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Sufficient Conditions for Compatibility of Unequal-replicate Component Designs†

DongKwon Park[1]

ABSTRACT

A multi-dimensional design is most easily constructed via the amalgamation of one-dimensional component block designs. However, not all sets of component designs are compatible to be amalgamated. The conditions for compatibility are related to the concept of a complete matching in a graph. In this paper, we give sufficient conditions for unequal-replicate designs. Two types of conditions are proposed; one is based on the number of vertices adjacent to at least one vertex and the other is on a degree of vertex, in a bipartite graph. The former is an extension of the sufficient conditions of equal-replicate designs given by Dean and Lewis (1988).

KEYWORDS: Multi-Dimensional Design, Graph Theory, Amalgamation, Compatibility.

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[1]Department of Statistics, Yonsei University, Wonju, Kangwon-Do, 222-701, Korea.

1. INTRODUCTION

Multi-dimensional designs are designs for experiments involving two or more crossed, non-interacting blocking factors. We restrict attention to experiments with v treatments, and one treatment is observed at each combination of levels of the blocking factors. Geometrically, a multi-dimensional design with d blocking factors having $b_1, b_2, \ldots, b_d$ levels can be represented by a d-dimensional lattice with each dimension representing a blocking factor, thus producing $\prod_{i=1}^{d} b_i$ nodes with one treatment label at each nodes.

Two dimensional designs are better known as row-column designs and many results, Latin square designs being the simplest case, exist in the literature in their construction and properties. Two series of designs based on Latin cubes for experiments in three crossed blocking factors are given by Preece, Pearce and Kerr (1973) who point out their usefulness for experiments in the food industry. Youden hyperrectangles (Cheng, 1979) are multi-dimensional designs which are analogues of generalized Youden designs.

When an experiment cannot be accommodated by the designs available in the literature, a suitable design must be constructed by the experimenter. A multi-dimensional design with d blocking factors having $b_1, b_2, \ldots, b_d$ levels is most easily constructed via the amalgamation of one-dimensional block designs, where the i-th design has b_i blocks, $i = 1, 2, \ldots, d$. These one-dimensional designs are called the *component* designs. The i-th component design can be recovered from the multi-dimensional design by ignoring all blocking factors except for the i-th. Not all sets of one-dimensional designs are able to be amalgamated, as shown by Freeman (1957). If d component designs can be amalgamated into a multi-dimensional design, they are called *compatible*. Dean an Lewis (1988) present sufficient conditions for compatibilty of component designs which each of the treatment labels occurs r times.

In this paper, we give sufficient conditions for compatibility of unequal-replicate component designs (i.e., each of the treatment labels occurs a different number of times in the design). The conditions for compatibility are related to the concept of a complete matching in a graph. Two types of conditions are proposed; one is based on the number of vertices adjacent to at least one vertex and the other is on a degree of vertex, in a bipartite graph. The former is modified from the sufficient conditions which are given by Dean and Lewis (1988) (the conditions contain incorrect parts ; see Remark 1 for detail) and extended to unequal-replicate designs from equal-replicate designs. The conditions are rather easy to check in practice.

2. DEFINITIONS AND RESULTS

Next paragraph contains a short summary of definitions and well-known results from the theory of graphs which will be useful for stating and proving the results to follow.

A graph is an ordered pair $G = (V; E)$ where V is a finite sets of vertices and E edge set. An edge $e = \{v_1, v_2\} \in E$ is called incident to v_1 and v_2, the vertices v_1 and v_2 are called adjacent. If an edge joines vertex to itself, the edge is called loop. The degree of vertex v, $d(v)$, is the number of two-element edges that touch v plus twice the number of loops that touch v. An open walk is a path if all vertices are distinct. $G^* = (V_1, V_2; E)$ is a *bipartite* if the set of vertices V can be partitioned into V_1 and V_2 with $V_1 \cap V_2 = \emptyset$, such that each edge in E is incident to one vertex in V_1 and one vertex in V_2. A *matching* in G^* is a set M of edges of G^* with no common vertices. A matching M in G^* is *complete* if every $x \in V_1$ is matched. The König-Hall Marriage Theorem (see Wilson (1979), Theorem 25A) is well-known for existence of a complete matching. The condition of the Marriage Theorem is rather difficult to check. A simple lemma is proposed.

Lemma 2.1. If there exists $k \geq 1$ such that $d(x) \geq k \geq d(y)$ for all $x \in V_1, y \in V_2$ in the bipartite graph $G^* = (V_1, V_2; E)$, then there exists a complete matching.

Now, let's translate from our compatibility problem into graphical problem. Consider a set of d component designs with v treatment labels, T, each observed $r_i (i = 1, \ldots, v)$ times and let B_{ix_i} be the set of treatment labels in block x_i of the i-th component designs $(i \leq x_i \leq b_i, i = 1, \ldots, d)$. Let r_i's have h $(1 \leq h \leq v)$ different values and let $r(p)$ be the p-th $(p = 1, \ldots, h)$ smallest values of r_i. Then a set of d component designs have a common set of $n(p)$ treatment labels replicate $r(p)$ times and the design consist of d block designs such that j-th design has b_j blocks of size $\left[\sum_{p=1}^{h} r(p)n(p)\right]/b_j (j = 1, \ldots, d)$.

Let $n_i(t, x_i)$ denotes the number of times that treatment label t $(t = 1, \ldots, v)$ occurs in B_{ix_i} $(i \leq x_i \leq b_i,\ i = 1, \ldots, d)$. The *block-intersection* $I(x) = B_{1x_1} \cap B_{2x_2} \cap \cdots \cap B_{dx_d}$ consists of $n(t, x) = \prod_i n_i(t, x_i)$ replicates of each treatment label $t \in T$. Let $Y = \{x_1\ x_2\ \ldots\ x_d\}$, then $x \in Y$ represents a selection of blocks, one from each component design. Construct the bipartite graph $G^* = (V_1, V_2; E)$, where V_1 contains $\sum_p r(p)n(p)$ vertices representing the r_i (or $\sum_{p=1}^{h} r(p)$) occurrence of the v (or $\sum_{p=1}^{h} n(p)$) treatment labels and V_2 contains $\prod_i b_i$ vertices representing the nodes of the d-dimensional lattice (i.e. $V_2 = Y$). The existence of a complete matching from V_1 to V_2 does not guarantee compatibility of the component designs. However, repeated deletion of a replicate of treatments leads to the compatibility.

3. SUFFICIENT CONDITIONS

Let t_{pq} denote the q-th treatment that has replicate $r(p)$ $(q = 1, \ldots, n(p), p = 1, \ldots, h, h \leq v; \sum_p n(p) = v)$ and $T_{r(p)}$ denote the set of treatment labels which have replicate $r(p)$, i.e., $T_{r(p)} = \{t_{pq}; q = 1, \ldots, n(p)\}$. The next results, Theorem 3.1 and 3.2, give sufficient conditions to unequal replicate designs. If a theorem is satisfied, for the first step, we can delete a single replicate of the treatment labels $T_{r(h)}$ from each component design together with a deletion of $x_1\ x_2\ \ldots\ x_d$ from Y when $T_{r(h)}$ is deleted from $B_{1x_1}, B_{2x_2}, \ldots, B_{dx_d}$. Next, if possible, we can also delete a single replicate of the treatment labels $T_{r(h)-1}$ and assign them to the nodes of the d-dimensional lattice. Whenever possible, we can continue this way.

After deleting some replicates of the treatment labels and assigning them to the nodes of the d-dimensional lattice, each nodes can be filled or already filled for the remaining parts of the component designs. Define F_x to be an indicator function which takes the value 0 if the node $x \in Y$ is already filled, and $F_x = 1$ otherwise. During the deletion process, $F_x = 0$ means that the corresponding node $x \in Y$ is already filled and cannot be assigned anymore. Define $m_i(t, x_i)$ to be an indicator function which takes the value 1 if $n_i(t, x_i) \geq 1$, that is treatment label t occurs at least once in block B_{ix_i}, and $m_i(t, x_i) = 0$ otherwise.

Theorem 3.1. Let $M_t(p) = \sum_{x \in V_2} F_x \prod_i m_i(t, x_i)$, where $t \in T_{r(p)}$. If

$$\min_t M_t(h) \geq n(h), \tag{3.1}$$

a deletion can be made of a single replicate of the treatment labels $T_{r(h)}$ from each component design together with a deletion of the corresponding $x \in Y$, where $r(h)$ is the largest value of r_i for the remaining replicates after deleting some replicates and $n(h)$ is the largest value among $n(p)$ where $p = 1, 2, \ldots, h$.

Proof. Construct the bipartite graph $Q(V_3, V_2; E)$, subgraph of G^*, where V_3 contains $n(h)$ vertices representing a single replicate of the treatment labels $t \in T_{r(h)}$. Let $|N_Q(S)|$ denote the number of vertices in V_2 which are collectively adjacent to all $t \in S (S \subseteq V_3)$. If $S = \{t\}$, we write $|N_Q(S)| = |N_Q(t)|$. Now,

$$|N_Q(t)| = \sum_{x \in V_2} F_x \prod_i m_i(t, x_i). \tag{3.2}$$

Therefore, using the equation (3.2), $|N_Q(S)| \geq \min_t |N_Q(t)| = \min_t M_t(h)$ for all $S \subseteq V_3$. If $\min_t M_t(h) \geq n(h)$, it follows that $|N_Q(S)| \geq \min_t M_t(h) \geq n(h) = |V_3| \geq |S|$ for all $S \subseteq V_3$. From the König-Hall Theorem, there exists a complete matching from the vertices V_3 to a subset of the vertices of V_2, and if $\min_t M_t(h) \geq n(h)$, deletion stated in the theorem can be therefore be achieved.

Define $q_i(t)$ is the number of blocks in the i-th component designs that contain at least one replicate of treatment $t \in T_{r(h)}$ and define $P_t(h) =$

$\prod_i q_i(t) - \left(\prod_{i=1}^d b_i - \sum F_x\right)$. Here, numbers of $\left(\prod_{i=1}^d b_i - \sum F_x\right)$ means that the vertices which is already filled in Y. In practice, the following conditions are rather easier to check than that of Theorem 3.1.

Corollary 3.1. If

$$\min_t P_t(h) \geq n(h), \tag{3.3}$$

a deletion can be made of a single replicate of the treatment labels T from the remaining parts of each component design.

Proof. The equation $\min_t M_t(h) \geq \min_t P_t(h)$ holds from $q_i(t) = \sum m_i(t, x_i)$ where the sum is over $x_i = 1, \ldots, b_i$, even though the treatment t which minimize $M_t(h)$ does not necessarily the same as t of $\min_t P_t(h)$, where $t \in T_{r(h)}$ and the proof follows.

Corollary 3.2. At some stage, suppose $r_1 = r_2 = \cdots = r_v = r$. Then, in the equation (3.1), $p = 1$, $q = v$, $h = 1$ and $n(h) = v$. Thus, if

$$\min_t M_t(1) \geq v \tag{3.4}$$

a deletion can be made of a single replicate of the treatment labels T from the remaining parts of each component design.

Remark 1. Note that $M_t(h) \leq \prod_i q_i(t)$. The equality only holds for the first deletion step (i.e., all $F_x = 1$). The equation sayes that some sufficient conditions given by Dean and Lewis (1988; Theorem 2.2 and Corollary 2.2) are incorrect and should be modified.

Theorem 3.2. Let $U_t(p) = \sum_x F_x n(t, x)$, where $t \in T_{r(p)}$. If

$$\min_t U_t(h) \geq \max_x F_x |I(x)| \tag{3.5}$$

a deletion can be made in such a same way in Theorem 3.1.

Proof. If $t \in V_3$, then the edge set E contains the edge (t, x) with frequency $F_x n(t, x)$, since $x \in Y$ appears in F_x times in V_2. And the degree of the vertex $x \in Y$ is $|I(x)|$. From lemma 2.1, the proof is completed.

In practice, Theorem 3.1 and 3.2 could be combined in way of easiness for applying. In applying Theorem 3.1 and 3.2, the deletion should be carried out in such a way that degrees of vertices in V_2, $|I(x)|$, are independent of x for all $x \in Y$ in order to minimize the maximum of $F_x|I(x)|$.

4. AN EXAMPLE

We shall now illustrate this procedure by an example. We take two component designs with 4 treatments as follows.

$$\begin{array}{lcllllll} B_{11} & : & 0 & 0 & 0 & 0 & 1 & 2 \\ B_{12} & : & 0 & 0 & 1 & 2 & 2 & 3 \\ B_{13} & : & 0 & 0 & 2 & 2 & 3 & 3 \\ B_{14} & : & 0 & 1 & 2 & 3 & 3 & 3 \end{array} \qquad \begin{array}{lcllll} B_{21} & : & 0 & 1 & 2 & 3 \\ B_{22} & : & 0 & 1 & 2 & 3 \\ B_{23} & : & 0 & 1 & 2 & 3 \\ B_{24} & : & 0 & 0 & 0 & 3 \\ B_{25} & : & 0 & 2 & 3 & 3 \\ B_{26} & : & 0 & 0 & 2 & 2 \end{array}$$

Treatment label 0 occurs 9 times, label 1 occurs 3 times and label 2, 3 occur 6 times, so that $r(1) =$ 1st smallest $[9,3,6] = 3$, $r(2) =$ 2nd smallest $[9,3,6] = 6$ and $r(3) =$ 3rd smallest $[9,3,6] = 9$ and $n(1) = 1$, $n(2) = 2$ and $n(3) = 1$. Since $T_{r(3)}$ contains only the treatment label $t = 0$ and $F_x = 1$ for all $x \in Y$ and

$$M_o(3) = P_o(3) = q_1(0) \times q_2(0) = 4 \times 6 = 24 \geq 1 = n(3).$$

From the equation (3.1), an assignment of treatment label 0 to the node of the lattice can be made as shown below.

$$\begin{array}{cccccc} \cdot & \cdot & \cdot & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

We can see that 0 is deleted from B_{11} and B_{24}. For next step, since $F_x = 0$ if $x = 14$ and $F_x = 1$ otherwise and $h = 3$,

$$M_o(3) = P_o(3) = q_1(0) \times q_2(0) - 1 = 4 \times 6 - 1 = 23 \geq 1 = n(3)$$

and therefore the Corollary 3.2 guarantees that a further assignment of the treatmemt 0 which is deleted from B_{12} and B_{24} can be made to the remaining nodes of the lattice as follows.

$$\begin{array}{cccccc} \cdot & \cdot & \cdot & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

Similarly, another assignment of 0 is possible since $P_o(3) = 22 \geq 1 = n(3)$ to the node $x = 34$. Then, the remaining parts of the component designs are

$$\begin{array}{llllllll} B_{11} & : & 0 & 0 & 0 & 1 & 2 & \\ B_{12} & : & 0 & 1 & 2 & 2 & 3 & \\ B_{13} & : & 0 & 2 & 2 & 3 & 3 & \\ B_{14} & : & 0 & 1 & 2 & 3 & 3 & 3 \end{array} \qquad \begin{array}{llllll} B_{21} & : & 0 & 1 & 2 & 3 \\ B_{22} & : & 0 & 1 & 2 & 3 \\ B_{23} & : & 0 & 1 & 2 & 3 \\ B_{24} & : & 3 & & & \\ B_{25} & : & 0 & 2 & 3 & 3 \\ B_{26} & : & 0 & 0 & 2 & 2 \end{array}$$

Now, $h = 2$ and $T_{r(2)} = \{t = 0, 2, 3\}$. We can verify that the condition in Corollary 3.2 met since $q_1(3) \times q_2(3) = 3 \times 5 = 15$ and $q_1(0) \times q_2(0) = q_1(1) \times q_2(1) = 4 \times 5 = 20$ and so $\min_t P_t(3) = 15 - 3 = 12 \geq 3 = n(2)$. Therefore, an assignment of one copy of each of the treatment label 0, 2 and 3 to nodes of the lattice as follows.

$$\begin{array}{cccccc} \cdot & \cdot & \cdot & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & 3 & 2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

We can continue to use the condition of Corollary 3.2 until all remaining treatment have same replicates 3 times and the partially completed design and the remaining parts of the component designs are following.

$$\begin{array}{cccccc} \cdot & \cdot & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & 0 & 2 & 0 \\ \cdot & \cdot & \cdot & 0 & 3 & 2 \\ \cdot & \cdot & \cdot & 3 & 3 & 2 \end{array}$$

$$\begin{array}{lllll} B_{11} & : & 0 & 1 & 2 \\ B_{12} & : & 1 & 2 & 3 \\ B_{13} & : & 0 & 2 & 3 \\ B_{14} & : & 0 & 1 & 3 \end{array} \qquad \begin{array}{llllll} B_{21} & : & 0 & 1 & 2 & 3 \\ B_{22} & : & 0 & 1 & 2 & 3 \\ B_{23} & : & 0 & 1 & 2 & 3 \end{array}$$

It can be verified that $|I(x)| = 34$ for all remaining $x \in Y$. Theorem 3.2 can be used since

$$\begin{aligned} &\min_t U_t(1) = U_t(1) = 9 \quad \text{for all } t \in T_{r(1)} \\ &\geq \max_x |I(x)| = 3 = |I(x)| \quad \text{for all remaining } x \in Y. \end{aligned}$$

The partially completed design and the remaining parts of the component designs are following.

$$\begin{array}{llll} B_{11} & : & 1 & 2 \\ B_{12} & : & 2 & 3 \\ B_{13} & : & 0 & 3 \\ B_{14} & : & 0 & 1 \end{array} \qquad \begin{array}{llllll} B_{22} & : & 0 & 1 & 2 & 3 \\ B_{23} & : & 0 & 1 & 2 & 3 \end{array}$$

$$\begin{array}{cccccc} 0 & \cdot & \cdot & 0 & 0 & 0 \\ 1 & \cdot & \cdot & 0 & 2 & 0 \\ 2 & \cdot & \cdot & 0 & 3 & 2 \\ 3 & \cdot & \cdot & 3 & 3 & 2 \end{array}$$

By using the Theorem 3.2 two more times, we can do the entire assignment as follows.

$$\begin{array}{cccccc} 0 & 1 & 2 & 0 & 0 & 0 \\ 1 & 2 & 3 & 0 & 2 & 0 \\ 2 & 3 & 0 & 0 & 3 & 2 \\ 3 & 0 & 1 & 3 & 3 & 2 \end{array}$$

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