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Bayes Estimation of a Reliability Function for Rayleigh Model

YeungHoon Kim[1] and JoongKweon Sohn[2]

ABSTRACT

This paper deals with the problem of obtaining some Bayes estimators and Bayesian credible regions of a reliability function for the Rayleigh distribution. Using several priors for a reliability function some Bayes estimators and Bayes credible sets are proposed and studied under squared error loss and Harris loss. Also the performances and behaviors of the proposed Bayes estimators are examined via Monte Carlo simulations and some numerical examples are given.

KEYWORDS: Bayesian estimation, Reliability, Rayleigh distribution, Squared error loss, Harris loss, Monte Carlo method.

1. INTRODUCTION

In the context of the lifetime of industrial equipments and components, the problem of estimating a reliability function plays an important role in many practical reliability analyses. In this paper we deal with the Bayesian point

[1]Department of Statistics, Andong National University, Andong 760-749, Korea.
[2]Department of Statistics, Kyungpook National University, Daegu 702-701, Korea.

estimation and the Bayesian interval estimation for the reliability function, at a specified mission time t, based upon a complete sample of failure times observed from the Rayleigh model. Siddiqui(1962) discussed the origin and the properties of the Rayleigh distribution. Polovko(1968) noted the importance of this distribution in electrovaccum devices. Dyer and Whisenand(1973a, 1973b) mentioned the important role of the Rayleigh distribution in the area of the communication engineering. Cheng(1980) investigated the optimum spacing for asymptotically best linear unbiased estimator of the parameter based upon order statistics.

Although the classical statistical estimation procedures have been applied for many situations, there are many cases in which the Bayesian procedures perform more satisfactory. The benefits of Bayesian approach in the reliability analysis were discussed by Evans(1969). For the Bayesian estimation, Sinha and Howlader(1983a, 1983b) obtained the Bayes estimator with respect to the Jeffreys' noninformative prior for the reliability under squared error loss function. They also proposed the Bayes credible sets and the highest posterior density credible intervals for reliability function. Under the Bayesian framework, a decision-maker can utilize some prior knowledge. Such knowledges can be usually translated to a prior distribution. For almost all cases of estimating the reliability, the prior informations about the unknown parameter of the given model were used rather than the prior knowledge of reliability itself. However in practice, it is more reasonable to assign a prior on the reliability than to assign a prior on the parameter which is related to the reliability. Futhermore, it is not easy to provide the equivalent priors on different unknown quantities, even though it can be taken into considerations the mathematical relationship between the unknown parameter and the reliability.

The purpose of this paper is to propose and to study some Bayes estimators and Bayesian credible regions of a reliability function with respect to the direct prior information about the reliability for the Rayleigh model. As prior distributions, we consider a locally uniform prior distribution and a beta prior distribution. The squared error loss function is used typically, but it is not always appropriate in analyzing reliability data. Thus the loss function suggested by Harris(1976) as well as the squared error loss function is considered.

In Section 2, we derive some Bayes estimators of the reliability function under squared error loss function and Harris loss function. In Section 3, we propose the Bayesian credible intervals and the highest posterior density (HPD) credible regions for the reliability function with respect to a locally uniform prior and a beta prior. In Section 4, we discuss the performances and behaviors of the Bayes estimators of the reliability function via Monte Carlo simulations. In Section 5, some numerical examples are presented. The proposed Bayes estimators are compared with the estimators given in the literature. The mean squared errors and biases of the proposed estimators are computed.

2. SOME PROPOSED BAYES ESTIMATORS

We consider the Bayesian approach of the estimation of reliability, at a specified time t, for the Rayleigh distribution, denoted by $\mathcal{R}(\sigma^2)$, with probability density function(pdf) given in (2.1).

$$f(x|\sigma) = \frac{x}{\sigma^2}\exp\Big(-\frac{x^2}{2\sigma^2}\Big), \quad 0 < x < \infty, \tag{2.1}$$

This is the special case of the two-parameter Weibull distribution. The hazard function of this distribution is an increasing function in x, which is interested in the life testing problem. Thus this could be suitable for life testing experiments on components which age with time in that way.

For the Rayleigh distribution, the reliability function θ, at a specified 'mission' time $t > 0$, is given by

$$\theta = P(X > t) = \exp\Big(\frac{-t^2}{2\sigma^2}\Big). \tag{2.2}$$

Then our goal is to derive a Bayes estimator of the reliability function θ.

Let $\underline{X} = (X_1, X_2, \dots, X_n)$ be a random sample from the Rayleigh distribution with pdf given in (2.1). Then the likelihood function is given by

$$L(\sigma|\underline{X}) = \frac{1}{\sigma^{2n}}\prod_{i=1}^{n} X_i \exp\Big(-\frac{\sum X_i^2}{2\sigma^2}\Big), \quad 0<\sigma,\ 0<X_i<\infty,\ i=1,2,\ldots,n.$$

Thus, based upon a random sample $\underline{X}$ from $\mathcal{R}(\sigma^2)$, the likelihood function of θ , which can be obtained from (2.2) by letting $\sigma^2 =. -t^2/2\ln\theta$, is

$$L(\theta|\underline{X}) = \Big(\frac{-2\ln\theta}{t^2}\Big)^n \prod_{i=1}^{n} X_i \exp\Big(\frac{\sum X_i^2}{t^2}\ln\theta\Big), \quad 0<\theta<1. \tag{2.3}$$

Here we consider the Jeffreys(1961)' noninformative prior for θ. When the 'mission' time t is given, the noninformative prior for θ is

$$\Pi(\theta) = I(\theta)^{\frac{1}{2}} = -\frac{1}{\theta\ln\theta}, \quad 0<\theta<1,$$

where $I(\theta) = -E\Big[\frac{\partial^2}{\partial\theta^2}\ln L(\theta|X)\Big] = \frac{1}{\theta^2(\ln\theta)^2}$ and $I(\theta)$ is the Fisher's information. This is also the implied prior of the noninformative prior for σ, which is given by Sinha and Howlader(1983a, 1983b), by the properties of the invariance under parametric transformations. Therefore the Bayes estimators under the noninformative prior for θ and σ are equivalent. Sinha and Howlader(1983a, 1983b) proposed the Bayes estimator $\hat{\theta}_{NS}$ of θ under squared error loss, which is given by $\hat{\theta}_{NS} = \Big(1+\frac{t^2}{\sum X_i^2}\Big)^{-n}$. Note that the Bayes estimator with respect to a noninformative prior is approximately equal to the classical maximum likelihood estimator(MLE) $\exp\Big(-\frac{nt^2}{\sum X_i^2}\Big)$ for large sample size.

Next, we consider a loss function suggested by Harris(1976) for the case $k=2$, which is given by

$$L(\theta,\delta) = \Big|\frac{1}{1-\delta} - \frac{1}{1-\theta}\Big|^2. \tag{2.4}$$

The reasons of considering such loss function were as follows :

"If the system reliability is 0.99, on the average it should fail one time in 100, whereas if the system reliability is 0.999, it should fail one time in 1000

and hence is ten times as good. Thus the loss function should depend on how well one estimates $(1-\delta)^{-1}$*.*"

Under Harris loss, the Bayes estimator $\hat{\theta}_H$ of θ can be drived from the following relation: For a given $\underline{X} = \underline{x}$,

$$\frac{1}{1-\delta} = E^{\theta|\underline{x}}\left[\frac{1}{1-\theta}\right] \equiv \gamma(\underline{x}).$$

Thus

$$\hat{\theta}_H = 1 - \frac{1}{\gamma(\underline{x})}, \tag{2.5}$$

where

$$\gamma(\underline{x}) = E^{\theta|\underline{x}}\left[\frac{1}{1-\theta}\right].$$

Then one can obtain the Bayes estimator of the reliability function as follows:

Theorem 2.1. If the Harris loss and the noninformative prior are used, then the Bayes estimator $\hat{\theta}_{NH}$ of the reliability function θ is given by

$$\hat{\theta}_{NH} = 1 - \frac{1}{\left(\frac{\sum X_i^2}{t^2}\right)^n \sum\limits_{m=0}^{\infty} \left(m + \frac{\sum X_i^2}{t^2}\right)^{-n}}.$$

Proof. The posterior density of θ given $\underline{X} = \underline{x}$ is

$$\Pi(\theta|\underline{x}) = K(-\ln\theta)^{n-1}\theta^{\frac{1}{t^2}\sum x_i^2 - 1}, \quad 0 < \theta < 1,$$

where K is the normalizing constant and is given by $\frac{1}{\Gamma(n)}\left(\frac{\sum x_i^2}{t^2}\right)^n$. And $\Gamma(a)$ is the gamma function defined by $\Gamma(a) = \int_0^\infty t^{a-1}e^{-t}dt$. Then

$$E^{\theta|\underline{x}}\left(\frac{1}{1-\theta}\right) = K\int_0^1 \frac{1}{1-\theta}\theta^{\frac{1}{t^2}\sum x_i^2 - 1}(-\ln\theta)^{n-1}\, d\theta.$$

Letting $Y = -\ln\theta\Big(1+\frac{\sum X_i^2}{t^2}\Big)$,

$$E^{\theta|\underline{x}}\Big(\frac{1}{1-\theta}\Big) = \frac{1}{\Gamma(n)\Big(1+\frac{t^2}{\sum x_i^2}\Big)^n}\int_0^\infty y^{n-1}\frac{\exp\Big(\frac{-y}{1+t^2/\sum x_i^2}\Big)}{1-\exp\Big(\frac{-y}{1+\sum x_i^2/t^2}\Big)}\,dy. \quad (2.6)$$

With the aid of the formula 4.5(10) in Erdélyi *et al.*(1955),

$$\int_0^\infty z^{\nu-1}(1-e^{-z/a})^{-1}e^{-pz}\,dz = a^\nu\Gamma(\nu)\zeta(\nu, ap) \quad \text{for } Re(p)>0,\ Re(\nu)>1,$$

where $\zeta(s,\nu) = \sum_{m=0}^{\infty}(\nu+m)^{-s}(Re(s)>0)$ is the generalized zeta function, the equation (2.6) becomes

$$\Big(\frac{\sum x_i^2}{t^2}\Big)^n\zeta\Big(n, \frac{\sum x_i^2}{t^2}\Big) = \Big(\frac{\sum x_i^2}{t^2}\Big)^n\sum_{m=0}^{\infty}\Big(m+\frac{\sum x_i^2}{t^2}\Big)^{-n}.$$

Hence the theorem follows from the equation (2.5).

Consider a locally uniform prior distribution $\mathcal{U}^\theta(0,1)$ for θ with pdf

$$\Pi(\theta) = 1, \qquad 0<\theta<1. \quad (2.7)$$

Note that this prior is the special case of a beta distribution. Then the posterior density of θ given $\underline{X} = \underline{x}$ is

$$\Pi(\theta|\underline{x}) = \frac{1}{\Gamma(n+1)}\Big(1+\frac{\sum x_i^2}{t^2}\Big)^{n+1}(-\ln\theta)^n\exp\Big(\frac{\sum x_i^2}{t^2}\ln\theta\Big), 0<\theta<1. \quad (2.8)$$

If the squared error loss function is applied, then the Bayes estimator of the reliability function is given as follows:

Theorem 2.2. If the squared error loss and a locally uniform prior are used, then the Bayes estimator $\tilde{\theta}_{US}$ of the reliability function is given by

$$\tilde{\theta}_{US} = \Big(1+\frac{t^2}{\sum X_i^2}\Big)^{n+1}\Big(1+\frac{2t^2}{\sum X_i^2}\Big)^{-(n+1)}.$$

Proof. Since the Bayes estimator of θ is the posterior mean,

$$\tilde{\theta}_{US} = E^{\theta|\underline{x}}(\theta) = \frac{1}{\Gamma(n+1)}\Big(1+\frac{\sum x_i^2}{t^2}\Big)^{n+1}\int_0^1 \theta^{\frac{1}{t^2}\sum x_i^2+1}(-\ln\theta)^n d\theta.$$

Letting $Y = -\ln\theta\Big(2+\frac{\sum X_i^2}{t^2}\Big)$,

$$\begin{aligned}\tilde{\theta}_{US} &= \frac{1}{\Gamma(n+1)}\Big(1+\frac{\sum x_i^2}{t^2}\Big)^{n+1}\Big(2+\frac{\sum x_i^2}{t^2}\Big)^{-(n+1)}\int_0^\infty y^n e^{-y}dy \\ &= \Big(1+\frac{t^2}{\sum x_i^2}\Big)^{n+1}\Big(1+\frac{2t^2}{\sum x_i^2}\Big)^{-(n+1)}.\end{aligned}$$

Also with the Harris loss function in (2.4) one can obtain the following theorem.

Theorem 2.3. If the Harris loss and a locally uniform prior distribution are used, then the Bayes estimator $\tilde{\theta}_{UH}$ of the reliability function is given by

$$\tilde{\theta}_{UH} = 1 - \frac{1}{\Big(1+\frac{\sum X_i^2}{t^2}\Big)^{n+1}\sum_{m=0}^{\infty}\Big(1+m+\frac{\sum X_i^2}{t^2}\Big)^{-(n+1)}}.$$

Proof. By transforming $Y = -\theta^{\frac{\sum x_i^2}{t^2}}$, one can obtain the following relation:

$$E^{\theta|\underline{x}}\Big(\frac{1}{1-\theta}\Big) = \frac{1}{\Gamma(n+1)}\Big(1+\frac{t^2}{\sum x_i^2}\Big)^{n+1}\int_0^\infty y^n \frac{\exp\Big[-y\Big(1+\frac{t^2}{\sum x_i^2}\Big)\Big]}{1-\exp\Big(-\frac{t^2}{\sum x_i^2}y\Big)}dy.$$

Using 4.5(10) in Erdélyi *et al.*(1955), the above expectation is equal to

$$\Big(1+\frac{\sum x_i^2}{t^2}\Big)^{n+1}\sum_{m=0}^{\infty}\Big(1+m+\frac{\sum x_i^2}{t^2}\Big)^{-(n+1)}.$$

Therefore the theorem follows immediately from the equation (2.5).

Remark. The generalized maximum likelihood estimator of the reliability is the largest mode of the posterior density function of θ given $\underline{X} = \underline{x}$. If the prior density of θ does not depend upon θ, then the posterior density function of θ is proportional to the likelihood function of σ. If a uniform prior $\mathcal{U}^{\theta}(0,1)$ is assigned on θ, then the generalized MLE(GMLE) is also the classical MLE, $\exp\left(-\frac{nt^2}{\sum X_i^2}\right)$, $t > 0$.

Now we consider a beta prior distribution $\mathcal{B}^{\theta}(\alpha, \beta)$ with pdf

$$\Pi(\theta) = \frac{1}{B(\alpha,\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}, \quad 0 < \theta < 1, \ 0 < \alpha, \ \beta,$$

where $B(\alpha,\beta)$ is the beta function with parameters α and β defined by $B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt$.

Combining the likelihood function in (2.3) and a beta prior density function, the posterior density function of θ given $\underline{X} = \underline{x}$ is

$$\Pi(\theta|\underline{x}) = \frac{1}{I(\alpha,\beta)}(-\ln\theta)^n \exp\left(\frac{\sum x_i^2}{t^2}\ln\theta\right)\theta^{\alpha-1}(1-\theta)^{\beta-1}, \ 0 < \theta < 1, \quad (2.9)$$

where

$$I(\alpha,\beta) = \int_0^1 (-\ln\theta)^n \exp\left(\frac{\sum x_i^2}{t^2}\ln\theta\right)\theta^{\alpha-1}(1-\theta)^{\beta-1}d\theta. \quad (2.10)$$

With the squared error loss, the Bayes estimator of the reliability is the posterior mean and is given as follows:

Theorem 2.4. If the squared error loss function is used and θ has a beta prior distribution with parameters α and β, then the Bayes estimator $\tilde{\theta}_{BS}$ of the reliability function given $\underline{X} = \underline{x}$ is given by

$$\tilde{\theta}_{BS} = \frac{I(\alpha+1,\beta)}{I(\alpha,\beta)},$$

where $I(a,b)$ is given by the equation (2.10).

Proof. This can be easily proved, so we omit the proof.

Also with the Harris loss function the following theorem can be easily obtained.

Theorem 2.5. If the Harris error loss function is used and θ follows a beta prior distribution with parameters α and β, then the Bayes estimator $\tilde{\theta}_{BH}$ of the reliability function given $\underline{X} = \underline{x}$ is given by

$$\tilde{\theta}_{BH} = 1 - \frac{I(\alpha, \beta)}{I(\alpha, \beta - 1)},$$

where $I(a, b)$ is given by the equation (2.10).

3. CREDIBLE INTERVALS AND HPD CREDIBLE REGIONS

In this section we derive the equal-tail credible intervals and the highest posterior density credible intervals for the reliability function. A subset C of the parameter space, with a degree of confidence $(1-\alpha)$ based upon a posterior probability density function $\Pi(\theta|\underline{x})$ is called a credible set, *i.e.*, $\int_C \pi(\theta|\underline{x})d\theta \geq 1 - \alpha$.

If the credible set is an interval, then it is known as a *credible interval* (Edwards, Lindman, and Savage(1963)) or a *Bayesian confidence interval* (Lindley(1965)). An equal-tail $100(1 - \alpha)$ percent confidence interval $(\hat{\theta}_L, \hat{\theta}_U)$ for the reliability function θ is obtained by solving $\int_{-\infty}^{\hat{\theta}_L} \pi(\theta|\underline{x})d\theta = \int_{\hat{\theta}_U}^{\infty} \pi(\theta|\underline{x})d\theta$.

It is natural to want to find the shorter credible set. A subset C of the parameter space Θ satisfying $C = \{\theta \in \Theta \mid \Pi(\theta|\underline{x}) \geq K(\alpha)\}$, where $K(\alpha)$ is the largest constant such that $P(C|\underline{x}) \geq 1-\alpha$, is called the $100(1-\alpha)$ percent highest probability density credible set (Berger(1985)).

First consider the Rayleigh distribution with pdf in (2.1). In this case, the likelihood function is given in (2.3). With the Jeffreys' noninformative prior distribution on the reliability θ, the equal-tail credible interval and the HPD credible interval were proposed by Sinha and Howlader (1983a, 1983b) and are given in the following: With a noninformative prior on θ, the $100(1-\alpha)$ percent credible interval $(\tilde{\theta}_{NL}, \tilde{\theta}_{NU})$ for θ is given by

$$\left(\exp\left(\frac{-\chi^2_{(2n;\alpha/2)}}{\frac{2\sum X_i^2}{t^2}}\right),\ \exp\left(\frac{-\chi^2_{(2n;1-\alpha/2)}}{\frac{2\sum X_i^2}{t^2}}\right)\right).$$

With the Jeffreys' noninformative prior, the $100(1-\alpha)$ percent HPD credible bounds $\tilde{\theta}_{NHL}$ and $\tilde{\theta}_{NHU}$ are the simultaneous solutions of

$$1-\alpha = P\left[\frac{2\sum X_i^2}{t^2}\ln\left(\frac{1}{\tilde{\theta}_{NHU}}\right) < \chi^2_{2n} < \frac{2\sum X_i^2}{t^2}\ln\left(\frac{1}{\tilde{\theta}_{NHL}}\right)\right],$$

$$\tilde{\theta}_{NHL}^{\frac{1}{t^2}\sum X_i^2-1}\left(\ln\tilde{\theta}_{NHL}\right)^{n-1} = \tilde{\theta}_{NHU}^{\frac{1}{t^2}\sum X_i^2-1}\left(\ln\tilde{\theta}_{NHU}\right)^{n-1}.$$

Now we consider a locally uniform prior distribution on θ to construct the credible interval for the reliability.

From the equations (2.3) and (2.7) the posterior density of θ is in (2.8). We transform θ to w, in the sense that

$$w = -2\left(1+\frac{\sum x_i^2}{t^2}\right)\ln\theta.$$

The Jacobian of transformation is given by

$$|J| = \frac{1}{2\left(1+\frac{\sum x_i^2}{t^2}\right)}\exp\left\{\frac{-w}{2\left(1+\frac{\sum x_i^2}{t^2}\right)}\right\},$$

so that the pdf of w given $\underline{X} = \underline{x}$ is given by

$$\Pi(w|\underline{x}) = \frac{1}{\Gamma(n+1)}\left(\frac{1}{2}\right)^{n+1} w^n e^{-w/2}, \qquad 0 < w < \infty,$$

where $\Gamma(a)$ is the gamma function. That is, the distribution of w is the chi-square distribution with 2(n+1) degrees of freedom. Therefore the $100(1-\alpha)$ percent equal-tail credible bounds $\tilde{\theta}_{CL}$ and $\tilde{\theta}_{CU}$ for θ in (2.2) are obtained by solving

$$-2\left(1+\frac{\sum X_i^2}{t^2}\right)\ln\tilde{\theta}_{CU} = \chi^2_{(2(n+1);1-\alpha/2)} \qquad (3.1)$$

and

$$-2\Big(1+\frac{\sum X_i^2}{t^2}\Big)\ln\tilde{\theta}_{CL}=\chi^2_{(2(n+1);\alpha/2)}, \tag{3.2}$$

where $\chi^2_{(\nu;\alpha)}$ is the $100(1-\alpha)th$ percentiles of a chi-square distribution with ν degrees of freedom.

Then we obtain the following theorems.

Theorem 3.1. With a locally uniform prior on θ, the $100(1-\alpha)$ percent equal-tail credible interval $(\tilde{\theta}_{CL},\ \tilde{\theta}_{CU})$ is

$$\left(\exp\Big(\frac{-\chi^2_{(2(n+1);\alpha/2)}}{2\big(1+\frac{\sum X_i^2}{t}\big)}\Big),\ \ \exp\Big(\frac{-\chi^2_{(2(n+1);1-\alpha/2)}}{2\big(1+\frac{\sum X_i^2}{t^2}\big)}\Big)\right).$$

Proof. The theorem is easily proved by solving the equations (3.1) and (3.2).

Theorem 3.2. If a locally uniform prior distribution for θ is used, the $100(1-\alpha)$ percent HPD credible bounds $\tilde{\theta}_{HL}$ and $\tilde{\theta}_{HU}$ of the reliability function θ in (2.2) are the simultaneous solutions of

$$P\Big[-2\Big(1+\frac{\sum X_i^2}{t^2}\Big)\ln\tilde{\theta}_{HU}<\chi^2_{2(n+1)}<-2\Big(1+\frac{\sum X_i^2}{t^2}\Big)\ln\tilde{\theta}_{HL}\Big]=1-\alpha,$$

$$(-\ln\tilde{\theta}_{HL})^n\exp\Big(\frac{\sum X_i^2}{t^2}\ln\hat{\theta}_{HL}\Big)=(-\ln\tilde{\theta}_{HU})^n\exp\Big(\frac{\sum X_i^2}{t^2}\ln\hat{\theta}_{HU}\Big).$$

Proof. From the posterior densities of θ and w given $\underline{X}=\underline{x}$, the theorem is easily proved.

We assume that the reliability function θ follows a beta prior distribution, $B^{\theta}(\alpha,\beta)$. Then the posterior density of θ given $\underline{X}=\underline{x}$ is given in (2.9). For this case the equal-tail credible interval and the HPD credible region can not be obtained in a closed form. Therefore in order to obtain the credible intervals, some numerical techniques should be used for numerical integrations. For example, the International Mathematical and Statistical Libraries(IMSL) subroutine DCADRE may be a useful computer routine for the numerical integrations.

4. MONTE CARLO SIMULATION STUDIES

In order to evaluate the performances of the proposed Bayes estimators for a given mission time t, some Monte Carlo simulations were carried out. IMSL subroutine GGWIB and the 50 points Gauss–Legendre quadratures were used for the numerical integration. Simulations were replicated 1000 times and the same seeds were used to compare the Bayes estimators. We consider both the squared error loss and the Harris loss with several prior distributions for the reliability. The estimated mean squared errors (MSE's) and biases of the Bayes estimators in Section 2 were computed. Some of the results are tabulated in Table 4.1 and 4.2 (biases are given in the parentheses) for $\sigma^2 = 1,\ n = 5, 10, 30$, and various values of time t.

From Tables 4.1 and 4.2, one can observe the followings:

1. The proposed Bayes estimators with respect to a beta prior distribution on the reliability θ perform better than the Bayes estimator with respect to a noninformative prior distribution for θ in terms of MSE's (biases) when the prior for θ spreads near the true value regardless of the type of a loss function. Note that a noninformative prior for θ is the implied prior of a noninformative prior for the parameter σ.

2. The Bayes estimator with respect to a locally uniform prior for θ performs better than the estimator with respect to a noninformative prior when the value of θ is not near the endpoints of the $(0, 1)$ interval.

3. The estimated MSE's (biases) decrease as sample size increases for both the squared error loss and the Harris loss.

4. The estimated MSE's (biases) become close to each other as the true value of θ approaches to 1 for both losses.

5. The performances of the proposed estimators are relatively sensitive to the prior density. That is, the MSE's with a proper prior are much smaller than those with a wrong prior distribution.

6. For a specified prior distribution, the Bayes estimator under Harris loss performs better than the Bayes estimator under squared error loss when the value of θ becomes larger.

Table 4.1. Estimated MSE's and Biases under Squared Error Loss for the Rayleigh Distribution($\sigma^2 = 1$).

n	t	θ	NI	LU^*	LU	$\mathcal{B}(1,9)$	$\mathcal{B}(3,7)$	$\mathcal{B}(5,5)$	$\mathcal{B}(7,3)$	$\mathcal{B}(9,1)$
5	.5	.8824	.00458	.00501	.00643	.04242	.02639	.01429	.00616	.00202
			(-.02136)	(-.02371)	(-.04102)	(-.17950)	(-.13729)	(-.09499)	(-.05262)	(-.01017)
	.8	.7261	.01418	.01679	.01563	.07427	.04038	.01734	.00552	.00520
			(-.03050)	(-.04096)	(-.05602)	(-.24524)	(-.17247)	(-.09905)	(-.02515)	(.04916)
	1.2	.4867	.02107	.02611	.01661	.04992	.01836	.00564	.01334	.04239
			(-.01338)	(-.04063)	(-.02392)	(-.20018)	(-.10301)	(-.00331)	(.09805)	(.20061)
	1.6	.2780	.01862	.02063	.01377	.01513	.00451	.01826	.05928	.12916
			(.02081)	(-.01931)	(.03541)	(-.09933)	(.00873)	(.12221)	(.23882)	(.35757)
	2.1	.1102	.01241	.00949	.01454	.00226	.01422	.05543	.12980	.23942
			(.04827)	(.00565)	(.08707)	(-.00236)	(.11000)	(.23189)	(.35870)	(.48862)
10	.5	.8824	.00188	.00195	.00238	.01332	.00866	.00510	.00264	.00129
			(-.01157)	(-.01250)	(-.02146)	(-.09851)	(-.07584)	(-.05316)	(-.03045)	(-.00773)
	.8	.7261	.00683	.00744	.00748	.03125	.01820	.00916	.00421	.00341
			(-.01820)	(-.02276)	(-.03250)	(-.15288)	(-.10864)	(-.06419)	(-.01956)	(.02523)
	1.2	.4867	.01187	.01343	.01055	.02697	.01193	.00536	.00779	.01960
			(-.01136)	(-.02483)	(-.01818)	(-.13855)	(-.07333)	(-.00688)	(.06052)	(.12870)
	1.6	.2780	.01061	.01147	.00885	.00999	.00438	.01040	.02948	.06261
			(.00748)	(-.01406)	(.01626)	(-.07192)	(.00319)	(.08179)	(.16283)	(.24571)
	2.1	.1102	.00588	.00505	.00673	.00209	.00787	.02766	.06413	.11899
			(.02434)	(.00083)	(.04755)	(-.00174)	(.07531)	(.16018)	(.24991)	(.34306)
30	.5	.8824	.00048	.00049	.00054	.00185	.00131	.00089	.00060	.00043
			(-.00410)	(-.00436)	(-.00736)	(-.03449)	(-.02671)	(-.01893)	(-.01114)	(-.00336)
	.8	.7261	.00199	.00205	.00210	.00588	.00392	.00253	.00173	.00152
			(-.00694)	(-.00828)	(-.01206)	(-.06009)	(-.04315)	(-.02619)	(-.00920)	(.00781)
	1.2	.4867	.00409	.00428	.00394	.00739	.00442	.00303	.00326	.00513
			(-.00550)	(-.00984)	(-.00837)	(-.06177)	(-.03365)	(-.00532)	(.02320)	(.05191)
	1.6	.2780	.00390	.00405	.00364	.00400	.00272	.00379	.00740	.01368
			(.00101)	(-.00649)	(.00426)	(-.03429)	(-.00012)	(.03490)	(.07065)	(.10706)
	2.1	.1102	.00188	.00178	.00201	.00124	.00245	.00623	.01308	.02337
			(.00773)	(-.00079)	(.01669)	(-.00075)	(.03313)	(.06955)	(.10807)	(.14835)

NI : Noninformative prior for θ
LU^* : Locally uniform prior for θ(GMLE)
LU : Locally uniform prior for θ
$\mathcal{B}(a,b)$: Beta prior for θ

Table 4.2. Estimated MSE's and Biases under Harris Loss for the Rayleigh Distribution($\sigma^2 = 1$).

n	t	θ	NI	LU	$\mathcal{B}(1,9)$	$\mathcal{B}(3,7)$	$\mathcal{B}(5,5)$	$\mathcal{B}(7,3)$	$\mathcal{B}(9,1)$
5	.5	.8824	.00311	.00417	.03696	.02184	.01087	.00409	.00154
			(.00245)	(-.01898)	(-.16507)	(-.12181)	(-.07846)	(-.03502)	(.00848)
	.8	.7261	.01177	.01175	.06631	.03368	.01290	.00440	.00848
			(.01146)	(-.01964)	(-.22809)	(-.15227)	(-.07576)	(.00128)	(.07875)
	1.2	.4867	.02318	.01698	.04516	.01504	.00614	.02022	.05836
			(.03557)	(.01679)	(-.18624)	(-.08358)	(.02186)	(.12912)	(.23770)
	1.6	.2780	.02534	.01924	.01377	.00541	.02493	.07567	.15951
			(.06121)	(.07020)	(-.08950)	(.02551)	(.14653)	(.27104)	(.39792)
	2.1	.1102	.01843	.02105	.00253	.01770	.06665	.15401	.28228
			(.07296)	(.11185)	(.00385)	(.12397)	(.25472)	(.39097)	(.53072)
10	.5	.8824	.00151	.00180	.01148	.00717	.00400	.00197	.00107
			(-.00037)	(-.01063)	(-.08971)	(-.06675)	(-.04378)	(-.02079)	(.00222)
	.8	.7261	.00606	.00624	.02749	.01528	.00729	.00362	.00431
			(.00223)	(-.01343)	(-.14032)	(-.09498)	(-.04942)	(-.00367)	(.04224)
	1.2	.4867	.01222	.01049	.02425	.01027	.00546	.01040	.02552
			(.01262)	(.00360)	(-.12720)	(-.05951)	(.00947)	(.07948)	(.15032)
	1.6	.2780	.01230	.01047	.00925	.00483	.01322	.03601	.07431
			(.02606)	(.03358)	(-.06409)	(.01442)	(.09668)	(.18159)	(.26849)
	2.1	.1102	.00722	.00838	.00227	.00932	.03195	.07318	.13499
			(.03381)	(.05763)	(.00251)	(.08335)	(.17265)	(.26726)	(.36560)
30	.5	.8824	.00044	.00047	.00161	.00112	.00076	.00052	.00040
			(-.00055)	(-.00384)	(-.03119)	(-.02338)	(-.01557)	(-.00775)	(.00007)
	.8	.7261	.00189	.00193	.00520	.00341	.00222	.00162	.00162
			(-.00027)	(-.00554)	(-.05446)	(-.03736)	(-.02025)	(-.00310)	(.01406)
	1.2	.4867	.00409	.00390	.00673	.00406	.00302	.00366	.00600
			(.00247)	(-.00066)	(-.05579)	(-.02723)	(.00156)	(.03054)	(.05972)
	1.6	.2780	.00409	.00384	.00382	.00282	.00428	.00839	.01531
			(.00693)	(.01005)	(-.03011)	(.00477)	(.04052)	(.07704)	(.11424)
	2.1	.1102	.00201	.00218	.00129	.00268	.00679	.01411	.02506
			(.01025)	(.01934)	(.00113)	(.03576)	(.07304)	(.11251)	(.15380)

NI : Noninformative prior for θ
LU : Locally uniform for θ
$\mathcal{B}(a,b)$: Beta prior for θ

5. NUMERICAL EXAMPLES

In this section, some numerical examples are given for the purpose of illustrating some results obtained in the previous sections. The data listed in

Table 5.1 are taken from Sinha(1986). They were generated from the Rayleigh distribution with $\sigma^2 = 2$.

Table 5.1. Random Samples of Size 25 from $\mathcal{R}(2)$

1.8487	0.3761	0.7500	3.0530	1.3545
1.8802	1.5700	1.7708	1.3592	3.0466
1.7961	1.5319	0.5903	0.6288	0.6461
1.6560	1.7172	1.9310	1.0509	1.6173
1.3162	0.7705	1.8889	1.8889	4.1505

The Bayes estimators proposed in Section 2 can be obtained for various values of time t. Using the data given in Table 5.1, the Bayes estimates with respect to a noninformative prior and a locally uniform prior for θ are calculated at a given misson time $t = 2$: $\hat{\theta}_{NS} = 0.30446$, $\hat{\theta}_{NH} = 0.31227$, $\tilde{\theta}_{US} = 0.30709$, $\tilde{\theta}_{UH} = 0.31464$, GMLE $= 0.29583$.

At a given time $t = 2$, we have the Bayes estimators with respect to a beta prior $\mathcal{B}(4,6)$ as follows :

$$\tilde{\theta}_{BS} = 0.317787, \tilde{\theta}_{BH} = 0.324345.$$

The 90% credible interval with respect to a beta prior $\mathcal{B}(4,6)$ at a specified mission time $t = 2$ is obtained and is given by (0.214992, 0.430183).

For a noninformative prior on θ, at a given $t = 2$, we obtain the HPD credible interval, given by

$$(\tilde{\theta}_{NHL}, \ \tilde{\theta}_{NHU}) = (0.186162, \ 0.420441).$$

Further $\tilde{\theta}_{NU} - \tilde{\theta}_{NL} = 0.23563 \quad > \quad \tilde{\theta}_{NHU} - \tilde{\theta}_{NHL} = 0.23428.$

For a locally uniform prior on θ, the HPD credible interval for θ, at a given $t = 2$, is obtained and is given by

$$(\tilde{\theta}_{HL}, \ \tilde{\theta}_{HU}) = (0.190871, \ 0.421115).$$

Note that $\tilde{\theta}_{CU} - \tilde{\theta}_{CL} = 0.23148 \quad > \quad \tilde{\theta}_{HU} - \tilde{\theta}_{HL} = 0.23024.$

For a beta prior $\mathcal{B}^{\theta}(4,6)$, the HPD credible interval at a given mission time $t = 2$ is given by (0.209743, 0.424107). Note that the length of the equal-tail credible interval is 0.21519 and that of the HPD interval is 0.21436.

As expected, the HPD credible intervals are shorter than the corresponding equal-tail credible intervals for all above cases. Note that the length of equal-tail credible interval for a locally uniform prior is shorter than that of the HPD interval for a noninformative prior and the length of equal-tail credible interval for a beta prior $\mathcal{B}(4,6)$ is shorter than that of the HPD interval for a locally uniform prior.

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REFERENCES

(1) Berger, J.O. (1985). *Statistical Decision Theory and Bayesian Analysis*, 2nd ed., Springer-Verlag, New York.

(2) Cheng, S.W. (1980). On the ABLUE and Optimum t-Test of the Parameter of Rayleigh Distribution. *Tamkang Journal of Mathematics*, **11**, **1–17**.

(3) Dyer, D.D. and Whisenand, C.W. (1973a). Best Linear Unbiased Estimator of the Parameter of the Rayleigh Distribution-Part I : Small Sample Theory for Censored Order Statistics, IEEE. *Transactions on Reliability*, **R-22**, **27–34**.

(4) Dyer, D.D. and Whisenand, C.W. (1973b). Best Linear Unbiased Estimator of the Parameter of the Rayleigh Distribution-Part II : Optimum Theory for Selected Order Statistics, IEEE. *Transactions on Reliability*, **R-22**, **229–231**.

(5) Edwards, W., Lindman, H., and Savage, L.J. (1963). Bayesian Statistical Inference for Psychological Research. *Psychological Review*, **70**, **193–242**.

(6) Erdélyi, A., Magnus, W., Oberhettinger, F., and Tricomi, F.G. (1955). *Tables of Integral Transforms*, Vol.**I**, McGraw-Hill, New York.

(7) Evans, R.A. (1969). Prior Knowledge, Engineers versus Statisticians, IEEE. *Transactions on Reliability*, **R-18**, **143**.

(8) Harris, B. (1976). A Survey of Statistical Methods in Systems Reliability using Bernoulli Sampling of Components. *Proceedings of the Conference on the Theory and Applications of Reliability with Emphasis on Bayesian and Nonparametric Methods*, Academic, New York.

(9) Jeffreys, H. (1961). *Theory of Probability*, 3rd ed., Oxford University Press, London.

(10) Lindley, D.V. (1965). *Introduction to Probability and Statistics from a Bayesian Viewpoint (Part 1:Probability and Part 2:Inference)*, Cambridge University Press, Cambridge.

(11) Polovko, A.M. (1968). *Fundamentals of Reliability Theory*, Academic, New York.

(12) Siddiqui, M.M. (1962). Some Problems Connected with Rayleigh Distribution. *Journal of Research of the National Bureau of Standards*-D, **66D**, **167–174**.

(13) Sinha, S.K. and Howlader, H.A. (1983a). Credible and HPD Intervals of the Parameter and Reliability of Rayleigh Distribution. *Technical Report No.* **120**, Department of Statistics, University of Manitoba, Winnipeg.

(14) Sinha, S.K. and Howlader, H.A. (1983b). Credible and HPD Intervals of the Parameter and Reliability of Rayleigh Distribution, IEEE. *Transactions on Reliability*, **R-32**, **217–220**.

(15) Sinha, S.K. (1986). *Reliability and Life Testing*, John Wiley, New York.