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A Law of Large Numbers for Maxima in $M/M/\infty$ Queues and INAR(1) Processes†

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ABSTRACT

Suppose that a stationary process $\{X_t\}$ has a marginal distribution whose support consists of sufficiently large integers. We are concerned with some analogous law of large numbers for such distribution function F. In particular, we determine a weak law of large numbers for maximum queueing length in $M/M/\infty$ system.

We also present a limiting behavior for the maxima based on AR(1) process with binomial thinning and poisson marginals(INAR(1)) introduced by E.Mckenzie. It turns out that the result of AR(1) process is the same as that of $M/M/\infty$ queueing process in limit when we observe the queues at regularly spaced intervals of time.

KEYWORDS: Law of large numbers, Maxima, $M/M/\infty$ queueing system, Binomial thinning, INAR(1) process.

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1. INTRODUCTION

Suppose $X_1, X_2, \ldots, X_n$ be *i.i.d.* random variables with distribution function F. Gnedenko(1943) proved that if for any $y > 0$,

$$\lim_{x \to \infty} \frac{1 - F(x)}{1 - F(x+y)} = \infty$$

then there exists a sequence of constant a_n such that the distribution of $M_n = \max\{X_1, X_2, \ldots, X_n\} - a_n$ converges to a distribution degenerate at 0 in which case M_n is said to obey the law of large numbers(LLN). The LLN, however, can not hold at all for integer valued random variables since $\frac{1-F(n)}{1-F(n+y)} = 1$ for each integer n and any $y \in (0,1)$. This problem stimulated Anderson(1970) to provide an analogus LLN for the case when F belongs to the class G of distribution functions whose support consists of all sufficiently large positive integers. Anderson defined a continuous distribution function F_c associated with each $F \in G$. Namely, define for any real number x

$$h_c(x) = h\Big([x]\Big) + \Big(x - [x]\Big)\Big(h\Big([x+1]\Big) - h\Big([x]\Big)\Big), \tag{1.1}$$

where $[x]$ is the largest integer not exceeding x and $h(n) = -\ln(1 - F(n))$ for integers n. Then set $F_c(x) = 1 - \exp\{-h_c(x)\}$. F_c has the properties that $F_c(x) = F(x)$ for integer x and $F(x) \le F_c(x) \le F(x+1)$. Furthermore, F_c is strictly increasing and for all sufficiently large n there exists an unique $\beta_n(\tau)$ such that for $\tau < \infty$

$$1 - F_c\Big(\beta_n(\tau)\Big) = n^{-1}\tau. \tag{1.2}$$

Theorem 1.1. (Anderson). Suppose $X_1, X_2, \cdots, X_n$ are *i.i.d* random variables with d.f F in G. Then

(i)
$$\limsup_{n \to \infty} P(M_n - \beta_n \le x) \le \exp(-e^{-ax})$$

and

$$\liminf_{n \to \infty} P(M_n - \beta_n \le x) \ge \exp(-e^{-a(x-1)})$$

for some $a > 0$, for all x if and only if

$$\lim_{n\to\infty} \frac{1-F(n)}{1-F(n+1)} = e^a.$$

Here, $\beta_n = \beta_n(1)$ in (1.2).

(ii) There exists a sequence of integers I_n such that

$$\lim_{n\to\infty} P\Big\{M_n = I_n \ or \ I_n + 1\Big\} = 1$$

if and ony if

$$\lim_{n\to\infty} \frac{1-F(n)}{1-F(n+1)} = \infty.$$

Anderson applied Theorem1.1,(i) to obtain bounds on the asymptotic distribution of the maximum queue length in a $M/M/1$ system with traffic intensity less than 1 by considering maxima between regeneration points. Similar works were considered for $M/M/s$ system by Serfozo(1988a and b) and McCormick and Park(1992a). With basically the same idea,we may apply the result (ii) in Theorem 1.1 to get an analogous LLN on the maximum queue length in $M/M/\infty$ queueing system. This is the first part of this paper.

Mckenzie(1988) introduced a family of models for discrete time processes with Poisson marginal distributions which are members of discrete self decomposible class(SDS)(cf.Steutel and Harn(1979)). Let $B_i, i \geq 1$ be an $i.i.d$ sequence of Bernoulli random variables with $P(B_i = 1) = \alpha$. The defining difference equation takes the form

$$X_n = \alpha * X_{n-1} + W_n, \tag{1.3}$$

where $\alpha * X$ denotes $\sum_{i=1}^{X} B_i$, and $\{W_n\}$ is a sequence of $i.i.d$ Poisson ($\bar{\alpha}\theta$), here $\bar{\alpha} = 1 - \alpha$. Then the sequence $\{X_n\}$ is stationary and has the same autocorrelation structures as for the usual linear AR(1) processes. Moreover, we easily see by probability generating function(p.g.f) that $\alpha * X$ is Poisson $(\alpha\theta)$ and the marginal distribution of X_n is Poisson (θ). In the second part of this paper we devote to extremes of sequences generated by (1.3). The approach used to establish an analogous LLN of M_n is technically similar to that of McCormick and Park(1992b) which generalizes Theorem1.1, (i).

2. A LAW OF LARGE NUMBERS FOR MAXIMUM QUEUEING LENGTH IN $M/M/\infty$ SYSTEM

Consider the queueing processes, so called $M/M/\infty$, that may be interpreted as the case where a new server is always available for each arriving customer in a birth and death processes with the birth rate $\lambda_k = \lambda$ and the death rate $\mu_k = k\mu$ where k is the queue size at which births occur or deaths occur. Let $Q(t)$ be the queue length at time t in a $M/M/\infty$ system. It can be checked by Chung(1967) that

$$F(n) = P\Big\{\max_{\tau_i \le t \le \tau_{i+1}} Q(t) \le n\Big\} = 1 - \Big(\sum_{k=0}^{n} k!\rho^{-k}\Big)^{-1}, \tag{2.1}$$

where τ_i is the ith visit time to state 0 and $\rho = \lambda/\mu$. Thus we have

$$\frac{1-F(n)}{1-F(n+1)} = \frac{(\sum_{k=0}^{n} k!\rho^{-k})^{-1}}{(\sum_{k=0}^{n+1} k!\rho^{-k})^{-1}} \to \infty \quad \text{as } n \to \infty. \tag{2.2}$$

We first present some preliminary results which are useful to find an unique $\beta_n(\tau)$ because, in practice, it is often not easy to get the β_n in (1.2).

Lemma 2.1. Suppose that $\frac{1-F(n)}{g(n)} \to 1$ as $n \to \infty$, where $F \in G$ and g is a continuous function. Then we may take $\beta_n(\tau)$ such that for some $\tau < \infty$,

$$n \cdot g\big(\beta_n(\tau)\big) \to \tau$$

if one of the following conditions is satisfied ;
A: $\lim_{\to\infty} \frac{1-F(n)}{1-F(n+1)} = e^a$ $0 < a < \infty$ and for $0 \le y \le 1$

$$\frac{g(x)}{g(x+y)} \to e^{ay} \quad \text{uniformly in } y \text{ as } x \to \infty, \tag{2.3}$$

or B: $\lim_{\to\infty} \frac{1-F(n)}{1-F(n+1)} = \infty$ and for some constant θ and $0 \le y \le 1$

$$\ln \frac{g(x+y)}{g(x)} \Big/ y \ln \frac{\theta}{x+y} \to 1 \quad \text{uniformly in } y \text{ as } x \to \infty. \tag{2.4}$$

Proof. We have from (1.1) that for large enough x

$$h_c(x) = -\ln g\Big([x]\Big) + \Big(x - [x]\Big)\Big(-\ln g\Big([x+1]\Big) + \ln g\Big([x]\Big)\Big) + o(1).$$

Assume that (2.3) holds. Then for large enough x

$$\begin{aligned} h_c(x) &= -\ln g(x) - \ln \frac{g([x])}{g(x)} + \Big(x - [x]\Big)\Big(\ln \frac{g([x])}{g([x+1])}\Big) + o(1). \\ &= -\ln g(x) + o(1). \end{aligned} \tag{2.5}$$

Next,when (2.4) is true,then for sufficiently large x

$$\begin{aligned} h_c(x) &= -\ln g(x) - \ln \frac{g([x])}{g(x)} + \Big(x - [x]\Big)\Big(\ln \frac{g([x])}{g([x+1])}\Big) + o(1). \\ &= -\ln g(x) + o(1). \end{aligned} \tag{2.6}$$

Choose $\beta_n(\tau)$ such that $n \cdot g(\beta_n(\tau)) \to \tau$ as $n \to \infty$. Let

$$\nu_n = \ln\Big\{n\Big(1 - F_c(\beta_n(\tau))\Big)\Big\} = \ln n - h_c(\beta_n(\tau)).$$

Then by (2.5) or (2.6) we have that

$$\begin{aligned} &\ln n \cdot g(\beta_n(\tau)) - \ln n + h_c(\beta_n(\tau)) \\ &= \ln g(\beta_n(\tau)) + h_c(\beta_n(\tau)) \to 0 \text{ as } n \to \infty \end{aligned} \tag{2.7}$$

Furthermore,

$$\ln n \cdot g(\beta_n(\tau)) \to \ln \tau.$$

Thus by (2.7)

$$\nu_n = \ln n - h_c(\beta_n(\tau)) \to \ln \tau.$$

Therefore,

$$n\Big(1 - F_c(\beta_n(\tau))\Big) \to \tau \text{ as } n \to \infty.$$

This completes the proof.

Lemma 2.2. Suppose that $Q(t)$ is the queue length at time t in $M/M/\infty$ queue with $\tau_0 = 0$. Then there exists a sequence of integers I_n such that

$$\lim_{n\to\infty} P\Big\{\max_{0\le t\le \tau_n} Q(t) = I_n \textit{ or } I_n + 1\Big\} = 1,$$

where

$$I_n = \Big[\frac{\ln n - \frac{1}{2}\ln 2\pi - \frac{1}{2}\ln\ln n}{\ln\ln n - 1 - \ln\rho} + \frac{1}{2}\Big].$$

Proof.. Let $Y_i = \max_{\tau_i \le t \le \tau_{i+1}} Q(t)$. Then, since $\tau_0 = 0$, Y_i's are $i.i.d$ random variables with d.f F defined in (2.1). Thus we have

$$\max_{0\le t\le \tau_n} Q(t) = \max\Big\{Y_1, Y_2, \dots, Y_n\Big\}.$$

The conclusion follows from (2.2) and Theorem 1.1, (ii) if we show that

$$\beta_n = \frac{\ln n - \frac{1}{2}\ln 2\pi - \frac{1}{2}\ln\ln n}{\ln\ln n - 1 - \ln\rho} + \frac{1}{2}, \tag{2.8}$$

where $\beta_n = \beta_n(1)$ in (1.2).

Now, by Stirling's formula, for large n

$$\begin{aligned} 1 - F([\beta_n]) &= \Big(\sum_{k=0}^{[\beta_n]} k!\rho^{-k}\Big)^{-1} \\ &\approx \Big\{\frac{[\beta_n]}{e}^{[\beta_n]}\sqrt{2\pi[\beta_n]}\rho^{-[\beta_n]}\Big\}^{-1} \end{aligned}$$

By taking $g(x) = \{(\frac{x}{e})^x\sqrt{2\pi x}\rho^{-x}\}^{-1}$ in Lemma 2.1, note that for $0 \le y \le 1$

$$\begin{aligned} &\ln\frac{g(x+y)}{g(x)}\Big/\Big\{y\ln\frac{\rho}{x+y}\Big\} \\ &= \frac{\ln(1+\frac{y}{x})^{-x} + \ln\sqrt{\frac{x}{x+y}} + y\ln\rho - y\ln(x+y) + y}{y\ln\rho - y\ln(x+y)} \to 1 \end{aligned}$$

uniformly in y as $x \to \infty$ since $(1 + y/x)^x \to e^y$ uniformly in y as $x \to \infty$.

Hence, by Lemma 2.1 and simply setting

$$\left(\frac{\beta_n}{e}\right)^{\beta_n} \sqrt{2\pi\beta_n}\rho^{-\beta_n} = n$$

the claim follows.

Note that the result of Lemma 2.2 may be useless in the practical sense because the time interval itself is a random variable. That is, τ_n cannot be observed in advance. Thus we consider the maximum queue length over a constant time interval $[0, t]$, i.e, the limiting behavior of $M_t = \max_{0 \le s \le t} Q(s)$.

Theorem 2.3. Suppose that $Q(t)$ is the same as in Lemma 2.2. Then there exists a sequence of integers I_t such that

$$\lim_{t \to \infty} P\Big\{M_t = I_t \ or \ I_t + 1\Big\} = 1,$$

where

$$I_t = \Big[\frac{\ln[m_t] - \frac{1}{2}\ln 2\pi - \frac{1}{2}\ln\ln[m_t]}{\ln\ln[m_t] - 1 - \ln\rho} + \frac{1}{2}\Big]$$

and

$$m_t = t \cdot \lambda e^{-\rho}.$$

Proof. We know from Theorem 2.2 of Hall(1988) that the expected length of busy cycles in a $M/M/\infty$ with arrival and service rate, λ and μ, respectively, is $\lambda^{-1}e^{\rho}$. Let N_t be the number of busy cycles up to time t. Then it is obvious by the renewal theory that

$$\frac{N_t}{m_t} \to 1 \text{ in probability as } t \to \infty. \tag{2.9}$$

Since

$$\Big\{M_t \le x_t\Big\} = \bigcup_{j=0}^{\infty} \Big[\Big\{N_t = j, \max_{0 \le s \le \tau_j} Q(s) \le x_t\Big\} \bigcap \Big\{\max_{\tau_j \le t < \tau_{j+1}} Q(s) \le x_t\Big\}\Big]$$

where $x_t \to \infty$ as $t \to \infty$, using the Markovian property, (2.1) and (2.9), one can show (cf. Berman(1986)) that for any $\delta > 0$ and for sufficently large t,

$$P\Big\{\max_{1\le k\le[m_t(1+\delta)]} Y_k \le x_t\Big\} + o(1) \tag{2.10}$$

$$\le P(M_t \le x_t)P\Big\{\max_{1\le k\le[m_t(1-\delta)]} Y_k \le x_t\Big\}, \tag{2.11}$$

where $Y_i = \max_{\tau_{i-1}\le t\le\tau_i} Q(t)$.
As shown by Anderson (1970), $\lim \frac{1-F(n)}{1-F(n+1)} = \infty$ for $F \in G$ implies that for any $\epsilon > 0$,

$$\lim_{n\to\infty} P\Big\{\max_{0\le s\le\tau_n} Q(s) \le \beta_n + \epsilon + 1\Big\} = 1$$

and

$$\lim_{n\to\infty} P\Big\{\max_{0\le s\le\tau_n} Q(s) \le \beta_n - \epsilon\Big\} = 0,$$

where β_n is $\beta_n(1)$ as given in (1.2).
Since,

$$\begin{aligned}\beta_{[m_t(1+\delta)]} &= \frac{\ln[m_t(1+\delta)] - \frac{1}{2}\ln 2\pi - \frac{1}{2}\ln\ln[m_t(1+\delta)]}{\ln\ln[m_t(1+\delta)] - 1 - \ln\rho} \\ &\le \beta_{[m_t]} + \frac{\ln[m_t(1+\delta)] - \ln[m_t]}{\ln\ln[m_t] - 1 - \ln\rho}\end{aligned}$$

for any $0 < \epsilon_1 < \frac{1}{2}$, there exists t_1 such that

$$\beta_{[m_t(1+\delta)]} \le \beta_{[m_t]} + \epsilon_1 \quad \text{whenever } t \ge t_1. \tag{2.12}$$

Similarly, there exists t_2 such that for any $0 < \epsilon_2 < \frac{1}{2}$

$$\beta_{[m_t(1-\delta)]} \ge \beta_{[m_t]} - \epsilon_2 \quad \text{whenever } t \ge t_2. \tag{2.13}$$

Let $\epsilon_0 = \max(\epsilon_1, \epsilon_2)$. Taking $X_t = \beta_{[m_t(1+\delta)]} + \epsilon + 1$ in (2.10), we have by (2.12)

$$1 \le \liminf P\Big\{M_t \le \beta_{[m_t(1+\delta)]} + \epsilon + 1\Big\} \le \liminf P\Big\{M_t \le \beta_{[m_t]} + \epsilon_0 + \epsilon + 1\Big\}$$

and thus

$$\lim P\Big\{M_t \le \beta_{[m_t]} + \epsilon_0 + \epsilon + 1\Big\} = 1. \tag{2.14}$$

Next, letting $X_t = \beta_{[m_t(1-\delta)]} - \epsilon$ in (2.11), we have by (2.13)

$$\limsup P\Big\{M_t \le \beta_{[m_t]} - \epsilon_0 - \epsilon\Big\} \le \limsup P\Big\{M_t \le \beta_{[m_t(1-\delta)]} - \epsilon\Big\} \le 0$$

and so

$$\lim P\Big\{M_t \le \beta_{[m_t]} - \epsilon_0 - \epsilon\Big\} = 0. \tag{2.15}$$

Thus (2.14) and (2.15) yield

$$\lim P\Big\{\Big|M_t - (\beta_{[m_t]} + \frac{1}{2})\Big| \le \frac{1}{2} + \epsilon_0 + \epsilon\Big\} = 1.$$

Since M_t is an integer valued random variable, the proof follows.

3. A LAW OF LARGE NUMBERS FOR AR(1) PROCESSES WITH POISSON MARGINALS

Let $\{X_n\}$ be a sequence of stationary processes with the marginal distribution function F in G. Obviously, some form of dependence restriction is necessary for a limiting distribution of $M_n = \max(X_1, X_2, \cdots, X_n)$ as for an usual stationary sequence. Leadbetter et. al.(1983) introduced distributional mixing conditions. Let

$$F_{1,2,\cdots,k}(u_n) = P(X_1 \le u_n, X_2 \le u_n, \cdots, X_k \le u_n),$$

where u_n is a sequence of constants such that $u_n \to \infty$ as $n \to \infty$. Now, define, so called, mixing conditions as follows :

The *condition* $D(u_n)$ is said to hold if for any integers $1 \le i_1 < \cdots < i_p < j_1 < \cdots < j_q \le n$ for which $j_1 - i_p \ge l$, we have

$$\Big|F_{i_1,\cdots,i_p,j_1,\cdots,j_q}(u_n) - F_{i_1,\cdots,i_p}(u_n)F_{j_1,\cdots,j_q}(u_n)\Big| \le \alpha_{n_l}, \tag{3.1}$$

where $\alpha_{n_l} \to 0$ as $n \to \infty$ for some sequence $l_n = o(n)$.

The *condition* $D'(u_n)$ is said to hold for the stationary sequence $\{X_n\}$ if

$$\limsup n \sum_{i=2}^{[\frac{n}{k}]} P(X_1 > u_n, X_i > u_n) \to 0 \quad \text{as} \quad k \to \infty.$$

If $n\{1 - F(u_n)\} \to \infty$, the condition $D'(u_n)$ is not satisfied even for i.i.d sequences. However, when we modify the mixing conditions in a natural manner for such sequences, we have the following result analogous to LLN which may be regarded as an extention of the result (ii) in Theorem 1.1.

Theorem 3.1. For a stationary sequence $\{X_n\}$ with marginal d.f $F \in G$ such that

$$\lim_{n\to\infty} \frac{1 - F(n)}{1 - F(n+1)} = \infty \tag{3.2}$$

if, for arbitrarily large $\tau(< \infty)$, there exists a sequence ν_n of constants such that $\lim n\{1 - F_c(\nu_n)\} = \tau$ and, $D(\nu_n)$ and $D'(\nu_n)$ hold, then there exists a sequence of b_n such that

$$\lim P\Big\{M_n = \Big[b_n + \frac{1}{2}\Big] \text{ or } \Big[b_n + \frac{1}{2}\Big] + 1\Big\} = 1.$$

In fact, a possible choice of b_n is $\beta_n(1)$ in (1.2).

Proof. Let $n' = [n/k]$ for fixed k and each n. Since

$$\{M_{n'} > u_n\} = \bigcup_{i=1}^{n'} \{X_i > u_n\},$$

we have, by stationarity

$$1 - n'\Big\{1 - F(u_n)\Big\} \le P\Big\{M_{n'} \le u_n\Big\} \le 1 - n'\Big\{1 - F(u_n)\Big\} + S_n, \tag{3.3}$$

where $S_n = S_{n,k} = n' \sum_{i=2}^{n'} P(X_1 > u_n, X_i > u_n)$.
Since (3.2) implies that

$$\lim_{x\to\infty} \frac{1 - F_c(x)}{1 - F_c(x+y)} = \infty \quad \text{for any } y > 0,$$

it follows that

$$n\Big\{1 - F_c(\beta_n + \epsilon)\Big\} = \frac{1 - F_c(\beta_n + \epsilon)}{1 - F_c(\beta_n)} \to 0 \tag{3.4}$$

and

$$n\Big\{1 - F_c(\beta_n - \epsilon)\Big\} = \frac{1 - F_c(\beta_n - \epsilon)}{1 - F_c(\beta_n)} \to \infty, \tag{3.5}$$

where $\beta_n = \beta_n(1)$.

Furthermore, by the property of $F_c(\cdot)$, we have

$$1 - F_c(u_n) \leq 1 - F(u_n) \leq 1 - F_c(u_n - 1). \tag{3.6}$$

Thus, if we take $u_n = \beta_n + \epsilon + 1$ in (3.3), it follows immediately by (3.4) and (3.6) that

$$P\Big\{M_n \leq \beta_n + \epsilon + 1\Big\} \to 1. \tag{3.7}$$

Next, we have by assumptions that for fixed τ

$$\limsup P\Big\{M_n \leq \nu_n\Big\} \leq e^{-\tau},$$

where we used (3.3) and (3.6).
Since $n \cdot \{1 - F_c(\beta_n - \epsilon)\} \to \infty$, clearly $\beta_n - \epsilon \leq \nu_n$ for sufficiently large n so that

$$\limsup P\Big\{M_n \leq \beta_n - \epsilon\Big\} \leq \limsup P\Big\{M_n \leq \nu_n\Big\} \leq e^{-\tau}.$$

Since this holds for arbitrarily large τ, by letting $\tau \to \infty$ we have

$$\lim_{n\to\infty} P\Big\{M_n \leq \beta_n - \epsilon\Big\} = 0. \tag{3.8}$$

Hence, (3.7) and (3.8) imply the proof.

We now present some preliminary results that will be needed in establishing our mixing conditions defined earlier. Note throughout this paper we define $\bar{\alpha}^n = 1 - \alpha^n$.

Lemma 3.2. For the stationary sequence $\{X_n\}$ given in (1.3), we have for each n,

$$(X_0, X_n) = \Big(X_0, \sum_{i=1}^{n} \alpha^{n-i} * W_i + \alpha^n * X_0\Big) \quad \text{in distribution.} \tag{3.9}$$

Proof. The equality in (3.9) is easily checked by p.g.f and mathematical induction.

Lemma 3.3. When X_n is according to Poisson distribution with parameter $\theta, \beta_n(1)$ in (1.2) is given by

$$\beta_n = \frac{\ln n + \frac{7}{2} - \theta + \frac{5}{2}\ln\theta - \frac{1}{2}\ln 2\pi}{\ln\ln n - \ln\ln\ln n - 1 - \ln\theta} + \frac{3}{2}.$$

Proof. First, observe that for large n

$$\frac{1-F(n)}{1-F(n+1)} = \sum_{x=n+1}^{\infty} \frac{e^{-\theta}\theta^x}{x!} \Big/ \sum_{x=n+2}^{\infty} \frac{e^{-\theta}\theta^x}{x!} \approx \frac{\frac{\theta^{n+1}}{(n+1)!}}{\frac{\theta^{n+2}}{(n+1)!}} = \frac{n+2}{\theta},$$

where we used the *Stirling*'s formula. Hence we have the tail probability for large β_n,

$$\begin{aligned} 1 - F([\beta_n + 1]) &= \sum_{[\beta_n+1]}^{\infty} \frac{e^{-\theta}\theta^x}{x!} \\ &\approx e^{-\theta}\theta^{[\beta_n+1]}\Big\{\Big(\frac{[\beta_n+1]}{e}\Big)^{[\beta_n+1]}\sqrt{2\pi[\beta_n+1]}\Big\}^{-1} \end{aligned} \tag{3.10}$$

Note that (3.10) is of the same form as for $M/M/\infty$ queues in Lemma 2.2. Thus by Lemma 2.1 and simply setting

$$e^{-\theta}\theta^{\beta_n}\Big\{\Big(\frac{\beta_n}{e}\Big)^{\beta_n}\sqrt{2\pi\beta_n}\Big\}^{-1} = \frac{1}{n},$$

the proof follows.

For the Markov process $\{X_n\}$ given in (1.3), let P_x and P be the distributions of X_n with initial distributions which are degenerate at x and Poisson(θ), respectively.

Lemma 3.4. The stationary sequence $\{X_n\}$ given in (1.3) satisfies condition $D(u_n)$ for any sequence $\{u_n\}$. Moreover, the mixing coefficient in (3.1) is

$$\alpha_{n_l} = O(\eta^{\sqrt{l}}) \quad \text{for some } 0 < \eta < 1.$$

Proof. By Lemma 2.5-2.7 in McCormick and Park (1992a), for $j - i \geq l$,some constant c, d and $0 \leq \delta, \epsilon \leq 1$

$$\begin{aligned} &\Big|F_{X_1,\cdots,X_i,X_j,\cdots,X_n}(u_n) - F_{X_1,\cdots,X_i}(u_n)F_{X_j,\cdots,X_n}(u_n)\Big| \\ &\leq \ (n+2)P(\alpha^n * X_0 \geq 1) + 2P(X_n \geq n) + c\epsilon^{\sqrt{n}} \\ &\leq \ \Big\{(n+2)\alpha^n + \delta n\Big\}d + c\epsilon^{\sqrt{n}} \end{aligned}$$

where we used $\alpha * X$ is Poisson($\alpha\theta$). Hence the proof is completed.
Note that since $n\{1 - F_c(\beta_n(1)\} \to 1$, we can always find the sequence of ν_n by nature of Poisson distribution such that

$$n\Big\{1 - F_c(\nu_n)\Big\} \to \tau$$

Moreover, one can easily see that

$$\nu_n = O\Big(\frac{\ln n}{\ln \ln n}\Big) \tag{3.11}$$

Lemma 3.5. For the sequance of $\{X_n\}$ given in (1.3), condition $D'(\nu_n)$ holds.

Proof. First observe by Lemma 3.2 that

$$P(X_1 > \nu_n, X_i > \nu_n)$$

$$= P\Big(X_1 > \nu_n, \sum_{j=2}^{i} \alpha^{i-j} * W_i + \sum_{j=1}^{X_1} B_j(\alpha^{i-1}) > \nu_n\Big)$$

$$\approx P\Big(\sum_{j=2}^{i} \alpha^{i-j} * W_i + \sum_{j=1}^{[\nu_n+1]} B_j(\alpha^{i-1}) > \nu_n\Big) P\Big(X_1 = [\nu_n + 1]\Big) \qquad (3.12)$$

where we used the tail behavior of Poisson distribution. And, one can see by the choice of ν_n in (3.11) that

$$n \cdot P\Big(X_1 = [\beta_n + 1]\Big) = O(\ln n). \qquad (3.13)$$

Since $\sum_{j=2}^{i} \alpha^{i-j} * W_i$ has Poisson distribution with parameter $\bar{\alpha}^{i-1}\theta$ by p.g.f and is independent of $\sum_{j=1}^{[\nu_n+1]} B_j(\alpha^{i-1})$, we have an upper bound by *Bernstein*'s inequality such that, for some canstant ζ and each n

$$P\Big(\sum_{j=2}^{i} \alpha^{i-j} * W_i + \sum_{j=1}^{[\nu_n+1]} B_j(\alpha^{i-1}) > \nu_n\Big)$$

$$\leq \zeta \cdot \exp(-\bar{\alpha}\theta(1 - s_n))(\bar{\alpha} + \alpha s_n)^{\nu_n} {s_n}^{-\nu_n}. \qquad (3.14)$$

Now, taking $s_n = \frac{\bar{\alpha}(\ln n)^{\gamma}}{1-\alpha(\ln n)^{\gamma}}, \gamma > 1$, (3.14) becomes $O(n^{\gamma})$.
Thus, (3.12) and (3.13) with the above yield, for some constant ζ^*

$$\lim_{n\to\infty} n \cdot \sum_{i=2}^{[\frac{n}{k}]} P(X_1 > \nu_n, X_i > \nu_n)$$

$$\leq \lim_{n\to\infty} \zeta^* n^{1-\gamma} \ln n.$$

This completes the proof.

Theorem 3.6. For the sequence of $\{X_n\}$ defined by (1.3), we have

$$\lim_{n\to\infty} P\Big\{M_n = \Big[b_n + \frac{1}{2}\Big] \text{ or } \Big[b_n + \frac{1}{2}\Big] + 1\Big\} = 1,$$

where

$$b_n = \frac{\ln n + \frac{7}{2} - \theta + \frac{5}{2}\ln\theta - \frac{1}{2}\ln 2\pi}{\ln\ln n - \ln\ln\ln n - 1 - \ln\theta} + \frac{3}{2}.$$

Proof. By Theorem 3.1, Lemma 3.4 and Lemma 3.5, the theorem is immediate.

Remark. : The norming constant b_n in Theorem 3.6 is asymptotically the same as that in Lemma 2.2. This result confirms comments of Steutel et.al (1983) and Mckenzie(1988) when we observe $M/M/\infty$ queueing system at regularly spaced intervals of time. Moreover, we can see that the Poisson parameter θ in (1.3) can be interpreted as the traffic intensity $\rho = \frac{\lambda}{\mu}$ in $M/M/\infty$ queues.

Discussion. : The results we have discussed here may be applied to determine design of systems such as telephone exchanges and super markets. For example, telephone engineers studying questions of fundamental importance for designing telephone exchanges may be confronted with the type of stochastic processes discussed in this paper. A typical concern is the queue length of customers waiting to be served. Large values of the queue may call for installation of auxillary exchanges , employee overtime, or redesigning telephone exchanges. A natural question is: What is the probability that the queue will exceed a specified critical value within a certain interval of time? Or how does one design telephone exchanges to manage a given percentage of calling demand for a long future time? These kinds of questions can be answered by considering the maximum value behaviour of queues generated from $M/M/\infty$ or INAR(1) processes.

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