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Random Central Limit Theorem of a Stationary Linear Lattice Process

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ABSTRACT

A simple proof for the random central limit theorem is given for a family of stationary linear lattice processes, which belong to a class of 2 dimensional random fields, applying the Beveridge and Nelson decomposition in time series context. The result is an extension of Fakhre-Zakeri and Farshidi (1993) dealing with the linear process in time series to the case of the linear lattice process with 2 dimensional indices.

KEYWORDS: Random central limit theorem, Stationary linear lattice processes, Random fields, The Beveridge and Nelson decomposition, Time series.

1. INTRODUCTION

Let us consider the stationary linear lattice process of the form

$$X(s,t)-\mu=\sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}a_{ij}\varepsilon(s-i,t-j),\quad (s,t)\in\mathcal{Z}\times\mathcal{Z}, \tag{1.1}$$

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where $\sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}|a_{ij}|<\infty$ and $\varepsilon(s,t)$ are iid random variables with mean zero, variance $\sigma^2>0$ and finite $(2+\delta)$-th moment for some $\delta>0$. The model as above is an extension of the linear process in time series to the lattice process. Much details regarding the statistical applications of lattice processes can be found in Cressie (1991, Chapters 6 and 7).

The main objective of this article is to establish the following:

Theorem 1. Let $\bar{X}_{m,n}=\Big\{(2m+1)(2n+1)\Big\}^{-\frac{1}{2}}\sum_{s=-m}^{m}\sum_{t=-n}^{n}X(s,t)$. Suppose that $\{M_m\}$ and $\{N_n\}$ are both nonnegative integer valued random variables such that $M_m/m\to M$ and $N_n/n\to N$ in probability as $m,n\to\infty$, respectively, where $P(0<M<\infty)=P(0<N<\infty)=1$. Then, it follows that

$$\Big\{(2M_m+1)(2N_n+1)\Big\}^{\frac{1}{2}}(\bar{X}_{M_m,N_n}-\mu)\xrightarrow{\mathcal{D}}\mathcal{N}\Big(0,\Big(\sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}a_{ij}\Big)^2\sigma^2\Big)$$

$$\text{as}\quad m,n\to\infty.$$

This theorem generalizes the random central limit theorem of Fakhre-Zakeri and Farshidi (1991) in time series to stationary lattice processes. They show that if the time series is of the form $X_t=\mu+\sum_{i=-\infty}^{\infty}a_i\varepsilon_{t-i}$, where $\sum_{i=-\infty}^{\infty}|a_i|<\infty$ and ε_t are random variables with mean 0 and finite variance $\sigma^2>0$, and if $\{N_n\}$ is a sequence of positive integer-valued random variables satisfying $N_n/n\to N$, $\Big[P(0<N<\infty)=1\Big]$, as $n\to\infty$, then

$$N_n^{1/2}(\bar{X}_{N_n}-\mu)\xrightarrow{\mathcal{D}}\mathcal{N}\Big(0,\Big(\sum_{i=-\infty}^{\infty}a_i\Big)^2\sigma^2\Big)\quad\text{as}\quad n\to\infty,$$

where $\bar{X}_n$ is the sample mean of $X_1,\dots,X_n$.

The crucial step to establish the result turns out to be the application of the Beveridge and Nelson decomposition that proves very useful to derive several asymptotic properties in time series (cf. Phillips and Solo (1992)). It will be shown later on that a similar approach can be taken to assert our theorem.

Since the work of Anscombe (1952), there have been a variety of developments in the weak convergence of randomly indexed sequences of random variables. Representative works include Renyi (1960), Blum, Hanson and Rosenblatt (1963) and Durret and Resnick (1977). Further references can be found in these papers. Applications to sequential analysis are also well-known in both iid random sample and time series. See Fakhre-Zakeri and Lee (1992, 1993) and the literatures cited in there. Finally, for the history of the central limit theorem in spacial sample (or random fields) we refer the reader to Goldie and Morrow (1985). As seen in the article, most results are obtained assuming a mixing condition (see also Bolthausen (1982)). However, the process in (1.1) does not necessarily satisfy desired mixing conditions like we can observe in time series literature. Therefore, our result covers the processes that may not satisfy the strong mixing condition.

2. PROOFS

Lemma 1. Let $\{\varepsilon(s,t)\}$ and $\{a_{ij}\}$ be the family of the iid random variables and real numbers in (1), respectively. Define $X_1(s,t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij}\varepsilon(s-i,t-j)$. Then if $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}|b_{ij}| < \infty$, where $b_{ij} = \sum_{u=i}^{\infty}\sum_{v=j}^{\infty} a_{uv}$, it holds that

$$
\begin{aligned}
X_1(s,t) &= \Big(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij}\Big)\varepsilon(s,t) + W(s,t) - W(s+1,t) \\
&\quad - W(s,t+1) + W(s+1,t+1) + U(s+1,t) \\
&\quad - U(s+1,t+1) + V(s,t+1) - V(s+1,t+1),
\end{aligned}
$$

where $W(s,t) = \sum_{(i,j)\neq(0,0)} b_{ij}\varepsilon(s-i,t-j)$, $U(s,t) = \sum_{j=1}^{\infty} b_{0j}\varepsilon(s,t-j)$ and $V(s,t) = \sum_{i=1}^{\infty} b_{i0}\varepsilon(s-i,t)$.

Proof. Write

$$
X_1(s,t) = a_{00}\varepsilon(s,t) + \sum_{(i,j)\neq(0,0)} a_{ij}\varepsilon(s-i,t-j)
$$

$$= b_{00}\varepsilon(s,t) + (-b_{10} - b_{01} + b_{11})\varepsilon(s,t) + \sum_{(i,j)\neq(0,0)} \Big\{b_{ij} - b_{i+1,j} - b_{i,j+1} + b_{i+1,j+1}\Big\}\varepsilon(s-i,t-j).$$

Observe that the following identifications hold:

$$\sum_{(i,j)\neq(0,0)} b_{i+1,j}\varepsilon(s-i,t-j) + b_{10}\varepsilon(s,t) = W(s+1,t) - U(s+1,t)$$

and

$$\sum_{(i,j)\neq(0,0)} b_{i,j+1}\varepsilon(s-i,t-j) + b_{01}\varepsilon(s,t) = W(s,t+1) - V(s,t+1).$$

Similarly, we have

$$\sum_{(i,j)\neq(0,0)} b_{i+1,j+1}\varepsilon(s-i,t-j) + b_{11}\varepsilon(s,t)$$
$$= W(s+1,t+1) - U(s+1,t+1) - V(s+1,t+1).$$

The lemma is then asserted by simple algebras.

Lemma 2. Let $\{M_m\}$, $\{N_n\}$, $U(s,t)$ and $V(s,t)$ be those in Theorem 1 and Lemma 1, respectively, and suppose that $a_{ij} = 0$ for $|i| > l$ or $|j| > l$ $(l > 0)$. Then, as $m, n \to \infty$,

(a) $(M_m N_n)^{-1/2} \sum_{s=1}^{M_m} U(s, N_n) \to 0$ in probability

(b) $(M_m N_n)^{-1/2} \sum_{t=1}^{N_n} V(M_m, t) \to 0$ in probability.

Proof. We only prove (a) since (b) can be treated similarly. Let $\lambda > 0$ and r be a positive integer. Then

$$\limsup_{m,n\to\infty} P\left[\left|(M_m N_n)^{-1/2} \sum_{s=0}^{M_m} U(s, N_n)\right| > \lambda\right]$$
$$\leq \limsup_{m,n\to\infty} P\left[\sum_{j=1}^{\infty} |b_{0j}|\left|(M_m N_n)^{-1/2} \sum_{s=0}^{M_m} \varepsilon(s, N_n - j)\right| > \lambda,\right.$$

$$\left|M_m/m - M\right| < 2^{-r}, |N_n/n - N| < 2^{-r}\Big]$$

$$\leq \limsup_{m,n\to\infty} P\left[\sum_{j=1}^{\infty} |b_{0j}|\left\{(M-2^{-r})m\cdot(N-2^{-r})n\right\}^{-1/2}\right.$$

$$\left.\times \max_{(u,v)\in J_r}\left|\sum_{s=0}^{u}\varepsilon(s,v-j)\right| > \lambda\right],$$

where $J_r = \{(u,v); |u/m - M| < 2^{-r}, |v/n - N| < 2^{-r}\}$. Now, put $S_{r,\alpha,\beta} = \left\{(\alpha-1)2^{-r} \leq M \leq \alpha 2^{-r}, (\beta-1)2^{-r} \leq N \leq \beta 2^{-r}\right\}$, $\alpha,\beta = r,\ldots,r2^r$. Since $P(\cup_{\alpha,\beta=r}^{r2^r} S_{r,\alpha,\beta}) \to 1$ as $r\to\infty$ by assumption, the argument in the last inequality is bounded by

$$\limsup_{r\to\infty}\sum_{\alpha,\beta=r}^{r2^r}\limsup_{m,n\to\infty} P\left[\left\{\sum_{j=1}^{\infty}|b_{0j}|\left\{(M-2^{-r})m\cdot(N-2^{-r})n\right\}^{-1/2}\right.\right.$$

$$\left.\left.\times \max_{(u,v)\in J_{r,m,n,\alpha,\beta}}\left|\sum_{s=0}^{u}\varepsilon(s,v-j)\right| > \lambda\right\}\cap S_{r,\alpha,\beta}\right]$$

$$\leq \limsup_{r\to\infty}\sum_{\alpha,\beta=r}^{r2^r}\limsup_{m,n\to\infty} P\left[\sum_{j=1}^{\infty}|b_{0j}|\left\{(\alpha-2)m2^{-r}\cdot(\beta-2)n2^{-r}\right\}^{-1/2}\right.$$

$$\left.\times \max_{(u,v)\in J_{r,m,n,\alpha,\beta}}\left|\sum_{s=0}^{u}\varepsilon(s,v-j)\right| > \lambda\,\middle|\, S_{r,\alpha,\beta}\right] P(S_{r,\alpha,\beta}),$$

where $J_{r,m,n,\alpha,\beta} = (m(\alpha-2)2^{-r}, m(\alpha+1)2^{-r}) \times (n(\beta-2)2^{-r}, n(\beta+1)2^{-r})$. This, due to the lemma of Blum et al. (1963, Lemma 3), equals to

$$\limsup_{r\to\infty}\sum_{\alpha,\beta=r}^{r2^r}\limsup_{m,n\to\infty} P\left[\sum_{j=1}^{\infty}|b_{0j}|\left\{(\alpha-2)m2^{-r}\cdot(\beta-2)n2^{-r}\right\}^{-1/2}\right.$$

$$\left.\times \max_{(u,v)\in J_{r,m,n,\alpha,\beta}}\left|\sum_{s=0}^{u}\varepsilon(s,v-j)\right| > \lambda\right] P(S_{r,\alpha,\beta}) \tag{2.1}$$

However, using Doob's maximal inequality and Corollary 2 of Chow and Teicher (1988, P. 368) we have

$$E\left[\left\{(\alpha-2)m2^{-r}\cdot(\beta-2)n2^{-r}\right\}^{-1/2}\max_{(u,v)\in J_{m,n,\alpha,\beta}}\left|\sum_{s=0}^{u}\varepsilon(s,v-j)\right|\right]^{2+\delta}$$

$$= \left\{\frac{m(\alpha+1)2^{-r}}{m(\alpha-2)2^{-r}}\cdot(\beta-2)n2^{-r}\right\}^{-1-\delta/2}$$

$$\times E\left\{(m(\alpha+1)2^{-r})^{-1/2}\max_{v\in I_{r,n,\beta}}\max_{u\in I_{r,m,\alpha}}\left|\sum_{s=0}^{u}\varepsilon(s,v)\right|\right\}^{2+\delta}$$

$$\begin{pmatrix} I_{r,m,\alpha}=\left(m(\alpha-2)2^{-r},m(\alpha+1)2^{-r}\right) \\ I_{r,n,\beta}=\left(n(\beta-2)2^{-r},n(\beta+1)2^{-r}\right) \end{pmatrix}$$

$$\leq K_{r,\alpha,\beta}\sup_{l}E\left|l^{-1/2}\sum_{s=0}^{l}\varepsilon(s,0)\right|^{2+\delta}n^{-2/\delta},$$

where $K_{r,\alpha,\beta}>0$ is the constant that only depends on r, α, β, because $\max_{u\in I_{r,m,\alpha}}\left|\sum_{s=1}^{u}\varepsilon(s,v)\right|$ are iid with respect to v and for each v, $\sum_{s=0}^{u}\varepsilon(s,v)$ is an iid sum of random variables. Therefore, (2.1) is no more than

$$\limsup_{r\to\infty}\sum_{\alpha,\beta=r}^{r2^r}\limsup_{m,n\to\infty}K'_{r,\alpha,\beta}\Big(\sum_{j=1}^{\infty}|b_{0j}|\lambda^{-1}\Big)^{2+\delta}n^{-\delta/2}P(S_{r,\alpha,\beta})=0\quad(K'_{r,\alpha,\beta}>0),$$

which completes the proof.

Proof of Theorem 1. Consider the process $X(s,t;l)=\sum_{i=-l}^{l}\sum_{j=-l}^{l}a_{ij}\varepsilon(s-i,t-j)$. Let $X_1(s,t;l)=\sum_{i=0}^{l}\sum_{j=0}^{l}a_{ij}\varepsilon(s-i,t-j)$, $X_2(s,t;l)=\sum_{i=0}^{l}\sum_{j=-l}^{-1}a_{ij}\varepsilon(s-i,t-j)$, $X_3(s,t;l)=\sum_{i=-l}^{-1}\sum_{j=0}^{l}a_{ij}\varepsilon(s-i,t-j)$ and $X_4(s,t;l)=\sum_{i=-l}^{-1}\sum_{j=-l}^{-1}a_{ij}\varepsilon(s-i,t-j)$. Note that $\{a_{ij};0\leq i,j\leq l\}$ satisfies the summability condition of Theorem 1. From Lemma 1, we can write

$$\sum_{s=-M_m}^{M_m}\sum_{t=-N_n}^{N_n}X_1(s,t;l)$$

$$=W(-M_m,-N_n)-W(M_m+1,-N_n)$$

$$+ W(-M_m, N_n + 1) - W(M_m + 1, N_n + 1)$$

$$+ \sum_{s=-M_m}^{M_m} \Big\{U(s+1, -N_n) - U(s+1, N_n + 1)\Big\} \tag{2.2}$$

$$+ \sum_{t=-N_n}^{N_n} \Big\{V(-M_m, t+1) - V(M_m + 1, t+1)\Big\}.$$

Note first that as $m, n \to \pm\infty$,

$$\Big\{(2m+1)(2n+1)\Big\}^{-1/2} W(m,n) \to 0 \quad \text{a.s.},$$

which can be shown easily by Markov's inequality. Thus, since $M_m, N_n \to \infty$ in probability as $m, n \to \infty$, respectively, it follows that $\Big\{(2M_m + 1)(2N_n + 1)\Big\}^{-1/2} W(s,t)$, where $s = -M_m, M_m + 1$, $t = -N_n, N_n + 1$, goes to 0 in probability.

This implies, in view of Lemma 2, that the random variables in (2.2) divided by $\{(2M_m + 1)(2N_n + 1)\}^{1/2}$ go to 0 in probability. Therefore, as $m, n \to \infty$,

$$\Big\{(2M_m + 1)(2N_n + 1)\Big\}^{-1/2} \sum_{s=-M_m}^{M_m} \sum_{t=-N_n}^{N_n} X_1(s,t;l)$$

$$= \Big(\sum_{i=0}^{l} \sum_{j=0}^{l} a_{ij}\Big)\Big\{(2M_m + 1)(2N_n + 1)\Big\}^{-1/2} \sum_{s=-M_m}^{M_m} \sum_{t=-N_n}^{N_n} \varepsilon(s,t) + o_P(1).$$

Applying essentially the same arguments to $X_i(s,t;l)$, $i = 2, 3, 4$, we can write that

$$\Big\{(2M_m + 1)(2N_n + 1)\Big\}^{-1/2} \sum_{s=-M_m}^{M_m} \sum_{t=-N_n}^{N_n} X(s,t;l)$$

$$= \Big(\sum_{i=-l}^{l} \sum_{j=-l}^{l} a_{ij}\Big)\Big\{(2M_m + 1)(2N_n + 1)\Big\}^{-1/2} \sum_{s=-M_m}^{M_m} \sum_{t=-N_n}^{N_n} \varepsilon(s,t) + o_P(1).$$

Using Anscombe's theorem, we conclude that as $m, n \to \infty$,

$$\Big\{(2M_m + 1)(2N_n + 1)\Big\}^{-1/2} \sum_{s=-M_m}^{M_m} \sum_{t=-N_n}^{N_n} X(s,t;l) \xrightarrow{\mathcal{D}} \mathcal{N}\Big(0, \Big(\sum_{i=-l}^{l} \sum_{j=-l}^{l} a_{ij}\Big)^2 \sigma^2\Big).$$

Now in order to establish the theorem, it suffices to show that: for all $\lambda > 0$,

$$\lim_{l\to\infty} \limsup_{m,n\to\infty} P\left[\left|\left\{(2M_m+1)(2N_n+1)\right\}^{-1/2} \sum_{i=-M_m}^{M_m} \sum_{j=-N_n}^{N_n} \left\{X(s,t)-\mu-X(s,t;l)\right\}\right| > \lambda\right] = 0$$

in view of Proposition 6.3.9 of Brockwell and Davis (1987). However, the above can be yielded in a similar fashion that we proved Lemma 2. Indeed, it can be shown that

$$\limsup_{m,n\to\infty} P\left[\left|\left\{(2M_m+1)(2N_n+1)\right\}^{-1/2} \sum_{s=-M_m}^{M_m} \sum_{t=-N_n}^{N_n} \left\{X(s,t)-\mu-X(s,t;l)\right\}\right| > \lambda\right]$$

$$= O\left(\Big(\sum_{|i|>l \text{ or } |j|>l} |a_{ij}|\Big)^2\right) \to 0 \quad \text{as} \quad l \to \infty.$$

Without detailing algebras, we complete the proof.

Remark. So far, we have assumed that $E|\varepsilon(0,0)|^{2+\delta}$ exists and this moment condition seems more or less inevitable unless $\varepsilon(s,t) = \eta(s)\zeta(t)$, where $\{\eta(t); t \in \mathcal{Z}\}$ and $\{\zeta(t); t \in \mathcal{Z}\}$ are independent iid sequences of random variables with finite variances. (This can be checked without difficulties and the proof is omitted for brevity). In time series, however, we only need the existence of variance.

An extension of Theorem 1 to the lattice process with indices of dimension$\geq$ 3 can be considered although we do not present here in detail. It is expected that Theorem 1 still holds and can be proved with no serious technical difficulties.

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