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Testing a Multivariate Process for Multiple Unit Roots

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Abstract

An asymptotic property of the estimated eigenvalues for multivariate AR(p) process which consists of vector of nonstationary process and vector of stationary process is developed. All components of the nonstationary process are assumed to reveal random walk behavior. The asymptotic property is helpful in understanding multiple unit roots. In this paper we show the stationary part in multivariate AR(p) process does not affect the limiting distribution of estimated eigenvalues associated with the nonstationary process. A test statistic based on the ordinary least squares estimator for testing a certain number of multiple unit roots is suggested.

1. Introduction

Recently many researchers are interested in multivariate time series. Useful models have been developed for observed data. Among them autoregressive models, which express the current vector as a linear function of its predecessors, are fundamental and simple. Usually one uses ordinary least squares estimation or maximum likelihood estimation to estimate coefficient matrices in autoregressive models.

Often data, especially in economics, appear to be nonstationary (unit root case), which means the roots of characteristic polynomial are one in magnitude. Johansen(1988, 1991) derived the conditional maximum likelihood estimators of the cointegration vectors for an multivariate AR(p) model with independent Gaussian errors and initial values fixed. He derived a likelihood ratio test for the hypothesis that there are a given number of unit roots. Fountis and Dickey (1989) investigated the miltivariate AR(p) model with one unit root and others less than one in magnitude. They assumed that the initial values are fixed and used the ordinary least squares estimators that arise from the conditional likelihood. Shin (1992) investigated the multivariate AR(p) model with one unit root and others less than one in magnitude. He assumed that the initial values follow normal distribution with mean 0 and variance V which satisfies Yule-Walker equations. He used the unconditional maximum likelihood estimators to estimate the coefficient matrices in the multivariate AR(p) model. Using Monte-Carlo study he showed that the unconditional maximum likelihood estimators have better power properties than the ordinary least squares estimators when mean is estimated.

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In this paper we study multivariate processes with multiple unit roots and the rest less than one in magnitude. Fountis and Dickey (1989) showed that in the limit the nonstationary part and the stationary part of the estimated matrix can be separated, which means the existence of the stationary processes dose not affect the limiting distribution of the estimated eigenvalues for the nonstationary process. We show that the limiting property also carries to the multiple unit roots case. Finally we develop a test statistic for the multiple unit roots based on the ordinary least squares estimators.

2. Multivariate autoregressive model

2.1 Preliminary statements

Consider the multivariate first-order autoregressive AR(1) process defined by the rule

$$Y_t = V\, Y_{t-1} + \varepsilon_t,\; t = 1, 2, \ldots \tag{2.1.1}$$

where $Y_t = [\, Y_{1,t}, Y_{2,t}, \ldots, Y_{k,t} \,]'$, $\varepsilon_t = [\, \varepsilon_{1,t}, \varepsilon_{2,t}, \ldots, \varepsilon_{k,t} \,]'$, $Y_0 = \phi$, and $\{ \varepsilon_t : t = 1, 2, \ldots \}$ is a sequence of independent and identically distributed multivariate normal variates with mean ϕ (a vector of 0's) and variance matrix Σ_V.

Assume that there exists a real matrix R such that

$$A = R^{-1}VR = \begin{bmatrix} I_{rr} & \phi' \\ \phi & A_{22}^{*} \end{bmatrix} \tag{2.1.2}$$

where I_{rr} is an r-dimensional identity matrix and A_{22}^{*} is a diagonal matrix with elements less than 1 in magnitude. When the V matrix satisfies (2.1.2) we usually say that V has multiple unit roots and the rest less than 1 in magnitude.

Fountis and Dickey (1989) showed that the nonstationary part and the stationary part can be separated in the limit for AR(p) model with one unit root and the rest less than 1 in magnitude. That is, the limiting distribution of the unit root in the multivariate AR(p) process is the same as that of the univariate process which was studied by Dickey and Fuller (1979). Shin (1992) showed that this separation also occurs when one uses the unconditional maximum likelihood estimators for AR(p) model.

2.2 Separation of the nonstationary part and the stationary part for multiple unit roots case

In this section we show an asymptotic property of the least squares estimator that we will refer to as separation of the nonstationary part from the stationary part of the coefficient matrix in the AR(1) model.

Consider the k-dimensional multivariate AR(1) model with $X_0 = 0$,

$$X_t = A X_{t-1} + e_t, \quad t = 1, 2, \ldots \tag{2.2.1}$$

where e_t's are i.i.d $N(0, \Sigma)$.

Partitioning (2.2.1) we have

$$\begin{bmatrix} X_{1t} \\ X_{2t} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} X_{1t-1} \\ X_{2t-1} \end{bmatrix} + \begin{bmatrix} e_{1t} \\ e_{2t} \end{bmatrix}, \quad t \geq 1$$

where X_{1t} and X_{2t} correspond to the nonstationary process and the stationary process respectively. Assume that data are generated by

$$\begin{bmatrix} X_{1t} \\ X_{2t} \end{bmatrix} = \begin{bmatrix} I_{rr} & \phi' \\ \phi & A^*_{22} \end{bmatrix} \begin{bmatrix} X_{1t-1} \\ X_{2t-1} \end{bmatrix} + \begin{bmatrix} e_{1t} \\ e_{2t} \end{bmatrix}, \quad t \geq 1 \tag{2.2.2}$$

where A^*_{22} is a diagonal matrix, $X_{10} = 0$, $X_{20} = 0$ and e_t's are i.i.d. $N(0, \Sigma)$. Here ϕ is a proper 0 matrix and I_{rr} is an r-dimensional identity matrix. Note that (2.1.1) is transformed to (2.2.1) by R^{-1}. Since we use the eigenvalues of the ordinary least squares estimators as test statistics transforming by R^{-1} does not affect the test statistics. Hence these models cover general multiple unit roots case which satisfies assumption (2.1.2).

The usual ordinary least squares estimator for the parameter matrix A is

$$\begin{aligned} \hat{A}_{ols} &= \begin{bmatrix} \hat{A}_{ols11} & \hat{A}_{ols12} \\ \hat{A}_{ols21} & \hat{A}_{ols22} \end{bmatrix} \\ &= \left(\sum_{t=1}^{n} X_t X'_{t-1}\right)\left(\sum_{t=1}^{n} X_{t-1} X'_{t-1}\right)^{-1} \\ &= \begin{bmatrix} \sum_{t=1}^{n} X_{1t} X'_{1t-1} & \sum_{t=1}^{n} X_{1t} X'_{2t-1} \\ \sum_{t=1}^{n} X_{2t} X'_{1t-1} & \sum_{t=1}^{n} X_{2t} X'_{2t-1} \end{bmatrix} \begin{bmatrix} \sum_{t=1}^{n} X_{1t-1} X'_{1t-1} & \sum_{t=1}^{n} X_{1t-1} X'_{2t-1} \\ \sum_{t=1}^{n} X_{2t-1} X'_{1t-1} & \sum_{t=1}^{n} X_{2t-1} X'_{2t-1} \end{bmatrix}^{-1} \end{aligned} \tag{2.2.3}$$

Hence

$$\widehat{A}_{ols}-A = \begin{bmatrix} \sum_{t=1}^{n} e_{1t}X_{1t-1}' & \sum_{t=1}^{n} e_{1t}X_{2t-1}' \\ \sum_{t=1}^{n} e_{2t}X_{1t-1}' & \sum_{t=1}^{n} e_{2t}X_{2t-1}' \end{bmatrix} \begin{bmatrix} \sum_{t=1}^{n} X_{1t-1}X_{1t-1}' & \sum_{t=1}^{n} X_{1t-1}X_{2t-1}' \\ \sum_{t=1}^{n} X_{2t-1}X_{1t-1}' & \sum_{t=1}^{n} X_{2t-1}X_{2t-1}' \end{bmatrix}^{-1} \tag{2.2.4}$$

where A is defined in (2.1.2).

Fountis and Dickey (1989) showed that when A has one unit root the properly normalized statistics $n(\widehat{A}_{ols11} - 1)$ and $n^{\frac{1}{2}}(\widehat{A}_{ols22} - A_{22}^{*})$ converge weakly to ξ / Γ and Φ respectively. (ξ, Γ) is the weak limit of $(\sum_{t=1}^{n} e_{1t}X_{1t-1}'/n, \sum_{t=1}^{n} X_{1t-1}X_{1t-1}'/n^2)$ and Φ is the weak limit of the normalized coefficient matrix in the regression of $X_{2t} - A_{22}^{*}X_{2t-1}$ on X_{2t-1}. See Fountis and Dickey (1989) for precise definition of (ξ, Γ) and Φ. This means the limiting distribution of the normalized largest eigenvalue of the $\widehat{A}_{ols}$ corresponding to nonstationary process does not change even though there exist stationary processes. Thus one says that the nonstationary part and the stationary part can be separated for the multivariate AR(1) model with one unit root. We can easily extend the orders of sums of squares and cross products developed by Fountis and Dickey (1989) to the vector cases to separate the nonstationary part of the $\widehat{A}_{ols}$ in the multiple unit roots case. That is, let

$$W_1 = \sum_{t=1}^{n} e_{1t}X_{1t-1}', \ W_2 = \sum_{t=1}^{n} e_{2t}X_{1t-1}', \ W_3 = \sum_{t=1}^{n} e_{1t}X_{2t-1}',$$

$$W_4 = \sum_{t=1}^{n} e_{2t}X_{2t-1}', \ W_5 = \sum_{t=1}^{n} X_{2t-1}X_{1t-1}' \text{ and } W_6 = \sum_{t=1}^{n} X_{2t-1}X_{2t-1}'$$

then

$$\begin{aligned} &W_1 = O_p(n), \ W_2 = O_p(n), \ W_3 = O_p(n^{\frac{1}{2}}), \\ &W_4 = O_p(n^{\frac{1}{2}}), \ W_5 = O_p(n) \text{ and } W_6 = O_p(n) \end{aligned} \tag{2.2.5}$$

Also by direct moment calculation we have

$$W_7 = \sum_{t=1}^{n} X_{1t-1}X_{1t-1}' = O_p(n^2). \tag{2.2.6}$$

Let $D_n = \text{diag}(n, \ldots, n, n^{\frac{1}{2}}, \ldots, n^{\frac{1}{2}})$. Then

$$[\hat{A}_{ols} - A] D_n = [\sum_{t=1}^{n} e_t X'_{t-1}] D_n^{-1} [D_n^{-1} \sum_{t=1}^{n} X_{t-1} X'_{t-1} D_n^{-1}]^{-1} = O_p(1). \quad (2.2.7)$$

Since $D_n^{-1} \sum_{t=1}^{n} X_{t-1} X'_{t-1} D_n^{-1}$ produces a block diagonal matrix in the limit we have

$$n(\hat{A}_{ols11} - I) = [\sum_{t=1}^{n} e_{1t} X'_{1t-1}/n] [\sum_{t=1}^{n} X_{1t-1} X'_{1t-1}/n^2]^{-1} + o_p(1). \quad (2.2.8)$$

Let

$$n(\hat{B}_{ols11} - I) = [\sum_{t=1}^{n} e_{1t} X'_{1t-1}/n] [\sum_{t=1}^{n} X_{1t-1} X'_{1t-1}/n^2]^{-1}. \quad (2.2.9)$$

Then $\hat{B}_{ols11}$ is the ordinary least squares estimator of the coefficient matrix in the r-dimensional AR(1) model with all unit roots. From (2.2.8) and (2.2.9) one can say that the stationary process $\{X_{2t}\}$ does not affect the limiting distribution of the estimated coefficient matrix $\hat{B}_{ols11}$ associated with the nonsationary part for the model transformed by R^{-1}. Again even though the limiting distribution of $\hat{B}_{ols11}$ depends on the matrix R, the limiting distributions of the estimated eigenvalues of $\hat{B}_{ols11}$ do not change as is stated in the following theorem.

Theorem Let $\hat{\lambda}_n = (\hat{\lambda}_{1n}, \hat{\lambda}_{2n}, \ldots, \hat{\lambda}_{rn}, \ldots, \hat{\lambda}_{kn})$ be the ordered eigenvalues of $\hat{A}_{ols}$ defined in (2.2.3) such that $\hat{\lambda}_{1n} \geq \hat{\lambda}_{2n} \geq \ldots \geq \hat{\lambda}_{kn}$.

Also let $\hat{\lambda}_n^* = (\hat{\lambda}_{1n}^*, \ldots, \hat{\lambda}_{rn}^*)$ and $\hat{\lambda}_n^{\otimes} = (\hat{\lambda}_{1n}^{\otimes}, \ldots, \hat{\lambda}_{rn}^{\otimes})$ be the ordered eigenvalues of $\hat{A}_{ols11}$ and $\hat{B}_{ols11}$ defined in (2.2.7) and (2.2.8) respectively. Then for $i = 1, \ldots, r$, $n(\hat{\lambda}_{in} - 1)$, $n(\hat{\lambda}_{in}^* - 1)$ and $n(\hat{\lambda}_{in}^{\otimes} - 1)$ have the same limiting distribution.

Proof For all i, $1 \leq i \leq r$,

$|\hat{A}_{ols} - \hat{\lambda}_{in} I| = |\hat{A}_{ols11} - \hat{\lambda}_{in} I + \hat{A}_{ols12}(\hat{A}_{ols22} - \hat{\lambda}_{in} I)^{-1} \hat{A}_{ols21}| \, |\hat{A}_{ols22} - \hat{\lambda}_{in} I| = 0.$

Since $\hat{A}_{ols} \rightarrow A$ in probability which is a diagonal matrix we have $\hat{\lambda}_{in} \rightarrow 1$.

However $|\hat{A}_{ols22} - \hat{\lambda}_{in} I| \rightarrow |A_{22}^* - I| \neq 0$ in probability. Hence for large n

$$|\hat{A}_{ols11} - \hat{\lambda}_{in} I + \hat{A}_{ols12}(\hat{A}_{ols22} - \hat{\lambda}_{in} I)^{-1} \hat{A}_{ols21}| = 0.$$

Since $\hat{A}_{ols12}(\hat{A}_{ols22} - \hat{\lambda}_{in} I)^{-1} \hat{A}_{ols21} = O_p(n^{\frac{-3}{2}})$, using expansion of determinant of sum of two matrices (see Searle p112) we have

$$| n(\hat{A}_{ols11} - I) - n(\hat{\lambda}_{in} - 1) I | = O_p(n^{\frac{-1}{2}}).$$

Using eigenvalue-eigenvector decomposition we have

$$\det\{\mathrm{diag}(n(\hat{\lambda}_{in}-1)-n(\hat{\lambda}_{1n}^{*}-1),\ n(\hat{\lambda}_{in}-1)-n(\hat{\lambda}_{2n}^{*}-1),\ \ldots ,\ n(\hat{\lambda}_{in}-1)-n(\hat{\lambda}_{rn}^{*}-1))\}$$
$$= O_p(n^{\frac{-1}{2}})$$

Hence at least one of diagonal elements must go to zero. Therefore for $i = 1, \ldots , r$, $n(\hat{\lambda}_{in}-1)$ and $n(\hat{\lambda}_{in}^{*}-1)$ have the same limiting distribution. Also (2.2.8) guarantees that $n(\hat{\lambda}_{in}^{\otimes}-1)$ and $n(\hat{\lambda}_{in}^{*}-1)$ have the same limiting distirbution.

Q.E.D

2.3 Derivation of a test statistic

In section 2.2 we have shown that the limiting distribution of the estimated eigenvalues associated with the nonstationary part of $\hat{A}_{ols}$ can be developed without regard to the stationary part. With this in mind we derive a test statistic to test for multiple unit roots in a multivariate AR(1) model with all roots one.

Consider the k-dimensional multivariate AR(1) model with $X_0 = 0$

$$X_t = B X_{t-1} + \varepsilon_t,\ t \geq 1 \qquad (2.3.1)$$

where ε_t's are i.i.d $N(0, \Sigma)$.

Assume data are generated by

$$X_t = I_{kk} X_{t-1} + \varepsilon_t,\ t \geq 1 \qquad (2.3.2)$$

where I_{kk} is an k-dimensional identity matrix.

Consider transformation $Z_t = \Omega X_t$. Then Z_t is a univariate AR(1) process with one unit root, that is, for any k-dimensional vector Ω, Z_t satisfies

$$Z_t = Z_{t-1} + \eta_t,\ t \geq 1 \qquad (2.3.3)$$

where $\eta_t = \Omega\varepsilon_t$ is a sequence of independent errors.

Note that the multivariate AR(1) model with all unit roots is in fact the only multivariate process for which every linear combination is a univariate AR(1) model with unit root.

Consider the general univariate AR(1) model

$$Z_t = \rho Z_{t-1} + \eta_t,\ t \geq 1. \qquad (2.3.4)$$

A test that the B matrix has all roots one is equivalent to a test that $\rho = 1$ in (2.3.4) for all Ω vectors. Now the ordinary least squares estimator of ρ is

$$\hat{\rho}_{ols} = \sum_{t=1}^{n}(Z_t Z_{t-1}) / \sum_{t=1}^{n} Z_{t-1}^2$$
$$= \Omega \sum_{t=1}^{n}(X_t X_{t-1}')\Omega' \ / \ \Omega \sum_{t=1}^{n}(X_{t-1} X_{t-1}')\Omega' \quad \text{for all } \Omega .$$

If the B matrix has all roots one then for any Ω , ρ must be one. Generally we are interested in testing whether B has all roots one or B has some unit roots and the rest less than one in magnitude. Therefore our strategy is to find $\min_{\Omega} \hat{\rho}_{ols}$, if it rejects $H_0 : \rho = 1$ then we can decide that the B matrix does not have all roots one.

We need to find $\min_{\Omega} \hat{\rho}_{ols}$. Now

$$\min_{\Omega} \hat{\rho}_{ols} = \min_{\Omega} \Omega \sum_{t=1}^{n}(X_t X_{t-1}')\Omega' \ / \ \Omega \sum_{t=1}^{n}(X_{t-1} X_{t-1}')\Omega'$$
$$= \text{Smallest eigenvalue of } \{(\sum_{t=1}^{n} X_t X_{t-1}' + \sum_{t=1}^{n} X_{t-1} X_t') \cdot [\sum_{t=1}^{n} X_{t-1} X_{t-1}']^{-1}/2\}.$$

Let $\tilde{B} = (\sum_{t=1}^{n} X_t X_{t-1}' + \sum_{t=1}^{n} X_{t-1} X_t') \cdot [\sum_{t=1}^{n} X_{t-1} X_{t-1}']^{-1}/2$. Phillips and Durlauf (1986) showed that $\tilde{B}$ converges to I_{kk} in probability. This is similar to the ideas proposed by Johansen (1988) and Stock and Watson (1988).

Now from section 2.2 when n is large with R^{-1} known we can directly apply this statistic to the general AR(1) model with multiple unit roots and the rest less than one in magnitude.

2.4 Multivariate AR(p) model

In this section we investigate the k-dimensional AR(p) model

$$Y_t = C_1 Y_{t-1} + \dots + C_p Y_{t-p} + \delta_t \qquad (2.4.1)$$

where δ_t's are i.i.d $N(0, \Sigma)$ with Σ positive definite. Using state space representation we can make k-dimensional AR(p) model pk-dimensional AR(1) model. That is let

$$Z_t = \begin{bmatrix} Y_t \\ \vdots \\ Y_{t-p+1} \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & C_2 & \cdots & C_p \\ I & \phi & \cdots & \phi \\ \vdots & \ddots & & \vdots \\ \phi & \phi & \dots I & \phi \end{bmatrix}, \quad \eta_t = \begin{bmatrix} \delta_t \\ \phi \\ \vdots \\ \phi \end{bmatrix}.$$

then (2.4.1) becomes $Z_t = C Z_{t-1} + \eta_t$

We assume that the characteristic equation

$$|\lambda^p I - \sum_{j=1}^{p} \lambda^{p-j} C_j| = 0 \quad (2.4.2)$$

has r multiple unit roots and the other pk − r roots less than 1 in magnitude.

Again we assume that there exists a real matrix S such that

$$E = S^{-1}CS = \begin{bmatrix} I_{rr} & \phi' \\ \phi & E^*_{22} \end{bmatrix} \quad (2.4.3)$$

where E^*_{22} is a diagonal matrix with elements less than 1 in magnitude.

Assumption (2.4.3) eliminates the possibility of multiple unit roots being any component series of the vector process.

Using state space representation and transformation by S^{-1} (2.4.1) become

$$X_t = E X_{t-1} + e_t \quad (2.4.4)$$

where $X_t = S^{-1}Z_t$, $E = S^{-1}CS$ and $e_t = S^{-1}\eta_t$.

Then the ordinary least squares estimator of E is

$$\hat{E}_{ols} = [\sum_{t=1}^{n} X_t X'_{t-1}][\sum_{t=1}^{n} X_{t-1} X'_{t-1}]^{-1}$$
$$= \begin{bmatrix} \hat{E}_{ols11} & \hat{E}_{ols12} \\ \hat{E}_{ols21} & \hat{E}_{ols22} \end{bmatrix}$$

where $\hat{E}_{ols11}$ and $\hat{E}_{ols22}$ correspond to the nonstationary part and the stationary part respectively.

Then by arguments similar to those in section 2.2 we have

$$n(\hat{E}_{ols11} - I) = [\sum_{t=1}^{n} e_{1t} X'_{1t-1}/n][\sum_{t=1}^{n} X_{1t-1} X'_{1t-1}/n^2]^{-1} + o_p(1)$$

where X_{1t} is a nonstationary vector process and $e_{1t} = X_{1t} - X_{1t-1}$. Therefore with known S^{-1} we can use results in section 2.3 to test for multiple unit roots for the multivariate AR(p) model.

3. References

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다변량 시계열 자료의 다중단위근 검정법

신기일[2)]

요약

본 논문에서는 비정상(단위근) 시계열이 포함된 다변량 시계열 자료에서 단위근에 해당되는 계수행렬 추정량의 극한분포가 정상시계열의 유무에 상관없이 일정하다는 것을 밝혔다. 또한 단위근만 존재하는 다변량 시계열에서 다중단위근을 검정하는 검정 통계량을 제안하였다.

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