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On the Weak Law of Large Numbers of Randomly Indexed Partial Sums

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ABSTRACT

The purpose of this note is to provide a general weak law of randomly indexed partial sums for arrays.

KEYWORDS: Weak law of large numbers, Randomly indexed partial sums.

1. INTRODUCTION

Previously Pyke and Root(1968), Chatterji(1969), Chow(1971), Gut(1974), Hall (1971) and Rosalsky and Teicher(1981) generalize the weak law of large numbers for i.i.d. random variables. Chandra(1989) and Gut(1991) provided a general weak law for arrays.

Recently, a fairly general weak law of large numbers was provided by Hong and Oh(1994) in the following form : Let $\{(X_{ni}, 1 \leq i < k_n), n \geq 1\}, k_n \to \infty$ as $n \to \infty$, be an array of random variables on $(\Omega, \mathcal{F}, P)$ and set $\mathcal{F}_{nj} = \sigma\{X_{ni}, 1 \leq i \leq j\}$, $1 \leq j \leq k_n$, $n \geq 1$, and $\mathcal{F}_{n_0} = \{\phi, \Omega\}, n \geq 1$. Suppose

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that $\frac{1}{k_n}\sum_{i=1}^{k_n} aP\{|X_{ni}|^p > a\} \to 0$ as $a \to \infty$ uniformly in n for some $0 < p < 2$, then ${k_n}^{-1/p}S_n \to 0$ in probability as $n \to \infty$ where $S_n = \sum_{i=1}^{k_n}(X_{ni} - E(X_{ni}I(|X_{ni}|^p \le k_n)|\mathcal{F}_{n,i-1})$. In this note, using Hájek-Rényi inequality(see Chow and Teicher(1988), Theorem 7.4.8) we will extend this theorem to the case of random indices for $1 \le p < 2$. As a corollary we have a result which generalize Theorem 5.2.6(Chow and Teicher(1988)).

2. MAIN RESULT

Theorem 1. Let $\{(X_{ni}, 1 \le i < \infty), n \ge 1\}$ be an array of random variables and set $S_{nm} = \sum_{i=1}^{m} X_{ni}, n \ge 1$, and let $\{\nu_n, n \ge 1\}$ be positive integer-valued random variables satisfying

$$\frac{\nu_n}{n^{1/p}} \longrightarrow \quad \text{in probability, where } 0 < c < \infty. \tag{2.1}$$

Let $k_n \to \infty$ as $n \to \infty$ and suppose for $1 \le p < 2$ that

$$\frac{1}{n}\sum_{i=1}^{n} aP\Big\{|X_{ni}|^p > a\Big\} \to 0 \quad \text{as } a \to \infty \quad \text{uniformly in } n. \tag{2.2}$$

Then $(S_{n\nu_{k_n}} - a_n)/\nu_{k_n} \to 0$ in probability as $n \to \infty$, where

$$a_n = \sum_{i=1}^{\nu_{k_n}} E\Big(X_{ni}I\Big(|X_{ni}|^p \le k_n\Big)\Big|\mathcal{F}_{n,i-1}\Big), \quad n \ge 1.$$

Proof. Define $X'_{ni} = X_{ni}I\{|X_{ni}|^p \le k_n\}$, $S'_{nm} = \sum_{i=1}^{m} X'_{ni}$ and $c_n = [c \cdot k_n^{1/p}]$ = largest integer $\le c \cdot k_n^{1/p}$. Then

$$\begin{aligned} P\Big\{S_{n\nu_{k_n}} \ne S'_{n\nu_{k_n}},\ \nu_{k_n} \le 2c_n\Big\} &\le P\Big\{\bigcup_{i=1}^{2c_n}\{X_{ni} \ne X'_{ni}\}\Big\} \\ &\le \sum_{i=1}^{2c_n} P\Big\{|X_{ni}|^p > k_n\Big\} \end{aligned}$$

$$= \frac{1}{k_n}\sum_{i=1}^{2c_n} k_n P\Big\{|X_{ni}|^p > k_n\Big\}$$

$$= \frac{2c_n}{k_n}\frac{1}{2c_n}\sum_{i=1}^{2c_n} k_n P\Big\{|X_{ni}|^p > k_n\Big\}$$

$$= \mathrm{o}(1)$$

by (2.2), whence, taking cognization of (2.1),

$$P\Big\{S_{n\nu_{k_n}} \neq S'_{n\nu_{k_n}}\Big\} \leq P\Big\{S_{n\nu_{k_n}} \neq S'_{n\nu_{k_n}}, \nu_{k_n} \leq 2c_n\Big\} + P\Big\{\nu_{k_n} > 2c_n\Big\} = o(1),$$

that is,

$$S_{n\nu_{k_n}} - S'_{n\nu_{k_n}} \longrightarrow 0 \quad \text{in probability.} \tag{2.3}$$

Define

$$B_j^n = \Big\{\Big|S'_{nj} - \sum_{i=1}^{j} E(X'_{ni}|\mathcal{F}_{n,i-1})\Big| > k_n^{1/p}\varepsilon\Big\},$$

$$D_n = \bigcup_{j=1}^{2c_n} B_j^n.$$

Since $X'_{ni} - E(X'_{ni}|\mathcal{F}_{n,i-1}), 1 \leq i \leq 2c_n$, form a martingale difference sequence and hence by Hájek-Rényi inequality(Chow and Teicher(1988), Theorem 7.4.8).

$$P\{D_n\} \leq \frac{1}{k_n^{2/p}\epsilon^2}\sum_{i=1}^{2c_n} E(X'_{ni})^2.$$

Now

$$\sum_{i=1}^{2c_n} E(X'_{ni})^2$$

$$= \sum_{i=1}^{2c_n}\sum_{j=1}^{k_n}\int_{\{j-1<|X_{ni}|^p<j\}} X_{ni}^2 dP$$

$$\leq \sum_{i=1}^{2c_n}\sum_{j=1}^{k_n} j^{2/p}\Big(P\Big\{|X_{ni}|^p > j-1\Big\} - P\Big\{|X_{ni}|^p > j\Big\}\Big)$$

$$
\begin{aligned}
&= \sum_{i=1}^{2c_n}\Big[P\Big\{|X_{ni}|^p > 0\Big\} - k_n^{2/p}P\Big\{|X_{ni}|^p > k_n\Big\} \\
&\qquad + \sum_{j=1}^{k_n-1}\Big((j+1)^{\frac{2}{p}} - j^{\frac{2}{p}}\Big)P\Big\{|X_{ni}|^p > j\Big\}\Big] \\
&\le k_n + \sum_{j=1}^{k_n}\Big((j+1)^{\frac{2}{p}} - j^{\frac{2}{p}}\Big)\sum_{i=1}^{2c_n}P\Big\{|X_{ni}|^p > j\Big\} \\
&\le k_n + \alpha\sum_{j=1}^{k_n}\Big((j+1)^{\frac{2}{p}-1} - j^{\frac{2}{p}-1}\Big)\sum_{i=1}^{2c_n}jP\Big\{|X_{ni}|^p > j\Big\} \\
&\le k_n + 2\alpha c_n\sum_{j=1}^{k_n}\Big((j+1)^{\frac{2}{p}-1} - j^{\frac{2}{p}-1}\Big)\sup_n\Big\{(2c_n)^{-1}\sum_{i=1}^{2c_n}jP\Big\{|X_{ni}|^p > j\Big\}\Big\} \\
&\le k_n + 2\alpha ck_n^{1/p}\sum_{j=1}^{k_n}\Big((j+1)^{\frac{2}{p}-1} - j^{\frac{2}{p}-1}\Big)\sup_n\Big\{(2c_n)^{-1}\sum_{i=1}^{2c_n}jP\Big\{|X_{ni}|^p > j\Big\}\Big\},
\end{aligned}
$$

where α is an unimportant positive constant and the second equality comes from Lemma 5.1.1 (4)(Chow and Teicher (1988)). By the hypothesis (2.2), $\sup_n\Big\{(2c_n)^{-1}\sum_{i=1}^{2c_n}jP\Big\{|X_{ni}|^p > j\Big\}\Big\}$ goes to zero as $j \to \infty$ and $\sum_{j=1}^{k_n}\Big((j+1)^{\frac{2}{p}-1} - j^{\frac{2}{p}-1}\Big) = (k_n+1)^{\frac{2}{p}-1} - 1$. Thus, by Toeplitz Lemma (Ash(1972), Lemma 7.1.2)

$$\sum_{i=1}^{2c_n}E(X'_{ni})^2 = k_n + o\Big(k_n^{(\frac{3}{p}-1)}\Big)$$

and hence $P\{D_n\} = o(1)$ for $1 \le p < 2$. Therefore,

$$
\begin{aligned}
P\Big\{B^n_{\nu_{k_n}}\Big\} &\le P\Big\{\nu_{k_n} \le 2c_n, B^n_{\nu_{k_n}}\Big\} + P\Big\{\nu_{k_n} > 2c_n\Big\} \\
&\le P\Big\{D_n\Big\} + P\Big\{\nu_{k_n} > 2c_n\Big\} = o(1),
\end{aligned}
$$

so that

$$\frac{S'_{n\nu_{k_n}} - \sum_{i=1}^{\nu_{k_n}}E\Big(X'_{ni}\Big|\mathcal{F}_{n,i-1}\Big)}{k_n^{1/p}} \longrightarrow 0 \quad \text{in probability.}$$

Hence, by (2.3) and (2.1)

$$\frac{S_{n\nu_{k_n}} - \sum_{i=1}^{\nu_{k_n}} E\Big(X_{ni} I(|X_{ni}|^p \le k_n)\Big|\mathcal{F}_{n,i-1}\Big)}{\nu_{k_n}}$$
$$= \frac{k_n^{1/p}}{\nu_{k_n}}\left(\frac{S_{n\nu_{k_n}} - \sum_{i=1}^{\nu_{k_n}} E\Big(X_{ni} I(|X_{ni}|^p \le k_n)\Big|\mathcal{F}_{n,i-1}\Big)}{k_n^{1/p}}\right) \longrightarrow 0.$$

in probability.

The following result is an immediate consequence of Theorem 1 and a generalization of Theorem 5.2.6(Chow and Teicher(1988)).

Corollary 2. If $\Big\{X_n, n \ge 1\Big\}$ are i.i.d. random variables obeying

$$nP\Big\{|X_1|^p > n\Big\} = o(1)$$

for some $1 \le p < 2$ and $\Big\{T_n, n \ge 1\Big\}$ are positive integer valued random variables satisfying

$$\frac{T_n}{n^{1/p}} \longrightarrow c \quad \text{in probability, where } 0 < c < \infty.$$

Then, setting $S_n = \sum_{i=1}^n X_i$,

$$(S_{T_n}/T_n) - EX_1 I\Big(|X_1|^p \le n\Big) \longrightarrow 0 \quad \text{in probability.}$$

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