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A Note on the Interchangeable Process

DugHun Hong[1]

ABSTRACT

Let $\{X_n\}$ be conditionally i.i.d. given $\mathcal{G} \subset \sigma(X_n, n \geq 1)$. We will prove that $\mathcal{G}$ is degenerate if and only if $\{X_n, n \geq 1\}$ are i.i.d. random variables(r.v.s). As a corollary the Hewitt–Savage zero–one law is obtained using the fact that interchangeable process is conditionally i.i.d. given the σ–algebra of permutable events.

KEYWORDS: Interchangeability, Ergodic theorem, Degenerate σ-algebra.

Let $\{X_n, n \geq 1\}$ be a stochastic process. The random variables comprising it or the process itself will be said to be interchangeable if, for any choice of distinct positive integers $i_1, i_2, i_3, \ldots, i_k$, the joint distribution of

$$X_{i1}, X_{i2}, \ldots, X_{ik}$$

depends merely on k and is independent of the integer $i_1, i_2, \ldots, i_k$. From this definition we can easily check that it is a stationary process (Breiman(1968),p118). Then by the Birkhoff's ergodic theorem $\lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} I_{[X_i<x]} = P[X_1 < x|\mathcal{I}]$ almost certainly(a.c.), where $\mathcal{I}$ is the σ–algebra of invariant events. If we

[1]Department of Statistics, Hyosung Women's University, Kyungbuk, 713–702, Korea.

follow the proof of theorem 7.3.2(Chow and Teicher(1988)) using the above fact and noting that the tail σ-algebra $\mathcal{T} \supset \mathcal{I}$, we have the following result.

Theorem 1. Random variables X_n, $n = 1, 2, \ldots$, on $(\Omega, \mathcal{F}, P)$ are interchangeable if and only if they are conditionally independent and identically distributed given either the σ-algebra $\mathcal{P}$ of permutable events or the σ-algebra $\mathcal{I}$ of invariant events.

Definition. A σ-algebra $\mathcal{G}$ of events is called degenerate if $P(G) = 0$ or 1, for all $G \in \mathcal{G}$.

It is noted that the conditional independence of $\{X_n\}$ given $\mathcal{G}$ does not imply that $\mathcal{G}$ is degenerate. A good example for this is sign-invariant random variables (see Berman(1965)). Now we consider the main part of this paper.

Theorem 2. Let $\{X_n, n \geq 1\}$ be conditionally i.i.d. given a σ-algebra $\mathcal{G} \subset \sigma(X_n, n \geq 1)$. Then $\mathcal{G}$ is degenerate if and only if $\{X_n, n \geq 1\}$ are i.i.d. r.v.s.

Proof. Suppose $\mathcal{G}$ is trivial. Then for any event A, a version of $P(A|\mathcal{G})$ is $P(A)$. Thus, any two conditionally independent events given $\mathcal{G}$ are independent. The independence of the sequence $\{X_n\}$ depends only on countably many events, so the sequence is independent if it is conditionally independent given $\mathcal{G}$. The distribution of the sequence $\{X_n\}$ also is determined by countably many events, so the conditional distribution is the same as the distribution.

For the other half, let $\mathcal{G}$ be a non-trivial sub-σ-algebra of $\sigma(X_1, X_2, \ldots)$ and let $X_1, X_2, \ldots$ be i.i.d. Let A be an event in $\mathcal{G}$ such that $\delta < P(A) < 1 - \delta$, where δ is some positive quantity. It is enough to show that the sequence $\{X_n\}$ is not identically distributed, conditioned on the occurrence of A. Choose an integer N and a Borel set $B \in R^N$ such that A is closely approximated by the event $C = \{(X_1, \ldots, X_N) \in B\}$. In other words, $P(A \Delta C)$ is small in comparison with δ. Note that C is independent of $C' = \{(X_{N+1}, \ldots, X_{2N}) \in B\}$. Thus, $P(C'|A) \approx P(C'|C) = P(C') = P(C) \approx P(A) < 1 - \delta$. But

$P(C|A) \approx P(C|C) = 1$. The desired result follows.

Remark 1. The second half of the proof of above theorem can be obtained easily from Theorem 7.3.4(Chow and Teicher(1988)), but we give simple direct proof here.

Corollary 1. If $\{X_n, n \geq 1\}$ are conditionally i.i.d. given a σ–algebra $\mathcal{G} \subset \sigma(X_n, n \geq 1)$ and is ergodic, then $\mathcal{G}$ is degenerate.

Corollary 2. If an interchangeable process $\{X_n, n \geq 1\}$ is ergodic, then the σ-algebra of permutable events is degenerate.

Corollary 3. (Hewitt–Savage Zero-One Law) If $\{X_n, n \geq 1\}$ are i.i.d. r.v.s, then the σ–algebra of permutable events is degenerate.

Remark 2. In Corollary 7.3.8 (Chow and Teicher(1988)), Theorem 7.3.4 (Chow and Teicher(1988)) and, additionally, Kolmogorov zero-one law are used.

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