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An Economic Design of the $\bar{X}$-R Chart with Variable Sample Size Scheme†

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ABSTRACT

An economic design of the $\bar{X}$-R chart using variable sample size (VSS) scheme is proposed in this paper. In this design the sample size at each sampling time changes according to the values of the previous two sample statistics, sample mean and range. The VSS scheme uses large sample if the sample statistics appear near inside the control limits and smaller sample otherwise. The set of process parameters, such as the sampling interval, control limits and the sample sizes, are chosen to minimize the expected cost per hour.

The efficiency of the VSS scheme is compared to the fixed sample size one for cases where there is multiple of assignable causes. Percent reductions of the expected cost in the VSS design are calculated for some given sets of cost parameters. It is shown that the VSS scheme is economically better than the fixed sample size scheme in terms of the expected cost per hour. Also it is shown that the VSS scheme improves the confidence of the procedure and performs statistically better in terms of the number of false alarms and the average time to signal, respectively.

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KEYWORDS: Expected cost, Average number of samples, Average number of observations, Average number of false alarms, Average time to signal.

1. INTRODUCTION

Control charts have been widely used in monitoring shifts in quality characteristics that cause deterioration in the quality of the productions. When the mean and the variance are to be monitored simultaneously, the simple $\bar{X}$-R chart is used as a standard.

The $\bar{X}$-R chart is used by plotting the mean and range of the sample taken from the process in time order on the chart. The procedure of the chart is to give an out-of-signal if a sample mean and/or a sample range falls outside the control limits. The traditional sampling scheme with respect to the sample size is to take samples of fixed size which is called the fixed sample size(FSS) chart.

Recently the scheme of varying the sample size during the operation of the chart is introduced by Sawalapurkar et al(1990) and Park and Reynolds(1994). The chart in which the sample size is varied is called the variable sample size(VSS) chart. The idea is to use a large size for the next sample if the sample statistics used in the chart appear near inside the control limits and a small size otherwise to improve the efficiency of the chart. In designing a chart, the sample size, the sampling interval, and the control limits must be specified. An approach in specifying these process parameters in the economic design of the control chart was developed by Duncan(1956). Since Duncan, the economic design approach has been studied by many others such as Montgomery(1980), Lorenzen and Vance(1980) and Saniga(1989).

Park and Reynolds(1994) proposed the economic design of the $\bar{X}$ chart with VSS scheme. Comparing the VSS scheme to the FSS, it was shown that the VSS scheme shows considerably better economical and statistical performances when only two sample sizes are used.

The purpose of this paper is to extend the application of the VSS scheme to the economic design of the X̄-R chart. The effectiveness of the VSS X̄-R chart will be evaluated by comparing to the FSS X̄-R chart in terms of the economical and statistical performances.

2. THE VSS X̄-R CHART

Consider a process in which the distribution of the observations is normal with mean μ and variance σ^2, and the objective is to detect shifts in μ and σ^2 from target values μ_o and variance σ_o^2, respectively. Let $\delta_1 = (\mu - \mu_o)/\sigma_o$ and $\delta_2 = \sigma/\sigma_o$ be the deviations of μ and σ^2 from μ_o and σ_o^2, respectively. Suppose that a sample of variable size is taken at every h hour of operation.

Let N_k be the size of the sample taken at k^{th} sampling time. Also let $\bar{X}_k$ and R_k be the mean and range of the k^{th} sample, respectively. Then VSS X̄-R chart is maintained by plotting the standardized sample mean $S_{1,k} = \sqrt{N_k}(\bar{X}_k - \mu_o)/\sigma_o$ and sample range $S_{2,k} = R_k/\sigma_o$ at each time k on the chart with control limits $\pm l_1$ for $S_{1,k}$ and l_2 for $S_{2,k}$. In the VSS scheme, the sample size N_k is determined by the previous sample statistics $S_{1,k-1}$ and $S_{2,k-1}$. Let the region $\{(x,y); -l_1 < x < l_1, 0 < y < l_2\}$ be partitioned into η regions $I_1, \cdots, I_\eta$ such that

$$N_k = n_j \text{ if } \left(S_{1,k-1}, S_{2,k-1}\right) \in I_j,\ j = 1, 2, \cdots, \eta,$$

where $n_1 \leq n_2 \leq \cdots \leq n_\eta$. The boundary points for $(S_{1,k-1}, S_{2,k-1})$ between the regions are denoted as $(\pm l_{11}, l_{21})$, $(\pm l_{12}, l_{22}), \ldots, (\pm l_{1\eta}, l_{2\eta})$ where $l_{1\eta} = l_1$, $l_{2\eta} = l_2$. The partition of the region and the corresponding sample size are shown in Figure 1.

Since there is no sample before the first sampling time it would be reasonable to always take the size of the first sample $N_1 = n_\eta$, the largest sample size. Note that the standard FSS scheme with sample size n is equivalent to the VSS scheme with $n_1 = \cdots = n_\eta = n$.

The sequence of pairs of the statistics $\left\{(S_{1,k}, S_{2,k}), k \geq 1\right\}$ generates a Markov chain according to the sample size with the transient state transition matrix

$$Q = [q_{ij}]_{\eta \times \eta},$$

where $q_{ij} = P\left[(S_{1,k}, S_{2,k}) \in I_j \mid N_k = n_i\right]$.

The pair of statistics $(S_{1,k}, S_{2,k})$ falling outside the control limits corresponds to an absorbing state of the Markov chain. Let $\{Z_k, k \geq 1\}$ be a sequence of iid standard normal random variables and $\{R_k(n_i), k \geq 1\}$ be a sequence of ranges R_k from the k-th sample of size n_i. Then the transition probability q_{ij} is expressed by using the independency of $S_{1,k}$ and $S_{2,k}$ as the following. For $k \geq 1,\ 1 \leq i, j \leq \eta$,

$$\begin{aligned} q_{ij} &= P\left[(S_{1,k}, S_{2,k}) \in I_j \mid N_k = n_i\right] \qquad (2.1) \\ &= \left\{\Phi(l_{1j}/\delta_2 - \sqrt{n_i}\delta_1) - \Phi(-l_{1j}/\delta_2 - \sqrt{n_i}\delta_1)\right\}\Psi_{n_i}(l_{2j}/\delta_2) \\ &\quad - \left\{\Phi(l_{1j-1}/\delta_2 - \sqrt{n_i}\delta_1) - \Phi(-l_{1j-1}/\delta_2 - \sqrt{n_i}\delta_1)\right\}\Psi_{n_i}(l_{2j}/\delta_2), \end{aligned}$$

where $l_{10} = l_{20} = 0$, and $\Phi(\cdot)$ and $\Psi_{\eta_i}(\cdot)$ denote the distribution function of the standard normal random variable and $R_k(n_i)/\sigma$, respectively. An algorithm for calculation of $\Psi_{\eta_i}(\cdot)$ was proposed by Pearson and Hartley(1942). Note that when the process is in control(i.e. $\delta_1 = 0,\ \delta_2 = 1$), q_{ij} reduces to

$$q_{ij} = \left\{\Phi(l_{1j}) - \Phi(-l_{1j})\right\}\Psi_{n_i}(l_{2j}) - \left\{\Phi(-l_{1j-1}) - \Phi(-l_{1j-1})\right\}\Psi_{n_i}(l_{2j-1}). \quad (2.2)$$

3. AN ECONOMIC DESIGN OF THE VSS $\bar{X}$-R CHART

Suppose that the process starts initially in the in-control state and then the process goes to an out-of-control state due to the occurrence of one of possible assignable causes. The occurrence of the assignable cause A_{ij} produces the process mean and standard deviation as $\mu_0+\delta_{1i}\sigma_0$ and $\delta_{2j}\sigma_0(i = 1, \ldots, m_1, \ j = 1, \ldots, m_2,$ except $i = j = 1$), respectively. For convenience, we use $\sum_{ij}$ as the sum for $i = 1, \ldots, m_1$ and $j = 1, \ldots, m_2$ except $i = j = 1$ throughout the paper.

We assume that the times until the occurrences of the assignable causes are independent exponential random variables where the mean time for the occurrence of A_{ij} is $1/\lambda_{ij}$, $i = 1, \ldots, m_1$, $j = 1, \ldots, m_2$, except $i = j = 1$. Then the time at which the process goes to an out-of-control state is distributed as the minimum of m($= m_1 m_2 - 1$) independent exponentials, and follows an exponential distribution with mean $1/\lambda$ where $\lambda = \sum_{ij} \lambda_{ij}$. We also assume that once an assignable cause has occurred, the process is free from the other assignable causes until after the current assignable cause is detected and removed. The use of the exponential as an in-control time distribution was justified by McWilliams(1989). He showed that the economic design is quite insensitive to different types of in-control time distributions such as the Weibull.

The operation of the process can be viewed as a series of cycles where a cycle consists of a period of the in-control state followed by a period of an out-of-control state. During the in-control period the control chart may give some number of signals, which we call false alarms.

Due to administrative convenience it is not desirable to use many number of sample sizes in the VSS scheme. Thus we consider only two sample size(i.e. $\eta = 2$, $n_1 \leq n_2$) and always use the large sample size at the beginning of a cycle and right after each false alarm.

Let S_0 and O_0 be the number of samples and observations taken during the in-control period, respectively. Then the average number of samples in control(ANSC) and the average number of observations in control(ANOC) taken during the in-control period are obtained in Appendix A.1 as

$$E(S_0) = \frac{e^{-\lambda h}}{1 - e^{-\lambda h}} \qquad , \qquad E(O_0) = e^{-\lambda h}[1\ 0](I - e^{-\lambda h}P_0)^{-1}\underline{n} \tag{3.1}$$

where $\underline{n}' = (n_1, n_2)$ and $P_0 = \begin{bmatrix} p_{11} & 1-p_{11} \\ p_{21} & 1-p_{21} \end{bmatrix}$ in which $p_{i1} = q_{i1}(i = 1, 2)$ when the process is in control.

The average number of false alarms(ANFA) is obtained as

$$E(S_0)\frac{1}{[1\ 0](I - Q_0)^{-1}\underline{1}}, \tag{3.2}$$

where Q_0 denotes the transition matrix Q when the process is in control and $\underline{1}' = (1\,1)$. Let S_{ij} and O_{ij} be the number of samples and observations, respectively, taken when the process is out of control due to assignable cause A_{ij}. Then the average number of samples to signal(ANSS) and average number of observations to signal(ANOS) after the occurrence of A_{ij} are obtained in Appendix A.2 as

$$\begin{aligned} E(S_{ij}) &= [1\ 0]\Big[I + e^{-\lambda h}(I - P_0)(I - e^{-\lambda h}P_0)^{-1}\Big](I - Q_{ij})^{-1}\underline{1} \\ E(O_{ij}) &= [1\ 0]\Big[I + e^{-\lambda h}(I - P_0)(I - e^{-\lambda h}P_0)^{-1}\Big](I - Q_{ij})^{-1}\underline{n} \end{aligned}, \tag{3.3}$$

where Q_{ij} denotes the transition matrix Q when the process is out of control due to A_{ij}.

It was shown by Duncan(1971) that the average time of occurrence of the assignable cause A_{ij} within the interval is

$$\tau_{ij} = \frac{1 - (1 + \lambda_{ij}h)e^{-\lambda_{ij}h}}{\lambda_{ij}(1 - e^{-\lambda_{ij}h})}. \tag{3.4}$$

Thus the average time until the chart gives an out-of-control signal after its occurrence is

$$B_{ij} = hE(S_{ij}) - \tau_{ij}. \tag{3.5}$$

The expected length of a cycle is the expected time until an assignable cause occurs, $1/\lambda$, plus the expected time until the chart gives an out-of-control signal after its occurrence. The conditional probability of A_{ij} given the occurrence of an assignable cause is λ_{ij}/λ. Thus the expected length of a cycle is

$$\frac{\sum_{ij}\lambda_{ij}(1/\lambda + B_{ij})}{\lambda} = \frac{1 + \sum_{ij}\lambda_{ij}B_{ij}}{\lambda}. \tag{3.6}$$

Let M_{ij} be the cost per hour due to operating under the assignable cause A_{ij}. Then the expected cost per cycle due to operating out of control is

$$L_1 = \sum_{ij} B_{ij} M_{ij} \lambda_{ij} / \lambda. \tag{3.7}$$

If T is the cost for a false alarm, the expected cost for false alarms per cycle is

$$L_2 = TE(S_0) \frac{1}{[1\ 0](I - Q_0)^{-1}\underline{1}}. \tag{3.8}$$

Let W_{ij} be the cost for discovering the assignable cause A_{ij}, then the expected cost for discovering an assignable cause is

$$L_3 = \sum_{ij} W_{ij} \lambda_{ij} / \lambda. \tag{3.9}$$

Let ANSS* and ANOS* be the ANSS and ANOS, respectively, when the process is out of control due to any of the assignable causes. Then

$$\begin{aligned} \text{ANSS}^* &= \sum_{ij} E(S_{ij}) \lambda_{ij} / \lambda \\ \text{ANOS}^* &= \sum_{ij} E(O_{ij}) \lambda_{ij} / \lambda \end{aligned} . \tag{3.10}$$

Suppose that the cost for taking a sample of size n be linear in the sample size , that is $a_1 + a_2 n$ for given constants $a_1 \geq 0, a_2 > 0$. We denote that a_1 is the cost for a sample and a_2 is the cost for an observation. Then the expected total cost per cycle for sampling is

$$L_4 = a_1(E(S_0) + \text{ANSS}^*) + a_2(E(O_0) + \text{ANOS}^*). \tag{3.11}$$

Summing the various expected costs, we have the expected cost per hour as

$$L = \frac{L_1 + L_2 + L_3 + L_4}{(1 + \sum_{ij} \lambda_{ij} B_{ij})/\lambda}. \tag{3.12}$$

4. COMPARISON OF VSS TO FSS

In this section, two types of sampling schemes VSS and FSS are compared in terms of the expected cost per hour. In the FSS scheme the expected cost per hour can be obtained by letting $n_1 = n_2 = n$ in (3.12). Notice that if $n_1 = n_2$ then l_{11} and l_{21} are of no use. In each scheme the optimal process parameters are obtained and the resulting expected costs per hour are compared.

We consider $m_1 = 4$ and $m_2 = 3$ number of shifts in the mean and variance and let δ_{1i} and δ_{2j} be the corresponding values of δ_1 and δ_2, where $\delta_{1i} = i - 1$ and $\delta_{2j} = j, i = 1, \ldots, 4, j = 1, \ldots, 3$. Let $s(\delta_{1i}, \delta_{2j})$ be the resulting increase in the percent of product outside of specification limits at $3\sigma_0$ from μ_0, that is

$$s(\delta_{1i}, \delta_{2j}) = \Big[1 - \Phi\Big\{(3 - \delta_{1i})/\delta_{2j}\Big\}\Big] + \Phi\Big\{(-3 - \delta_{1i})/\delta_{2j}\Big\} - 2\Big\{1 - \Phi(3)\Big\}. \quad (4.1)$$

Now we consider new shifts in the mean, δ_k^*, $k = 1, \ldots, 11$, which gives the same amount of the resulting increase when $\sigma = \sigma_0$ as $s(\delta_{1i}, \delta_{2j})$. Then δ_k^*, $(\delta_{1i}, \delta_{2j})$ and $s(\delta_{1i}, \delta_{2j})$ are listed in Table 1.

For each (i, j) except for $i = j = 1$ the parameters λ_{ij} is chosen to be proportional to $(1/2)e^{-\delta_k^*/2}$ where k and δ_k^* are the values in Table 1 corresponding to the given (i, j) and $(\delta_{1i}, \delta_{2j})$. λ is set to be 0.01, 0.005, or 0.001. For each (i, j) the cost M_{ij} is chosen to be proportional to $s(\delta_{1i}, \delta_{2j})$. The cost for a false alarm T is set to be 50, 100, 200 and the W_{ij}'s are chosen as $W_{ij} = T \cdot e^{-\delta_k^*}$ where k is given by (i, j) in Table 1. The cost for a sample a_1 is set to be 0 or 1 and the cost for an observation a_2 is set to be 0.1 or 0.5. For each combination of parameter values, we find n_1, n_2, h, l_{11}, l_{12}, l_{21}, l_{22} for the VSS chart and n, h, l_{12}, l_{22} for the FSS chart which minimize the expected cost per hour in section 3. In finding the optimal parameters we use a generalized reduced gradient procedure using finite difference approximations to the partial derivatives which is explained in detail by Lasdon et al(1978).

The optimal sampling and chart parameters for both the VSS and FSS charts for the various combinations of the process and cost parameters are given in Table 2-13. The percent reduction in cost(denoted as %) of the VSS chart relative to the FSS chart is also given in the tables. To examine the statistical performance of the VSS chart, other values such as the ANFA per cycle and the average time to signal(ATS) after an assignable cause has occurred are also given in Table 2-13. The ATS is calculated by

$$ATS = \sum_{ij}(h \cdot E(S_{ij}) - \tau_{ij})\lambda_{ij}/\lambda. \tag{4.2}$$

According to the results in Tables 2-13, the optimal control limits l_{12} and l_{22} for the VSS chart are always larger than l_{12} and l_{22} for the FSS chart. The optimal sampling intervals of the VSS chart are always shorter than those of the FSS chart, which means that we take samples more often in the VSS chart than in the FSS chart. The percent reduction in cost range from 2.85 to 25.49. The larger percent reductions occur where a_1, a_2, and λ are smaller.

In addition, the ANFA as well as the ATS of the VSS chart are always smaller than those of the FSS chart. Thus we see that the VSS feature improves the confidence and the statistical performance of the control procedure.

5. CONCLUSIONS AND REMARKS

This paper developed an application of the VSS scheme to the $\bar{X}$-R chart in the context of an economic design model. It was shown that the VSS chart can be considerably more efficient than the FSS chart in terms of the expected cost per hour when only two sample sizes are used.

The characteristics of the VSS $\bar{X}$-R chart have the same trend as those of the VSS $\bar{X}$-chart from the results of Park and Reynolds(1994) when compared to the corresponding FSS chart. That is, both VSS charts are optimized by shorter sampling intervals, and larger control limits than the corresponding optimal values of the FSS chart. Also both VSS charts have similar amount of reduction in cost, ANFA's, and ATS's.

APPENDIX

A.1. Proof of (3.1).

Let U be the duration of the in-control state, then U follows an exponential distribution with mean $1/\lambda$. The probability function of S_0 is obtained as, for $x = 0, 1, 2, \ldots,$

$$\begin{aligned} P(S_0 = x) &= P(hx \leq U < h(x+1)) \\ &= e^{-\lambda hx} - e^{-\lambda h(x+1)}. \end{aligned}$$

Thus

$$\begin{aligned} E(S_0) &= \sum_{x=0}^{\infty} x(e^{-\lambda hx} - e^{-\lambda h(x+1)}) \\ &= \frac{e^{-\lambda h}}{1 - e^{-\lambda h}}. \end{aligned}$$

When the process is in control, we divide the space of the sample statistics $(S_{1,k}, S_{2,k})$ into two regions I_1^*, I_2^*, where $I_1^* = \{(x, y) : |x| < l_{11} \text{ and } 0 < y < l_{21}\}$ and $I_2^* = \{(x, y) : |x| \geq l_{11} \text{ or } y \geq l_{21}\}$. If $(S_{1,k}, S_{2,k})$ falls into the region I_2^* we use size n_2 for the next sample, otherwise we use n_1. Remember that we use larger sample after each false alarm.

Now we can think another Markov chain according to the sample size with the transition matrix P_0. By using the Markov chain property,

$$E(O_0 \mid S_0 = x) = \begin{cases} \sum_{k=0}^{x-1} [1\ 0] P_0^k \underline{n}, & \text{if } x \geq 1 \\ 0, & \text{if } x = 0 \end{cases}.$$

Thus,

$$\begin{aligned} E[O_0] &= \sum_{x=1}^{\infty} \sum_{k=0}^{x-1} [1\ 0] P_0^k \underline{n} (e^{-\lambda hx} - e^{-\lambda h(x+1)}) \\ &= \sum_{k=0}^{\infty} [1\ 0] P_0^k \underline{n} \sum_{x=k+1}^{\infty} (e^{-\lambda hx} - e^{-\lambda h(x+1)}) \end{aligned}$$

$$= \sum_{k=0}^{\infty}[1\ 0]P_0^k \underline{n} e^{-\lambda h(k+1)}$$

$$= e^{-\lambda h}[1\ 0]\sum_{k=0}^{\infty}(e^{-\lambda h}P_0)^k \underline{n}$$

$$= e^{-\lambda h}[1\ 0](I - e^{-\lambda h}P_0)^{-1}\underline{n}.$$

A.2. Proof of (3.3).

The conditional probability function of S_j given that $S_0 = y$ can be obtained as, for $x = 1, 2, \ldots,$

$$\begin{aligned} P\Big[S_j = x \mid S_0 = y\Big] &= \Big[S_j \geq x \mid S_0 = y\Big] - P\Big[S_j \geq x + 1 \mid S_0 = y\Big] \\ &= [1\ 0]P_0^y Q_j^{x-1}\underline{1} - [1\ 0]P_0^y Q_j^x \underline{1} \\ &= [1\ 0]P_0^y \Big[Q_j^{x-1} - Q_j^x\Big]\underline{1}. \end{aligned}$$

Thus,

$$\begin{aligned} E\Big[S_j \mid S_0 = y\Big] &= \sum_{x=1}^{\infty} x[1\ 0]P_0^y \Big[Q_j^{x-1} - Q_j^x\Big]\underline{1} \\ &= [1\ 0]P_0^y \sum_{x=1}^{\infty} x\Big[Q_j^{x-1} - Q_j^x\Big]\underline{1} \\ &= [1\ 0]P_0^y (I - Q_j)^{-1}\underline{1}. \end{aligned}$$

Therefore, by using $P(S_0 = x)$ from A.1.

$$\begin{aligned} E\Big[S_j\Big] &= \sum_{y=0}^{\infty}[1\ 0]P_0^y \Big[I - Q_j\Big]^{-1}\underline{1}(e^{-\lambda hy} - e^{-\lambda h(y+1)}) \\ &= [1\ 0]\sum_{y=0}^{\infty} P_0^y (e^{-\lambda hy} - e^{-\lambda h(y+1)})\Big[I - Q_j\Big]^{-1}\underline{1} \\ &= [1\ 0]\Big[I + e^{-\lambda h}(P_0 - I)(I - e^{-\lambda h}P_0)^{-1}\Big]\Big[I - Q_j\Big]^{-1}\underline{1}. \end{aligned}$$

Similarly,

$$E[O_j] = [1\ 0]\Big[I + e^{-\lambda h}(P_0 - I)(I - e^{-\lambda h}P_0)^{-1}\Big]\Big[I - Q_j\Big]^{-1}\underline{n}.$$

Figure 1. Partition of the region $\Big\{(x, y); -l_1 < x < l_1,\ 0 < y < l_2\Big\}$ and the corresponding sample sizes.

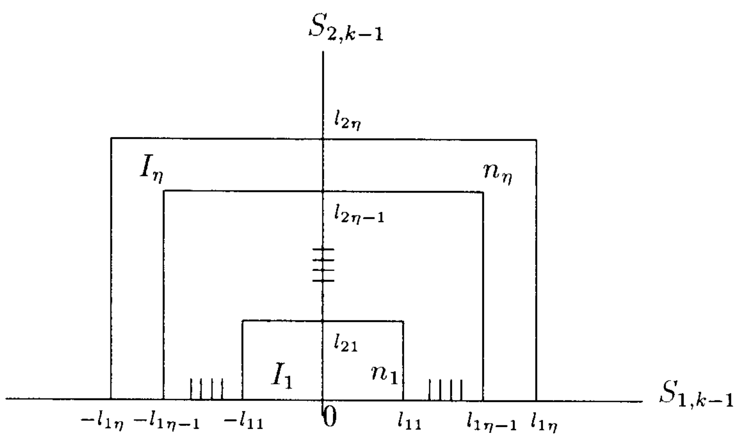

Table 1. δ_k^*, $(\delta_{1i}, \delta_{2j})$ and $s(\delta_{1i}, \delta_{2j})$.

(i,j)	(2,1)	(1,2)	(3,1)	(2,2)	(3,2)	(1,3)	(2,3)	(3,3)	(4,1)	(4,2)	(4,3)
$(\delta_{1i},\ \delta_{2,j})$	(1,1)	(0,2)	(2,1)	(1,2)	(2,2)	(0,3)	(1,3)	(2,3)	(3,1)	(3,2)	(3,3)
k	1	2	3	4	5	6	7	8	9	10	11
δ_k^*	1.0	1.891	2.0	2.09	2.518	2.525	2.598	2.791	3.0	3.003	3.057
$s(\delta_{1i},\ \delta_{2,j})$	0.0201	0.1310	0.1560	0.1788	0.3121	0.3147	0.3411	0.4146	0.4974	0.4987	0.520

Table 2. Optimal process parameters and the corresponding characteristics for $a_1 = 0.0$, $a_2 = 0.1$ and $T = 50.0$.

λ	$n_1(n_2)$	h	l_{11}	l_{12}	l_{21}	l_{22}	cost (%)	ANSC ANOC	ANSS ANOS	ATS ANFA
	2(11)	0.478	2.2454	3.6409	4.1810	5.8654	1.11949	208.37	3.025	1.208
							(15.88)	484.29	13.091	0.074
0.010	3	0.861		3.2205		5.4209	1.33090	162.96	3.972	2.124
								488.90	11.916	0.2688
	2(12)	0.472	2.2801	3.6705	4.2505	5.9146	1.08085	423.20	2.983	1.1724
							(17.17)	975.87	13.285	0.1345
0.005	3	0.605		3.2168		5.4269	1.30496	329.73	3.954	2.092
								989.20	11.863	0.5473
	2(12)	0.469	2.2786	3.6782	4.2344	5.9225	1.04910	2130.4	2.987	1.167
							(18.27)	4875.9	13.342	0.6593
0.001	3	0.600		3.2146		5.4315	1.28368	1663.7	3.943	2.096
								4991.1	11.831	2.7713

* In each cell, upper numbers denote values for the VSS and lower ones for the FSS.

Table 3. Optimal process parameters and the corresponding characteristics for $a_1 = 0.0$, $a_2 = 0.1$ and $T = 100.0$.

λ	$n_1(n_2)$	h	l_{11}	l_{12}	l_{21}	l_{22}	cost (%)	ANSC ANOC	ANSS ANOS	ATS ANFA
	2(12)	0.467	2.2834	3.8378	4.2504	6.1126	1.21350	213.54	3.159	1.2423
							(18.66)	497.34	14.902	0.0352
0.010	4	0.732		3.4097		5.7756	1.49197	135.96	3.512	2.2073
								543.86	14.048	0.1233
	2(13)	0.460	2.3143	3.8662	4.3117	6.1590	1.13869	433.75	3.109	1.201
							(20.47)	1001.3	15.05	0.064
0.005	4	0.724		3.4046		5.7813	1.43265	275.44	3.491	2.168
								1101.7	13.965	0.2521
	2(14)	0.455	2.3461	3.8903	4.3556	6.2092	1.07799	2195.2	3.085	1.177
							(22.11)	5023.4	15.262	0.2926
0.001	4	0.718		3.4014		5.7855	1.38411	1390.4	3.478	2.141
								5561.7	13.915	0.2790

Table 4. Optimal process parameters and the corresponding characteristics for $a_1 = 0.0$, $a_2 = 0.1$ and $T = 200.0$.

λ	$n_1(n_2)$	h	l_{11}	l_{12}	l_{21}	l_{22}	cost (%)	ANSC ANOC	ANSS ANOS	ATS ANFA
	2(14)	0.454	2.3529	4.0435	4.3807	6.3830	1.37538	219.45	3.230	1.241
							(20.05)	512.17	16.814	0.015
0.010	4	0.707		3.5531		5.9818	1.72043	140.87	4.310	2.695
								563.48	17.242	0.0728
	2(14)	0.450	2.3480	4.0525	4.3627	6.3944	1.23017	443.70	3.228	1.2280
							(22.94)	1024.7	16.888	0.0305
0.005	5	0.827		3.5845		6.0924	1.59639	241.21	3.221	2.252
								1206.0	16.108	0.1198
	2(15)	0.445	2.3763	4.0768	4.4092	6.4373	1.11222	2246.1	3.199	1.201
							(25.49)	5137.8	17.062	0.1398
0.001	5	0.820		3.5805		6.0965	1.49287	1218.7	3.207	2.220
								6093.6	16.037	0.6097

Table 5. Optimal process parameters and the corresponding characteristics for $a_1 = 0.0$, $a_2 = 0.5$ and $T = 50.0$.

λ	$n_1(n_2)$	h	l_{11}	l_{12}	l_{21}	l_{22}	cost (%)	ANSC ANOC	ANSS ANOS	ATS ANFA
	2(6)	1.182	2.0120	3.0463	3.6951	5.0027	2.27027	84.03	2.808	2.730
							(5.21)	191.43	8.512	0.2525
0.010	2	1.132		2.8094		4.6421	2.39519	87.82	3.587	3.495
								175.65	7.174	0.5256
	2(7)	1.156	2.0669	3.0912	3.8242	5.0979	2.23588	172.47	2.733	2.582
							(6.31)	392.92	8.762	0.444
0.005	2	1.114		2.8040		4.6537	2.38670	178.98	3.566	3.416
								357.96	7.132	1.0809
	2(8)	1.134	2.1182	3.1327	3.9280	5.1804	2.20479	881.31	2.689	2.482
							(7.33)	1998.6	9.023	1.9673
0.001	2	1.100		2.8004		4.6627	2.37929	908.11	3.552	3.359
								1816.2	7.104	5.5163

Table 6. Optimal process parameters and the corresponding characteristics for $a_1 = 0.0$, $a_2 = 0.5$ and $T = 100.0$.

λ	$n_1(n_2)$	h	l_{11}	l_{12}	l_{21}	l_{22}	cost (%)	ANSC ANOC	ANSS ANOS	ATS ANFA
	2(8)	1.143	2.1192	3.3068	3.9353	5.3858	2.41374	86.94	2.929	2.778
							(9.05)	202.65	10.452	0.1067
0.010	2	1.082		2.9811		4.9072	2.65396	91.87	4.576	4.412
								183.75	9.1522	0.3114
	2(9)	1.115	2.1617	3.3474	4.0277	5.4651	2.33917	178.72	2.860	2.6344
							(10.80)	202.65	10.452	0.1067
0.005	2	1.060		2.9745		4.9215	2.62236	188.03	4.537	4.282
								376.07	9.074	0.6454
	2(10)	1.093	2.204	3.3846	4.1009	5.5355	2.27447	913.9	2.822	2.540
							(12.35)	2089.1	10.919	0.8443
0.001	2	1.044		2.9706		4.9328	2.59508	956.8	4.513	4.192
								1913.7	9.027	3.3082

Table 7. Optimal process parameters and the corresponding characteristics for $a_1 = 0.0$, $a_2 = 0.5$ and $T = 200.0$.

λ	$n_1(n_2)$	h	l_{11}	l_{12}	l_{21}	l_{22}	cost (%)	ANSC ANOC	ANSS ANOS	ATS ANFA
	2(10)	1.106	2.2146	3.5450	4.1289	5.7189	2.61920	89.84	3.022	2.792
							(12.28)	212.06	12.345	0.0461
0.010	3	1.420		3.1826		5.3312	2.98591	69.88	3.756	4.626
								209.66	11.268	0.1353
	2(11)	1.079	2.2464	3.5822	4.1944	5.7890	2.47035	184.83	2.964	2.6590
							(14.56)	430.19	12.558	0.0823
0.005	3	1.389		3.1720		5.3450	2.89151	143.44	3.709	4.458
								430.33	11.127	0.2829
	2(12)	1.057	2.2805	3.6161	4.2453	5.851	2.34436	945.00	2.936	2.576
							(16.61)	2166.2	12.817	0.3695
0.001	3	1.366		3.1659		5.3550	2.81157	731.39	3.683	4.349
								2194.1	11.049	1.4569

Table 8. Optimal process parameters and the corresponding characteristics for $a_1 = 1.0$, $a_2 = 0.1$ and $T = 50.0$.

λ	$n_1(n_2)$	h	l_{11}	l_{12}	l_{21}	l_{22}	cost (%)	ANSC ANOC	ANSS ANOS	ATS ANFA
	5(13)	1.621	2.1498	3.3380	4.3016	5.6509	2.1074	61.17	1.545	1.695
							(2.96)	341.26	10.836	0.1018
0.010	6	1.684		3.1429		5.4984	2.17171	58.86	1.711	2.040
								353.20	10.268	0.1813
	5(14)	1.603	2.1766	3.3468	4.3460	5.6817	2.078	124.20	1.536	1.662
							(3.31)	688.30	10.940	0.1970
0.005	6	1.668		3.1343		5.5036	2.14929	119.39	1.703	2.008
								716.35	10.222	0.3714
	5(15)	1.589	2.2040	3.3595	4.3905	5.7056	2.05353	628.78	1.533	1.641
							(3.62)	3456.8	11.070	0.951
0.001	6	1.655		3.1283		5.5076	2.13084	603.52	1.698	1.984
								3621.1	10.190	1.889

Table 9. Optimal process parameters and the corresponding characteristics for $a_1 = 1.0$, $a_2 = 0.1$ and $T = 100.0$.

λ	$n_1(n_2)$	h	l_{11}	l_{12}	l_{21}	l_{22}	cost (%)	ANSC ANOC	ANSS ANOS	ATS ANFA
	5(15)	1.598	2.2165	3.533	4.3806	5.9152	2.21067	62.05	1.593	1.748
							(3.84)	350.20	12.091	0.0496
0.010	7	1.734		3.3546		5.8257	2.29910	57.15	1.728	2.1313
								400.08	12.102	0.088
	5(16)	1.579	2.2314	3.5479	4.4282	5.9359	2.14615	126.10	1.586	1.716
							(4.33)	704.35	12.214	0.096
0.005	7	1.716		3.3445		5.8306	2.24338	116.02	1.719	2.093
								812.19	12.037	0.1811
	5(17)	1.564	2.2513	3.5587	4.4640	5.9592	2.09307	638.57	1.582	1.694
							(4.76)	3530.2	12.353	0.4661
0.001	7	1.702		3.3379		5.8344	2.19771	536.80	1.713	2.066
								4107.6	11.995	0.9227

Table 10. Optimal process parameters and the corresponding characteristics for $a_1 = 1.0$, $a_2 = 0.1$ and $T = 200.0$.

λ	$n_1(n_2)$	h	l_{11}	l_{12}	l_{21}	l_{22}	cost (%)	ANSC ANOC	ANSS ANOS	ATS ANFA
	5(17)	1.576	2.2731	3.7265	4.4622	6.1570	2.28268	62.91	1.640	1.798
							(4.62)	358.15	13.425	0.0242
0.010	8	1.782		3.5528		6.119	2.4981	55.58	1.749	2.227
								444.71	13.992	0.043
	5(18)	1.556	2.2822	3.7362	4.4949	6.183	2.24867	127.99	1.633	1.7630
							(5.32)	719.51	13.551	0.0470
0.005	8	1.762		3.5402		6.123	2.3751	113.60	1.737	2.179
								904.01	13.896	0.0897
	6(21)	1.625	2.4058	3.7883	4.7119	6.276	2.1383	614.58	1.535	1.684
							(5.98)	3978.5	14.283	0.2028
0.001	8	1.747		3.5328		6.1275	2.27445	571.88	1.730	2.149
								4575.5	13.841	0.4581

Table 11. Optimal process parameters and the corresponding characteristics for $a_1 = 1.0$, $a_2 = 0.5$ and $T = 50.0$.

λ	$n_1(n_2)$	h	l_{11}	l_{12}	l_{21}	l_{22}	cost (%)	ANSC ANOC	ANSS ANOS	ATS ANFA
	2(6)	1.753	1.7739	2.8901	3.4470	4.8591	2.95394	56.53	2.264	3.093
							(2.85)	140.10	7.335	0.292
0.010	3	2.014		2.7344		4.6666	3.04065	49.13	2.235	3.496
								147.41	6.706	0.4427
	2(6)	1.723	1.7535	2.8928	3.4413	4.8791	2.93467	115.52	2.256	3.028
							(3.38)	284.03	7.342	0.5887
0.005	3	1.982		2.7250		4.677	3.03733	100.38	2.220	3.410
								301.15	6.660	0.914
	2(7)	1.685	1.8289	2.930	3.6029	4.9773	2.9166	592.96	2.225	2.907
							(3.86)	1453.1	7.595	2.650
0.001	3	1.958		2.7188		4.686	3.03385	510.22	2.210	3.348
								1530.6	6.630	4.6844

Table 12. Optimal process parameters and the corresponding characteristics for $a_1 = 1.0$, $a_2 = 0.5$ and $T = 100.0$.

λ	$n_1(n_2)$	h	l_{11}	l_{12}	l_{21}	l_{22}	cost (%)	ANSC ANOC	ANSS ANOS	ATS ANFA
	2(7)	1.706	1.8297	3.1229	3.5893	5.1931	3.11926	58.11	2.431	3.296
							(4.56)	147.15	8.864	0.1399
0.010	3	1.958		2.9062		4.9152	3.26857	50.55	2.662	4.236
								151.66	7.987	0.2592
	2(8)	1.659	1.8872	3.1578	3.7289	5.2846	3.06231	120.03	2.378	3.117
							(5.36)	301.59	9.076	0.2536
0.005	4	2.201		2.9691		5.1354	3.23593	90.35	2.144	3.620
								361.40	8.578	0.414
	3(10)	1.930	2.1021	3.2074	4.0571	5.3788	3.01124	517.39	1.996	2.889
							(6.10)	1746.9	9.491	1.0460
0.001	4	2.172		2.9612		5.1436	3.20714	459.90	2.132	3.546
								1839.6	8.530	2.1272

Table 13. Optimal process parameters and the corresponding characteristics for $a_1 = 1.0$, $a_2 = 0.5$ and $T = 200.0$.

λ	$n_1(n_2)$	h	l_{11}	l_{12}	l_{21}	l_{22}	cost (%)	ANSC ANOC	ANSS ANOS	ATS ANFA
	3(10)	1.963	2.1211	3.3827	4.0583	5.5802	3.33695	50.43	2.134	3.210
							(5.95)	176.16	10.730	0.0553
0.010	4	2.190		3.1420		5.3484	3.54839	45.15	2.535	4.459
								180.61	10.143	0.1160
	3(11)	1.916	2.1479	3.4094	4.1263	5.6372	3.20558	103.84	2.098	3.0630
							(7.29)	358.24	10.877	0.1027
0.005	4	2.143		3.1270		5.3611	3.45784	92.79	2.503	4.294
								371.16	10.012	0.2436
	3(12)	1.881	2.1802	3.4346	4.1802	5.6859	3.09318	531.11	2.080	2.972
							(8.51)	1805.6	11.060	0.4774
0.001	4	2.109		3.1187		5.3703	3.38103	473.49	2.485	4.187
								1893.9	9.940	1.2568

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