

$$\lambda^p + \phi_1 \lambda^{p-1} + \cdots + \phi_p = 0 \tag{2.2}$$

lie inside the unit circle. This means that $\{x_t\}$ is stationary if a stationarity condition on x_0 is imposed.

In Harvey(1981, p44), we know that $y_t = \sum_{i=1}^{m} x_{tm-m+i}$ has a representation of an ARMA(p,q) process. In fact, one may easily show that y_t has the following ARMA(p,q) representation

$$y_t + \alpha_1 y_{t-1} + \cdots + \alpha_p y_{t-p} = e_t + \beta_1 e_{t-1} + \cdots + \beta_q e_{t-q}, t = 1, \ldots, n \quad (2.3)$$

where $\{e_t\}$ is an iid(0, σ_e^2) process, the autoregressive coefficients $(\alpha_1, \ldots, \alpha_p)$ are coefficients of $(B, \ldots, B^p)$ in

$$1 + \sum_{i=1}^{p} \alpha_i B^i = (1 - \lambda_1^m B)(1 - \lambda_2^m B) \cdots (1 - \lambda_p^m B), \quad (2.4)$$

the moving average coefficients $(\beta_1, \beta_2, \ldots, \beta_q)$ are determined by

$$\sigma_e^2 \sum_{i=0}^{q-h} \beta_{i+h} \beta_i = \sigma_\epsilon^2 \sum_{i=0}^{r-hm} a_{i+hm} a_i, h = 0, \ldots, q, \quad (2.5)$$

$r = (p+1)(m-1)$ and $(a_1, \ldots, a_r)$ is the coefficient of $(B^1, \ldots, B^r)$ in

$$1 + \sum_{i=1}^{r} a_i B^i = (1 + B + \cdots + B^{m-1}) \Pi_{j=1}^{p} (1 + \lambda_j B + \cdots + \lambda_j^{m-1} B^{m-1}). \quad (2.6)$$

See Theorem 2.1 of Shin & Pantula(1993). Amemiya & Wu(1972) point out that the moving average coefficients can be chosen in the invertibility region, that is, the region in which all the moving average roots associated with (2.3) have absolute values less than one. In fact, by Wilson(1969) which can be found in Box & Jenkins (1976, p203), we can compute such β's in the invertibility region.

3. EFFECTS OF AGGREGATION ON ESTIMATION AND PREDICTION

In this section, we describe the effects of aggregation through AR(2) model of $\{x_t\}_1^{nm}$. In estimating the model for $\{y_t\}_1^n$, one may try maximum likelihood estimation for the ARMA(p,q) model of y_t. However, as seen later, the ARMA(p,q) fit for y_t contains an unidentifiability or near unidentifiability problem. Let us consider an AR(2) process for $\{x_t\}$

$$(1-\lambda_1 B)(1-\lambda_2 B)x_t = \epsilon_t. \tag{3.1}$$

From (2.3), the temporal aggregate y_t has an ARMA(2,2) representation

$$(1-\nu_1 B)(1-\nu_2 B)y_t = (1-\xi_1 B)(1-\xi_2 B)e_t, \tag{3.2}$$

for some ν_1, ν_2, ξ_1, ξ_2. In Table 1, we tabulate ν_1, ν_2, ξ_1, ξ_2 for λ_1, $\lambda_2 \in \{0.8, 0.4, 0.0, -0.4, -0.8\}$ and $m = \{3, 6, 7, 12\}$ computed from (2.4) - (2.6) for $p = 2$.

As seen in Table 1, there are many cases in which both ν_1 and ξ_2 are close to zero or almost same for $m = 6, 7, 12$. Moreover, even in those cases the other autoregressive root ν_2 has near zero root. However, this phenomena are unclear for $m = 3$. This causes the parameters associated with the roots, which are close to zero, almost unidentifiable. Also, one of the autoregressive roots is almost same as ξ_2 for $\lambda_1 = -\lambda_2$ and even m. Hence this cancellation effect makes the model (3.2) reduce to ARMA(1,1). In general, if $\lambda_1^m = \lambda_2^m$ and $\lambda_1 \neq \lambda_2$, then the orders for y_t can be reduced to lower orders(See Stram and Wei(1986)). Therefore, estimation of the roots becomes hard. In both cases, if we estimate ARMA(2,2) model, we are confronted with numerical instability caused by unidentifiability or near unidentifiability.

For several AR(2) processes of x_t, we compare three estimation procedures for y_t's through the prediction of y_{n+1}; ARMA(2,2) fitting using estimated parameters, ARMA(1,1) fitting using estimated parameters, and ARMA(2,2) fitting using true parameters obtained from the roots in Table 1. Corresponding predictions of y_{n+1} are $\tilde{y}_{n+1}$, $\bar{y}_{n+1}$, $\hat{y}_{n+1}$, which are given below for some m, n, λ_1 and λ_2. In estimating the parameters of the ARMA(2,2) model and ARMA(1,1) model, we use the estimator from the second stage of Hannan & Rissanen procedure with $p = 2$ and $q = 2$. See stage 2 in section 4.

Table 2. The forecasts of y_{n+1} for several simulated series y_t's

m	n	λ_1	λ_2	y_{n+1}	$\tilde{y}_{n+1}$	$\bar{y}_{n+1}$	$\hat{y}_{n+1}$
3	25	.8	.4	−12.4800	-518.5286×10^3	−10.5569	−2.5624
6	50	.4	.8	3.1817	-728.1209×10^7	4.2133	0.7333
7	100	−.4	−.8	−1.5643	438.3446×10^5	−0.0520	0.5702
12	25	.8	.8	25.5937	927.2959	29.3678	13.7317
12	100	−.8	.8	7.3593	-495.9974×10^{15}	0.9571	0.2288

From the above example, we know that $\tilde{y}_{n+1}$ have very large forecast values. This dues to the fact that, as seen in Table 1, for example, for $(m, n, \lambda_1, \lambda_2) = (3, 25, .8, .4)$, one of the AR roots and one of the MA roots are 0.0640 and −0.0132, respectively and hence are all close to zero. Therefore, the true model for $\{y_t\}$ is almost ARMA(1,1). If we try to estimate ARMA(2.2) model for $\{y_t\}$, the parameter estimate would have very large variance and hence estimated AR roots or MA roots fall outside the stationrity or invertibility region making $\tilde{y}_{n+1}$ very large. It seems that the reduced order model provides reasonable prediction for y_{n+1}. We will discuss these facts in more detail in section 5, a Monte-Carlo simulation. Therefore, estimating a lower order ARMA model gets around the unidentifiability problem and the numerical problem when we have near cancellation or near zero in the characteristic roots. However, when we have neither near cancellation nor near zero in the roots, the Hannan & Rissanen procedure would automatically select ARMA(2,2), which would give good forecast of y_{n+1}.

In particular, we have the following theorem which tells us that the temporally aggregated series become a white noise as m increases.

Theorem 1. Let x_t and y_t be defined in (2.1) and (1.1) respectively. Assume that all the roots $\lambda_1, \ldots, \lambda_p$ of the characteristic polynomial (2.2) lie inside the unit circle. Then the autocovariance function of y_t tends to that of white noise as m goes to infinity.

Proof. Note that all the roots $\lambda_1^m, \ldots, \lambda_p^m$ of the characteristic polynomial associated with autoregressive part of y_t decay to zero as m increases. Also from (2.6), $a_1, \ldots, a_{m-1}$ all tend to one and $a_m, \ldots, a_r$ all tend to zero. Therefore, equation (2.5) becomes

$$\sigma_e^2 \sum_{i=0}^{q} \beta_i^2 = \sigma_\epsilon^2, \sigma_e^2 \sum_{i=0}^{q-h} \beta_{i+h}\beta_i = 0, h = 1, 2, \ldots, q.$$

Hence, $\beta_1, \ldots, \beta_q$ all tend to zero and the theorem follows.

4. ESTIMATION FOR TIME SERIES MODEL OF TEMPORAL AGGREGATION

In section 3, we see that there are many situations in which a low order ARMA model is a good approximation to the true ARMA(p, q) model for y_t. However, the question that to what extent the orders should be reduced remains. Actually, the orders are affected by the values $\lambda_1, \ldots, \lambda_p$ and the order of aggregation, m.

We investigate the Hannan & Rissanen (1982) with Kavalieris(1991)'s modification. The procedure, by dropping insignificant parameters, avoids instability problem and yields smaller prediction mean squares error than that based on full orders model estimation. We state the procedure below.

Stage 1. An autoregression of order k is estimated by regressing y_t on $-y_{t-1}, \ldots, -y_{t-k}$ and applying the Akaike information criterion. The autoregressive coefficients $\bar{\phi}_j(k)$ of $-y_{t-j}, j = 1, \ldots, k$, are used to estimate the innovations

$$\bar{e}_t(k) = \sum_{j=0}^{k} \bar{\phi}_j(k) y_{t-j},$$

$t = 1, 2, \ldots, n$, where $\bar{\phi}_0(k) = 1$, $y_t = 0$, $t \leq 0$.

Stage 2. For each $(p,\ q)$, estimate the parameters $\{\bar{\alpha}_j,\ j = 1, \ldots, p\}$ and $\{\bar{\beta}_j,\ j = 1, \ldots, q\}$, by regressing y_t on $\{y_{t-j},\ j = 1, \ldots, p\}$ and $\{\bar{e}_{t-j}(k_0),\ j = 1, \ldots, q\}$, where k_0 is the regression order obtained in Stage 1, $p \leq p_0$ and $q \leq q_0$, p_0 is the true value of p and $q_0 = [(p_0+1)(m-1)/\ m]$. These parameters are used to determine order (p, q) of model by an information criterion, BIC below.

Stage 3. With $(p,\ q)$ selected in Stage 2, apply the maximum likelihood estimation for model (2.3) and compute estimates of $\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q$. In Stage 1, we select the order k by minimizing the Akaike information criterion

$$AIC(k) = \log \bar{\sigma}_k^2 + 2k\ /\ n, \bar{\sigma}_k^2 = \sum_{t=1}^{n} \bar{e}_t^2(k)/\ n.$$

In Stage 2, we choose model order (p, q) which minimizes the information criterion

$$BIC(s_{p,q}^2) = \log\ s_{p,q}^2 + (p+q) \log\ n\ /\ n,$$

where $s_{p,q}^2$ is an estimate of residual variance σ_e^2 from the regression in stage 2.

Hannan & Kavalieris(1984) show that the use of Hannan & Rissanen's estimator of σ_e^2 tends to introduce common factors into the estimates of autoregressive polynomial and moving average polynomial. Such an overparametrized model may also lead to instability in the stage 3 of the procedure, the maximum likelihood estimation stage. Hannan & Kavalieris(1984), Kavalieris(1991), and many others propose a number of modifications to the order estimation procedure in order to overcome the overparametrization. We use Kavalieris's modification $\bar{\sigma}_{p,q}^2 = \sum_{t=1}^{n} \bar{e}_t^2/\ n$, where

$$\bar{e}_t = y_t + \bar{\alpha}_1 y_{t-1} + \cdots + \bar{\alpha}_p y_{t-p} - \bar{\beta}_1 \bar{e}_{t-1} - \cdots - \bar{\beta}_q \bar{e}_{t-q},$$

$t = 1, 2, \ldots, n$, $\bar{e}_t = 0$, $t \leq 0$. Kavalieris's estimator produces less overestimation of order (p, q).

The procedure underestimates the order in the aggregated autoregressive process because some roots of the characteristic polynomials of the aggregated process are close to zero or nearly canceled out. The procedure, by dropping insignificant parameters, avoids the instability problem and yields smaller prediction mean squares error than the predictor based on full model estimation.

This study indicates that we should also consider small model order for estimating temporally aggregated ARMA when we use methods such as the maximum likelihood estimation and the least squares estimation.

5. MONTE CARLO SIMULATION

In this section, we analyze Hannan & Rissanen procedure by a Monte Carlo experiment. Let x_t be an AR(2) process given in equation (3.1) and y_t be a temporal aggregate of x_t of order m. Assume we have $\{y_1, \ldots, y_n\}$ as a data set, from which we estimate model for y_t. Note that the true model for y_t is (3.2).

We first compare two estimation procedure; ARMA(p, q) fitting $p = 2$ with $q = q_0 = [3(m-1)/m]$, and our estimation procedure with $p \leq 2$ and $q \leq q_0$. We consider λ_1, $\lambda_2 \in \{.8, .4, .0, -.4, -.8\}$. Since λ_1 and λ_2 have modulus less than one, we expect that a good procedure produces estimates of autoregressive roots ν_1 and ν_2 and moving average roots ξ_1 and ξ_2 inside the unit circle. In Table 3, we report numbers of cases in which some estimates of ν_1, ν_2, ξ_1, ξ_2 have modulus greater than or equal to one based on 1000 replications. In the cases, the parameter estimates are outside the stationarity region($|\nu_1| \geq 1$ or $|\nu_2| \geq 1$) or the invertibility region($|\xi_1| \geq 1$ or $|\xi_2| \geq 1$). The error sequence $\{\epsilon_t\}$ is simulated by RNNOA of IMSL library. The error variance σ_ϵ^2 is set to 1. In Hannan & Rissanen approach, the estimates are from the second stage with $p \leq 2$ and $q \leq q_0$.

In Table 3, ARMA(2, q_0) fitting for $(\lambda_1, \lambda_2) = (.8, .8)$ and $(-.8, -.8)$ with odd m yields very large number of cases outside the stationarity or invertibility region. The number of these cases decreases as the order m of aggregation increases, but for each m it tends to increase as the number n of observations of y_t increases. For the above (λ_1, λ_2) combination, the approach reduces it dramatically as m increases for each n. For $n = 100$ and $m = 12$, the number reduces from 274 of ARMA(2, q_0) fitting to 9 of the fitting based on Hannan & Rissanen procedure. Also, for $(\lambda_1, \lambda_2) = (.8, .4)$ and $(.4, .8)$ the proportion of the reduction tends to increase as m increases for each n. In case that the autoregressive roots and the moving average roots are both close to zero or almost same, the approach yields much less cases outside the stationarity or invertibility region than ARMA(2, q_0) fitting. In fact, the ARMA(2, q_0) fitting causes unidentifiability problem and so the estimators from ARMA(2, q_0) fitting are very unstable. For $\lambda_1^m = \lambda_2^m$ and $\lambda_1 \neq \lambda_2$, as Stram and Wei (1986) indicates, one of $\{\lambda_1, \lambda_2\}$ coincides with one of $\{\xi_1, \xi_2\}$ and hence our procedure is much better than ARMA(2, q_0) fitting. It is more clear as m increases.

Hannan & Rissanen approach tends to underestimate the orders of ARMA model for y_t. However, the underestimated model is also a good estimation for the true model. We investigate the performance of the model estimated from Hannan & Rissanen procedure by studying the behavior of the predictors of y_{n+1}. Denote the model for y_t estimated from the procedure by

$$y_t + \bar{\alpha}_1 y_{t-1} + \bar{\alpha}_2 y_{t-2} = e_t + \bar{\beta}_1 e_{t-1} + \bar{\beta}_2 e_{t-2}.$$

A predictor for y_{n+1} is

$$\bar{y}_{n+1} = -\bar{\alpha}_1 y_n - \bar{\alpha}_2 y_{n-1} + \bar{\beta}_1 \bar{e}_n + \bar{\beta}_2 \bar{e}_{n-1},$$

$t = 1, 2, \ldots, n$, where

$$\bar{e}_t = y_t + \bar{\alpha}_1 y_{t-1} + \bar{\alpha}_2 y_{t-2} - \bar{\beta}_1 \bar{e}_{t-1} - \bar{\beta}_2 \bar{e}_{t-2},$$

$t = 1, \ldots, n$, and $y_t = \bar{e}_t = 0$ for $t \leq 0$. For simplicity of computation, we use $(\bar{\alpha}_1,\ \bar{\alpha}_2,\ \bar{\beta}_1,\ \bar{\beta}_2)$ obtained in the second stage of the procedure instead of the estimator in the third stage. Note that, for example, $\bar{\beta}_2 = 0$ if order selected for moving average part is less than 2. For comparison, we consider another predictor based on the true model. Let $\hat{y}_{n+1}$ be a predictor of y_{n+1} based on the true model for y_t,

$$\hat{y}_{n+1} = -\alpha_1 y_n - \alpha_2 y_{n-1} + \beta_1 \hat{e}_n + \beta_2 \hat{e}_{n-1},$$

where

$$\hat{e}_t = y_t + \alpha_1 y_{t-1} + \alpha_2 y_{t-2} - \beta_1 \hat{e}_{t-1} - \beta_2 \hat{e}_{t-2},$$

$t = 1, \ldots, n$, and $y_t = \hat{e}_t = 0$ for $t \leq 0$. Note that we use the true value of $(\alpha_1,\ \alpha_2,\ \beta_1,\ \beta_2)$ in computing $\hat{y}_{n+1}$. We do not study the predictor based on parameters of ARMA(2, q_0) because as seen in Table 2, the predictor is very wild and hence we do not need to include in our comparative study.

In Table 4, we report sample mean squares error of the predictors based on 1000 replications. The sample mean squares error for $\bar{y}_{n+1}$ is smaller or slightly larger than that for $\hat{y}_{n+1}$ except $n = 25$. When the number of observation of the aggregated series y_t is small ($n = 25$), the underlined nineteen cases out of 100 combinations of (λ_1, λ_2) have large sample mean squares errors. This is because there are few samples which produces very large predictor which makes the sample mean squares error very large. However, except the underlined cases, the sample mean squares error for $\bar{y}_{n+1}$ is not much larger than that for $\hat{y}_{n+1}$. Recalling that the mean squares error for $\hat{y}_{n+1}$ is based on true model, we see that $\bar{y}_{n+1}$ is a reasonable predictor of y_{n+1}. From this investigation we know that the model estimated by Hannan & Rissanen procedure is a good approximation to the true model even though the estimated model have underestimated orders.

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Table 1. Autoregressive roots (ν_1, ν_2) and moving average roots (ξ_1, ξ_2) of temporally aggregated series y_t of order m when x_t has an AR(2) process with roots (λ_1, λ_2).

autoregressive roots

m	λ_1 \ λ_2	ν_1: .8	.4	.0	−.4	−.8	ν_2: .8	.4	.0	−.4	−.8
3	.8	.5120	.0640	.0000	−.0640	−.5120	.5120	.5120	.5120	.5120	.5120
3	.4	.0640	.0640	.0000	−.0640	−.5120	.5120	.0640	.0640	.0640	.0640
3	.0	.0000	.0000	.0000	−.0640	−.5120	.5120	.0640	.0000	.0000	.0000
3	−.4	−.0640	−.0640	−.0640	−.0640	−.5120	.5120	.0640	.0000	−.0640	−.0640
3	−.8	−.5120	−.5120	−.5120	−.5120	−.5120	.5120	.0640	.0000	−.0640	−.5120
6	.8	.2615	.0041	.0000	.0041	.2615	.2628	.2621	.2621	.2621	.2628
6	.4	.0041	.0041	.0000	.0041	.0041	.2621	.0041	.0041	.0041	.2621
6	.0	.0000	.0000	.0000	.0000	.0000	.2621	.0041	.0000	.0041	.2621
6	−.4	.0041	.0041	.0000	.0041	.0041	.2621	.0041	.0041	.0041	.2621
6	−.8	.2615	.0041	.0000	.0041	.2615	.2628	.2621	.2621	.2621	.2628
7	.8	.2091	.0016	.0000	−.0016	−.2097	.2103	.2097	.2097	.2097	.2097
7	.4	.0016	.0016	.0000	−.0017	−.2097	.2097	.0016	.0016	.0017	.0016
7	.0	.0000	.0000	.0000	−.0016	−.2097	.2097	.0016	.0000	.0000	.0000
7	−.4	−.0016	−.0017	−.0016	−.0016	−.2097	.2097	.0017	.0000	−.0016	−.0016
7	−.8	−.2097	−.2097	−.2097	−.2097	−.2103	.2097	.0016	.0000	−.0016	−.2091
12	.8	.0681	.0000	.0000	.0000	.0681	.0693	.0687	.0687	.0687	.0693
12	.4	.0000	.0000	.0000	.0000	.0000	.0687	.0000	.0000	.0000	.0687
12	.0	.0000	.0000	.0000	.0000	.0000	.0687	.0000	.0000	.0000	.0687
12	−.4	.0000	.0000	.0000	.0000	.0000	.0687	.0000	.0000	.0000	.0687
12	−.8	.0681	.0000	.0000	.0000	.0681	.0693	.0687	.0687	.0687	.0693

moving average roots

m	λ_1 \ λ_2	ξ_1: .8	.4	.0	−.4	−.8	ξ_2: .8	.4	.0	−.4	−.8
3	.8	−.2632	−.3199	−.2138	−.0998 − .07i	−.2256 − .16i	−.0366	−.0132	.0000	−.0998 + .07i	−.2256 + .16i
3	.4	−.3199	−.1977	−.1394	−.0560 − .07i	−.1522 − .11i	−.0132	−.0119	.0000	−.0560 + .07i	−.1522 + .11i
3	.0	−.2138	−.1394	.0000	.0000	.0000	.0000	.0000	.0000	.0716	.0147
3	−.4	−.0998 − .07i	−.0560 − .07i	.0000	.0562 − .10i	.1753 − .10i	−.0998 + .07i	−.0560 + .07i	.0716	.0562 + .10i	.1753 + .10i
3	−.8	−.2256 − .16i	−.1522 − .11i	.0000	.1753 − .10i	.0217 − .21i	−.2256 + .16i	−.1522 + .11i	.0147	.1753 + .10i	.0217 + .21i
6	.8	−.1212	−.2827	−.2300	−.2072	−.1951	−.0113	−.0088	.0000	.0015	.2621
6	.4	−.2827	−.1446	−.0902	−.0571	−.0553	−.0088	.0009	.0000	.0041	.2927
6	.0	−.2300	−.0902	.0000	.0000	.0000	.0000	.0000	.0000	.0724	.4006
6	−.4	−.2072	−.0571	.0000	.0045	−.0481	.0015	.0041	.0724	.2471	.6020
6	−.8	−.1951	−.0553	.0000	−.0481	.0588	.2621	.2927	.4006	.6020	.3819
7	.8	−.1268	−.2682	−.2255	−.2089	−.1971 − .08i	.0090	−.0064	.0000	−.0033	−.1971 + .08i
7	.4	−.2682	−.1278	−.0773	−.0492	−.0902 − .05i	−.0064	.0007	.0000	−.0024	−.0902 + .05i
7	.0	−.2255	−.0773	.0000	.0000	.0000	.0000	.0000	.0000	.0586	.0570
7	−.4	−.2089	−.0492	.0000	−.0031	.0597	−.0033	−.0024	.0586	.1713	.3122
7	−.8	−.1971 − .08i	−.0902 − .05i	.0000	.0597	−.1044	−.1971 + .08i	−.0902 + .05i	.0570	.3122	.1610
12	.8	−.1581	−.2071	−.1882	−.1824	−.1777	.0240	−.0013	.0000	−.0002	.0687
12	.4	−.2071	−.0725	−.0432	−.0289	−.0248	−.0013	.0000	.0000	.0000	.0902
12	.0	−.1882	−.0432	.0000	.0000	.0000	.0000	.0000	.0000	.0368	.1910
12	−.4	−.1824	−.0289	.0000	.0001	−.0136	−.0002	.0000	.0368	.1279	.4027
12	−.8	−.1777	−.0248	.0000	−.0136	.0653	.0687	.0902	.1910	.4027	.5639

$i^2 = -1$.

Table 3. Number of cases in which parameter estimates are outside the stationarity or invertibility region out of 1,000 replications

ARMA(2, q_0) Fitting

m	λ_2 / λ_1	.8	.4	.0 (n = 25)	−.4	−.8	.8	.4	.0 (n = 50)	−.4	−.8	.8	.4	.0 (n = 100)	−.4	−.8
3	.8	822	432	235	169	138	900	563	312	199	146	915	640	389	238	178
3	.4	393	156	104	105	94	536	172	94	86	75	652	200	77	77	119
3	.0	234	101	86	110	115	274	86	78	85	109	394	86	58	86	101
3	−.4	189	96	81	135	355	212	82	72	138	472	242	81	76	150	552
3	−.8	126	84	127	350	756	163	83	120	489	808	198	97	109	577	828
6	.8	510	180	166	169	152	638	255	228	166	172	653	294	237	223	193
6	.4	204	116	92	95	93	214	78	76	68	82	319	86	68	67	83
6	.0	169	89	91	91	93	171	81	86	72	74	226	77	58	75	75
6	−.4	164	79	78	87	146	171	84	80	86	138	215	62	52	93	129
6	−.8	159	85	100	144	200	167	92	90	130	182	209	64	87	153	127
7	.8	409	190	163	147	139	531	207	184	153	138	610	249	195	165	154
7	.4	185	107	80	101	83	223	88	89	80	65	254	64	58	64	60
7	.0	152	102	92	86	103	153	76	83	77	92	201	74	70	57	82
7	−.4	143	86	87	115	201	176	83	73	85	257	190	60	73	85	276
7	−.8	138	74	92	183	600	140	70	104	231	647	150	63	77	289	661
12	.8	203	114	118	115	123	212	118	105	87	106	274	109	104	104	97
12	.4	125	81	99	88	93	103	74	74	73	71	133	60	60	53	69
12	.0	123	92	88	90	108	109	69	91	73	83	112	77	69	58	68
12	−.4	127	76	85	85	121	126	78	76	76	102	96	62	60	65	134
12	−.8	113	89	89	142	203	95	74	80	136	192	94	62	71	159	122

Hannan & Rissanen Fitting

m	λ_2 / λ_1	.8	.4	.0 (n = 25)	−.4	−.8	.8	.4	.0 (n = 50)	−.4	−.8	.8	.4	.0 (n = 100)	−.4	−.8
3	.8	563	231	104	72	27	545	244	108	61	15	573	237	98	36	6
3	.4	227	63	14	17	10	218	27	10	8	7	272	5	0	0	3
3	.0	109	27	18	21	50	84	9	8	5	14	79	6	1	5	2
3	−.4	77	10	12	52	271	59	7	3	14	251	31	0	1	6	277
3	−.8	35	9	43	255	639	18	6	20	284	689	12	3	6	304	797
6	.8	297	87	66	53	66	293	58	35	33	32	305	26	8	9	4
6	.4	96	29	14	14	12	59	6	6	8	8	27	6	1	2	4
6	.0	65	18	19	21	20	43	7	0	5	7	14	1	1	1	1
6	−.4	65	12	8	23	38	34	5	3	6	13	7	3	1	3	6
6	−.8	59	15	20	44	62	28	6	2	11	56	11	1	3	4	50
7	.8	219	76	52	57	51	236	47	25	21	19	226	12	5	4	7
7	.4	75	19	14	12	17	41	3	5	3	1	12	0	1	0	1
7	.0	61	22	10	20	27	27	4	4	6	5	7	2	2	0	4
7	−.4	49	18	13	26	126	14	3	2	3	83	7	2	4	2	35
7	−.8	50	9	28	98	416	20	2	13	74	310	9	0	3	37	203
12	.8	99	27	34	22	25	42	9	11	7	5	9	4	6	3	5
12	.4	38	17	12	16	14	8	7	7	1	1	4	2	0	3	2
12	.0	25	15	27	18	18	8	3	6	6	4	1	1	1	2	3
12	−.4	27	11	8	11	44	10	7	5	4	12	4	2	1	2	3
12	−.8	29	11	12	35	95	5	2	4	9	50	1	1	3	5	19

Table 4. Sample mean square error of the predictors of y_{n+1} based on TRUE fitting and Hannan & Rissanen fitting with 1,000 replications

TRUE Fitting

m	λ_2 / λ_1	n = 25: .8	.4	.0	−.4	−.8	n = 50: .8	.4	.0	−.4	−.8	n = 100: .8	.4	.0	−.4	−.8
3	.8	77.80	29.22	13.04	7.28	7.86	64.76	28.30	13.39	8.40	7.63	66.82	29.84	13.55	7.56	7.97
3	.4	28.86	12.50	5.47	3.46	3.33	28.85	12.54	6.05	3.51	3.38	30.34	11.51	5.94	3.55	3.51
3	.0	13.07	5.91	2.94	2.01	2.02	13.49	6.25	3.02	2.04	2.10	12.63	5.81	3.06	2.03	2.08
3	−.4	7.35	3.39	2.17	1.86	3.23	6.85	3.52	1.92	1.81	3.38	7.82	3.47	2.05	1.69	3.67
3	−.8	7.49	3.49	2.19	3.48	5.73	8.04	3.26	2.36	3.40	6.12	8.00	3.32	2.15	3.49	5.78
6	.8	618.69	165.19	64.53	29.96	17.06	617.23	175.60	55.09	28.99	17.35	671.09	162.47	56.85	33.59	17.38
6	.4	151.66	37.11	13.28	7.51	5.55	169.21	34.82	13.79	7.48	5.47	155.78	35.62	14.91	7.52	5.73
6	.0	61.43	14.07	5.91	3.56	4.16	57.11	13.85	6.41	3.56	4.07	57.93	13.82	6.12	3.50	3.59
6	−.4	30.89	7.75	3.39	2.67	5.10	29.38	7.80	3.51	2.54	5.36	31.19	7.72	3.31	2.68	5.76
6	−.8	16.53	5.79	3.74	4.93	11.56	16.20	5.62	3.66	5.40	11.43	18.15	5.71	3.84	5.45	11.49
7	.8	991.34	241.29	79.01	41.00	31.91	999.95	228.00	82.64	41.45	30.04	1055.64	219.91	77.11	41.92	33.46
7	.4	212.78	43.41	15.98	9.37	6.60	222.45	41.20	16.85	9.36	6.52	221.67	42.03	16.63	9.04	6.70
7	.0	81.35	16.54	7.46	3.83	3.89	75.12	16.84	6.89	3.85	3.47	80.66	17.57	7.03	4.15	3.68
7	−.4	42.46	8.30	4.31	2.85	5.01	40.02	8.73	3.95	2.96	5.16	38.27	9.32	4.21	3.01	4.93
7	−.8	29.51	6.55	3.50	4.86	16.56	33.30	6.16	3.61	5.14	16.10	33.08	6.64	3.69	5.66	15.71
12	.8	3348.25	532.29	221.74	101.38	60.62	3919.65	541.07	191.05	97.26	58.82	3589.48	547.94	203.04	95.69	55.53
12	.4	508.00	86.23	33.87	16.67	9.87	527.96	79.08	31.33	15.96	11.07	565.71	76.88	29.29	16.14	11.10
12	.0	199.71	31.88	12.95	6.02	5.49	194.97	28.77	11.53	6.94	5.23	189.14	29.15	10.56	6.88	5.53
12	−.4	106.50	14.94	6.56	4.47	6.13	102.61	16.79	7.01	4.62	5.78	107.05	16.72	6.17	4.27	5.92
12	−.8	59.18	10.03	5.00	5.54	25.90	58.24	10.20	5.66	5.62	25.49	58.14	10.41	5.72	5.91	25.72

Hannan & Rissanen Fitting

m	λ_2 / λ_1	n = 25: .8	.4	.0	−.4	−.8	n = 50: .8	.4	.0	−.4	−.8	n = 100: .8	.4	.0	−.4	−.8
3	.8	93.65	26.12	13.70	7.54	6.82	55.02	23.51	12.91	7.96	5.88	56.42	24.35	12.78	7.20	5.44
3	.4	40.77	223.99	8.55	3.63	3.10	24.67	11.76	6.11	3.48	3.01	25.09	10.63	5.80	3.54	3.20
3	.0	13.18	6.21	49.89	2.15	2.14	12.97	6.21	3.09	2.57	2.19	11.26	5.49	3.09	2.05	2.11
3	−.4	9.37	3.50	2.24	1.96	6.82	6.72	3.52	1.97	1.80	2.74	7.60	3.45	2.04	1.64	2.66
3	−.8	6.05	3.17	2.50	2.91	303.05	5.51	2.84	2.47	2.85	10.00	5.78	2.91	2.17	2.87	4.57
6	.8	1046.43	157.98	78.32	30.81	56.48	570.22	155.86	52.74	28.25	18.33	587.23	141.78	52.92	32.44	17.78
6	.4	459.02	38.09	14.19	8.16	5.53	151.17	35.04	13.61	7.56	5.11	143.09	34.34	15.04	7.64	5.23
6	.0	197.83	24.33	6.34	3.93	3.08	53.88	14.21	6.56	3.60	3.25	53.36	13.72	6.17	3.52	2.68
6	−.4	53.00	8.48	3.51	2.63	8.78	28.43	8.07	3.55	2.46	3.11	27.68	7.76	3.30	2.44	3.11
6	−.8	18.11	5.64	3.10	298348.70	8.44	16.56	5.04	2.77	3.13	9.55	18.42	5.25	2.84	2.97	7.53
7	.8	1003.28	225.57	191.57	51.74	297.95	904.18	209.29	77.23	39.30	23.08	909.08	193.75	71.67	39.21	24.82
7	.4	227.91	48.15	17.43	10.22	6.89	199.13	40.87	17.10	9.39	6.33	201.92	40.87	16.51	9.08	11.26
7	.0	78.53	17.50	10.99	4.36	4.55	73.75	16.83	7.19	3.94	3.54	71.91	17.65	7.17	4.19	3.72
7	−.4	1809.07	8.94	6.74	3.05	30.01	38.12	8.94	3.93	2.88	4.09	35.70	9.45	4.31	2.88	3.56
7	−.8	25.86	6.66	3.71	7.35	21.07	25.26	6.04	3.60	4.10	12.04	25.15	6.31	3.64	4.20	10.90
12	.8	743160.8	553.80	293.51	104.75	65.29	3778.69	527.34	183.66	93.13	57.99	3297.44	494.55	191.01	92.60	54.94
12	.4	1019.30	99.18	35.79	17.64	10.51	496.09	82.46	32.55	16.36	11.34	538.22	77.63	29.51	16.17	11.01
12	.0	352.59	42.98	16.11	7.32	6.59	186.46	29.54	11.94	7.15	5.20	183.00	29.42	10.73	6.86	5.25
12	−.4	104.84	15.45	6.89	5.85	38.06	96.76	17.46	7.50	4.52	4.45	100.18	16.75	6.25	4.26	4.65
12	−.8	62.63	10.82	180.93	5.87	258.06	59.99	11.19	5.42	4.23	12.42	58.12	10.47	5.29	4.47	11.73