⊙論 文

A Three-Dimensional Dynamic Analysis of Towed Systems Part 1. A Mathematical Formulation

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(1993년 9월 1일 접수)

수중예인시스템의 3차원 동역학 해석 1부 : 수학모델 정식화

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Key Words : towed system(예인시스템), three-dimensional dynamic analysis(3차원 동역학 해석), nonlinear(비선형), axial flow(축류), hydroelastic instability(유탄성 불안정성

초 록

수중 예인시스템의 동적 거동 해석을 위한 3차원 비선형 수학모델이 제시되었다. 수중 예인체는 세장보로 이상화되었으며, 보요소의 굽힘강성 및 비틀강성의 영향이 수학모델에 포함되었다. 축류가 지배적인 비정상 상대유동장내의 세장예인체의 횡방향 운동에 따른 유체동역학적 반력과 기진력에 관한 비선형 3차원 수학 정식화가 수행되었다.

Introduction

Most of the recent investigations on the dynamics of towed systems are focused on the dynamic analysis of instrumented long flexible cylinders[1,5,6,12,16,19], which are used for geophysical exploration and research, and for detection of sound signals under water etc. Research works on the dynamic behaviour of towed cables are reported in 3), 4).

The hydroelatstic behaviours of the instrumented long towed cylinders and of the towed cables connecting the deep-sea-equipment to tow vessels differ, in spite of the same category of towed systems, from each other. The latter is towed inclined at the critical angle balancing the cable weight and the normal drags. The induced tension

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is relatively sufficient for stability of the towed system. The former is designed to be neutrally buoyant so that it is towed at the horizontal position. The tensions induced by tangential friction in axial flow are low, and at sufficiently high flow velocity hydroelastic instabilities are possible.[6,13,20]

The cylinder carrying an array of hydrophones is connected through towcable to tow vessel — surface vessels, submarines, or helicopters in air. To provide sufficient tensions through the entire region of towed cylinder in practice, a drogue is attached at the downstream end. The models by many investigators for prediction of the hydroelastic behaviours of instrumented towed arrays assume them as an idealized cable or a string without flexural rigidity.

The hydroelastic behaviour of a long cylindrical structure in axial flow is, with slight difference, analogous to that of a pipe conveying fluid. Stability analysis of a slender cylinder in axial flow[6] showed that hydroelastic instabilities occur when $T-m_aU^2\leq 0$, where T is the tension in the cylinder, m_a the sectional added mass and U the mean flow velocity.

In 20) it is shown that the structural stability of a string in axial flow can be assured by involving the influence of flexural rigidity in the low tension region. The works on the dynamics of cylindrical structures in axial flow involving flexural rigidity are reported in 6), 13), 14). The investigations are, however, restricted to purely axial, uniform and steady flow. Furthermore, quantitative investigation of the effects of flexural rigidity on the structural dynamics for a flexible cylinder subjected to low tension is very limited yet.

The purpose of this paper is to provide a general three-dimensional nonlinear mathematical model for towed cylindrical structures possessing torsional and flexural rigidity effects.

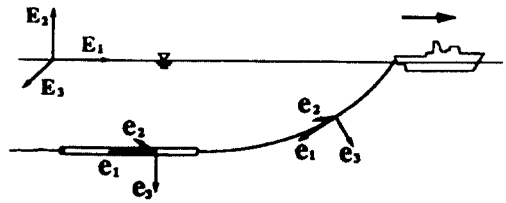

Fig. 1 Configuration of a towed system and coordinates systems

Mathematical Model

The assumptions for the formulation of the problem are made as

- material is homogeneous, isotropic and linear elastic
- towed body is idealized as a beam element
- shear deformation is neglected
- plane cross-sections remain plane after deformation
- thermal effects are not considered

The configuration of a towed system is shown in Fig. 1. Two coordinates systems are introduced to describe the instantaneous configuration of the towed system. The inertial frame is defined by space-fixed Cartesian coordinates with base vector (E_1, E_2, E_3). The moving local coordinates(e_1, e_2, e_3) are attached at the mass points of body.

The two base vectors (E_1, E_2, E_3) and (e_1, e_2, e_3) are to be transformed to each other through the rotation tensor T. Using notation of tensor summation the transformation relation can be written as follows

$$e_i=\tau_{ij}E_j \quad \cdots\cdots (1)$$

, where τ_{ij} are the elements of rotation tensor. The rotation tensor **T** is orthonormal. Euler angles are used to describe the rotations of the local base vectors. The Euler angles are employed in the following sequence,

$$\theta_2 E_2 \rightarrow \theta_3 E_3^1 \rightarrow \theta_1 e_1$$

Fig. 2 shows the applied Euler angles system. The rotation tensor determined using the Euler angles is given in appendix.

A complete description of the towing geometry requires the knowledge of the rates of change of the local coordinates in space and time. The two change rates are expressed by curvature and angular velocity tensor, respectively.

$$e_{i,s} = K_{ij} e_j \quad \cdots\cdots (2)$$
$$e_{i,t} = \Omega_{ij} e_j \quad \cdots\cdots (3)$$

, where the comma followed by subscript s or t denotes the partial derivative with respect to the Lagrangian coordinate s or time t. The components K_{ij} and Ω_{ij} are given using permutation tensor ε_{ijk} which gives values 1, -1 and 0 for cyclic, anticyclic and acyclic permutation of subscripts i, j and k, respectively, as

$$K_{ij} = \varepsilon_{ijk} \kappa_k \quad \cdots\cdots (4)$$
$$\Omega_{ij} = \varepsilon_{ijk} \omega_k \quad \cdots\cdots (5)$$

The κ_i and ω_i are the curvatures and angular velocities about the local unit vector e_i, respectively, and dependent on the Euler angles. They are given in appendix. The relations between the curvature and the angular velocity tensor is obtained from eq. (2) and (3) as

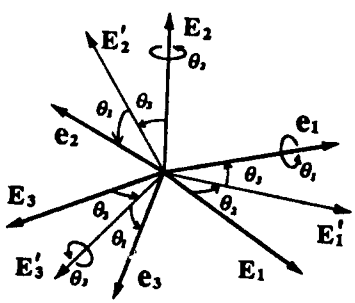

Fig. 2 Euler angles

$$\kappa_{i,t} = \omega_{i,s} + \varepsilon_{ijk} \kappa_j \omega_k \quad \cdots\cdots (6)$$

The position vector of a mass point of the towed body is defined in global coordinates system as

$$\mathbf{r} = x_i E_i \quad \cdots\cdots (7)$$

The tangential unit vector of local coordinates system e_1 is determined by the change rate of the position vector along the chord length s' in deformed state, i.e.

$$\mathbf{r}_{,s'} = e_1 \quad \cdots\cdots (8)$$
$$\mathbf{r}_{,s} = (1+\varepsilon) e_1 \quad \cdots\cdots (9)$$

The equation (9) above is obtained from the relation $ds' = (1+\varepsilon)ds$, where ε is the axial strain. The two principal axes of the cross section perpendicular to the tangential axis are selected as the other two local unit vectors in according to the right-hand rule. The change rates of the global coordinates are, therefore, determined by the rotation tensor as

$$x_{i,s} = (1+\varepsilon)\tau_{1i} \quad \cdots\cdots (10)$$

With definition of the local velocity vector

$$V = v_i e_i = \mathbf{r}_t \quad \cdots\cdots (11)$$

, the compatibility equations are obtained by considering the change rate of the velocity vector along the cord length as

$$v_{1,s} - v_2\kappa_3 + v_3\kappa_2 = \varepsilon_{,t} \quad \cdots\cdots (12)$$
$$v_{2,s} - v_3\kappa_1 + v_1\kappa_3 = (1+\varepsilon)\omega_3 \quad \cdots\cdots (13)$$
$$v_{3,s} - v_1\kappa_2 + v_2\kappa_1 = -(1+\varepsilon)\omega_2 \quad \cdots\cdots (14)$$

The accelerations of mass points in local coordinates systems are determined by time derivative of the local velocity vectors and their components are obtained in the following form

$$a_i = v_{i,t} + \varepsilon_{jik} v_j \omega_k \quad \cdots\cdots (15)$$

The linear elastic stress-strain relationship for a Poisson-type element is given as

$$E\varepsilon = \frac{Q_1^e}{A} + (2\mu-1)p \quad \cdots\cdots (16)$$

, where E is the modulus of elasticity, A the section area, μ the Poisson ratio, p the hydrostatic pressure and Q_1^e the effective tension resulted from using the effective weight w_e[17]. The effective tension is related to the real one T through

$$Q_1^e = T + pA \quad \cdots\cdots (17)$$

It is shown that the effective stress can be used for the criterion of the material safety[17].

For deformation preserving volume, i.e. $\mu=0.5$, the constitutive equation between tension and strain is written as

$$Q_1 = EA\varepsilon \quad \cdots\cdots (18)$$

, where the superscript e indicating the effective one is abbreviated for the brevity of expression. The constitutive equations between moment and curvature are given

$$M_i = J_{ii}\kappa_i \quad \cdots\cdots (19)$$

, where J_{ii} are the torsional and the flexural rigidities, respectively.

One obtains the components of the force and moment equilibrium equations in an infinitesimal segment of the towed body, based on the Kirchhoff's beam theory[11], as eqs. (20) and (21)

$$ma_i = Q_{i,s} + \varepsilon_{jik}Q_j\kappa_k + \Delta_i + w_e\tau_{i2} + f_{ri} + f_{ei} \quad \cdots (20)$$

$$I_{ii}\omega_{i,t} + \varepsilon_{jik}I_{jj}\omega_j\omega_k = M_{i,s} + \varepsilon_{jik}M_j\kappa_k + \varepsilon_{1ji}Q_j + \Theta_i + M_{ri} \quad \cdots\cdots (21)$$

, hereby m and I_{ii} are the mass and the mass moments of inertia per unit length of towed body, Q_i and M_i the internal forces and moments, Δ_i and Θ_i the structural damping forces and moments, f_{ri} and M_{ri} the hydrodynamic reaction forces and moments, and f_{ei} are the remaining external exciting forces — inertia and drag forces — from the surrounding fluid flow, respectively.

The formulation for the hydrodynamic reaction forces on the slender body section moving with the velocity V in the fluid flow of velocity field U is generalized from the two dimensional linear formulation proposed by Lighthill[10]

$$f_{rn} = -m_a(a_n + 2\varepsilon_{1nm}v_r\omega_m + \varepsilon_{1nm}v_r^2\kappa_m) \quad \cdots\cdots (22)$$

$$v_r = u_1 - v_1 : \text{relative axial velocity}$$

, where m_a is the added mass of the circular body section and the subscript n for $n=2$, 3 indicates the normal components. The first term is the translatory inertial reaction force, the second and the third terms ars the Coriolis and centrifugal force, the sources of possible hydroelastic instabilities.

Eq. (22) becomes, in the case of a two dimensional linear problem, identical to the equations used by many authors for analysis of stability or the dynamics of a cylinder in axial flow. Eq. (22) is fully nonlinear in the sense that in which an ambient in-flow relative to the towed body movement and the effect of the curvatures and angular velocities due to large deformations of structure configuration are taken into account.

For a fluid particle passing the towed cylinder with relative axial velocity v_r, the time change rate of the base vector **e** of local coordinate system is given by employing substantial derivative

$$\mathbf{e}_{i,t} = \Omega_{ij}^c \ \mathbf{e}_j \quad \cdots\cdots (23)$$

The angular velocity tensor Ω^c of a convective fluid particle is dependent on the cylinder curvatures and angular velocities, and the elements are determined as

$$\Omega_{ij}^c = \varepsilon_{ijk}\omega_k^c \quad \cdots\cdots (24)$$

$$\omega_i^c = \omega_i + v_r\kappa_i \quad \cdots\cdots (25)$$

The hydrodynamic reactional moments on the towed body are obtained by substantial derivative of the angular momentum of the surrounding fluid induced by the motion of body.

$$-M_{ri}\ \mathbf{e}_i = (I^a_{ii}\ \omega^c_i\ \mathbf{e}_i)_{,t} + v_r(I^a_{ii}\ \omega^c_i\ \mathbf{e}_i)_{,s} \quad \cdots\cdots (26)$$

, where I^a_{ii} are the added mass moments of inertia about local unit vectors $\mathbf{e}_i$. The components are given as

$$\begin{aligned} M_{ri} = & -I^a_{ii}\omega_{i,t} + \varepsilon_{ijk} I^a_{jj}\omega_j\omega_k \\ & -(v_r^2\ I^a_{ii}\ \kappa_i)_{,s} + \varepsilon_{ijk} v_r^2\ I^a_{jj}\ \kappa_j\kappa_k \\ & -v_r I^a_{ii}(2\omega_{i,s} + \varepsilon_{ijk}\kappa_j\omega_k) \\ & +\varepsilon_{ijk}\ v_r\ I^a_{jj}(\omega_j\kappa_k + \kappa_j\omega_k) \\ & -I^a_{ii}(v_{r,t} - v_r v_{r,s})\kappa_i - v_r\ I^a_{ii,s}\omega_i \quad \cdots\cdots\cdots (27) \end{aligned}$$

Substituting eqs. (22) and (27) into eqs. (20) and (21), and using the constitutive equations between moments and curvatures the equilibrium equations are given in the following forms

$$\begin{aligned} ma_1 - 2m_a v_r v_{r,s} = & Q_{1,s} - \kappa_3 Q_2 + \kappa_2 Q_3 + \Delta_1 + \\ & w_e\tau_{12} + f_{e1} \quad \cdots\cdots (28) \end{aligned}$$

$$\begin{aligned} m_v\ a_2 - 2m_a v_r \omega_3 = & Q_{2,s} + \kappa_3 Q_1 - \kappa_1 Q_3 + \Delta_2 + \\ & w_e\tau_{22} + f_{e2} \quad \cdots\cdots (29) \end{aligned}$$

$$\begin{aligned} m_v\ a_3 - 2m_a v_r \omega_2 = & Q_{3,s} - \kappa_2 Q_1 + \kappa_1 Q_2 + \Delta_3 + \\ & w_e\tau_{32} + f_{e3} \quad \cdots\cdots (30) \end{aligned}$$

$$\begin{aligned} & (J^v_{11}\kappa_1)_{,s} + (J^v_{33} - J^v_{22})\kappa_3\kappa_2 + \Theta_1 = \\ & \quad I^v_{11}\omega_{1,t} + (I^v_{33} - I^v_{22})\omega_2\omega_3 + 2v_r I^a_{11}\omega_{1,s} + \\ & \quad v_r(I^a_{33} - I^a_{22} + I^a_{11})\kappa_2\omega_3 + \\ & \quad v_r(I^a_{33} - I^a_{22} + I^a_{11})\kappa_3\omega_2 + \\ & \quad I^a_{11}(v_{r,t} - v_r v_{r,s})\kappa_1 + v_r I^a_{11,s}\omega_1 \quad \cdots\cdots (31) \end{aligned}$$

$$\begin{aligned} & (J^v_{22}\kappa_2)_{,s} + (J^v_{11} - J^v_{33})\kappa_1\kappa_3 + Q_3 + \Theta_2 = \\ & \quad I^v_{22}\omega_{2,t} + (I^v_{11} - I^v_{33})\omega_1\omega_3 + 2v_r I^a_{22}\omega_{2,s} + \\ & \quad v_r(I^a_{11} - I^a_{33} + I^a_{22})\kappa_3\omega_1 + \\ & \quad v_r(I^a_{11} - I^a_{33} + I^a_{22})\kappa_1\omega_3 + \\ & \quad I^a_{22}(v_{r,t} - v_r v_{r,s})\kappa_2 + v_r I^a_{22,s}\omega_2 \quad \cdots\cdots (32) \end{aligned}$$

$$\begin{aligned} & (J^v_{33}\kappa_3)_{,s} + (J^v_{22} - J^v_{11})\kappa_1\kappa_2 + Q_2 + \Theta_3 = \\ & \quad I^v_{33}\omega_{3,t} + (I^v_{22} - I^v_{11})\omega_1\omega_2 + 2v_r I^a_{33}\omega_{3,s} + \\ & \quad v_r(I^a_{22} - I^a_{11} + I^a_{33})\kappa_1\omega_2 + \\ & \quad v_r(I^a_{22} - I^a_{11} + I^a_{33})\kappa_2\omega_1 + \\ & \quad I^a_{33}(v_{r,t} - v_r v_{r,s})\kappa_3 + v_r I^a_{33,s}\omega_3 \quad \cdots\cdots (33) \end{aligned}$$

, where the subscript or superscript v indicates the virtual ones, i.e. $m_v = m + m_a$ and $I^v_{ii} = I_{ii} + I^a_{ii}$. For a circular section I^a_{ii} vanishes, whereas I^a_{22} and I^a_{33} become identical. Δ_i and Θ_i, the structural damping forces and moments, are neglected for a linear elastic model. The effective tension in eqs. (28−30) becomes to

$$Q_1 = T + pA - m_a v_r^2 \quad \cdots\cdots\cdots\cdots (34)$$

The effective rigidities in eqs. (31−33) defined as

$$J^v_{ii} = J_{ii} - v_r^2\ I^a_{ii} \quad \cdots\cdots\cdots\cdots (35)$$

are analogous to those of a pipe conveying fluid.[8, 15] As shown in eqs. (34) and (35), the influences of the centrifugal force and moment on the hydroelastic behaviour of a slender towed body in axial flow can be understood as decrease of the tension and rigidites.[8] These effects, together withe the Coriolis force and moment, can cause structural instabilities, buckling or flutter, depending on boundary conditions.

The external forces on the towed body are decomposed into inertia force by the acceleration of fluid particles and drag force due to the fluid viscosity. The inertia forces on slender body is often formulated using the mass coefficient C_M in Morison's equation.

The viscous effects on the interaction between slender body and the neighbouring fluid are often considered by means of the "crossflow" principle.[7, 14] The effect of the crossflow separation is neglected by many authors for the linear analysis of a horizontal cylinder or string in axial flow, for example 20). The validity of this assumption of no-separation is limited in the case of small angle of attack less than 2 deg.

The cross-flow separation from an inclined stationary object is observed when the inclination of the object to the external flow exceeds an angle of 2−3 deg.[2] For a transient dynamics of towed systems, therefore, the both terms of viscous drag force — friction part and pressure part — must be considered.

The drag force on the towed body moving with the relative velocity $V_r = v_i^r e_i$ is to be decomposed into the tangential force and the normal force.[18] The decomposition of the drag force shows

$$f_{d1,s} = \frac{\varrho}{2} \pi D C_f \mid V_r \mid v_1^r \quad \cdots\cdots (36)$$

$$f_{d2,s} = \frac{\varrho}{2} D(\pi C_n \mid V_r \mid + C_d \mid V_r^n \mid) v_2^r \quad \cdots (37)$$

$$f_{d3,s} = \frac{\varrho}{2} D(\pi C_n \mid V_r \mid + C_d \mid V_r^n \mid) v_3^r \quad \cdots (38)$$

, hereby ϱ is the mass density of water, D the diameter of circular section, V_r^n the normal part of the relative velocity vector, C_f the viscous friction coefficient, C_n the coefficient varying in the range of $0 \le C_n \le C_f$ depending on the roughness on the towed cylinder[18] and C_d the pressure drag coefficient depending on the angle of attack. The dependence of the pressure drag coefficient on the angle of attack α may be expressed using a loading function $f(\alpha)$

$$C_d = C_D f(\alpha) \quad \cdots\cdots (39)$$

The C_D in eq. (39) is the value of C_d at $\alpha = 90°$. Various formulae are suggested for the non-dimensional loading function by many investigators. An example is given in 9). The C_n is often identified as C_f.[13,19] Typically, C_f for a straight array is less than 0.01, whereas C_D is about 1. Eqs. (36−38) become identical with the formulations used for the linear dynamics of a long cylider in axial flow in 6), 13), 18) etc.

The complete solution of the set of differential equations(12−14), (18, 19) and (28−33) requires the boundary conditions at the well upstream connection poing and at the downstream end.

Concluding Remarks

A general fully three-dimensional mathematical model is developed, which is applicable for dynamic analysis of a towed system in an ambient flow and at an unsteady towing condition. The towed body retains torsional and flexural rigidities.

A lumped-mass model on the basis of the presented mathematical model is developed, and unmerical simulations of towed arrays in unsteady tow manoeuvres are undergoing.

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Appendix

The matrix of rotation tensor **T** obtained using the Euler angles of Fig. 2 is

$$\mathbf{T} = \begin{pmatrix} c_2c_3 & s_3 & -s_2c_3 \\ s_1s_2 - c_1c_2s_3 & c_1c_3 & s_1c_2 + c_1s_2s_3 \\ c_1s_2 + s_1c_2s_3 & -s_1c_3 & c_1c_2 - s_1s_2s_3 \end{pmatrix}$$

, where for the brevity of expression the definitions $c_i = \cos\theta_i$ and $s_i = \sin\theta_i$ are used. This matrix becomes singular at $\theta_3 = \pi/2$.

The element components of the curvature and angular velocity tensors are determined by the relations

$$K_{ij} = \tau_{ik,s}\ \tau_{jk}$$
$$\Omega_{i\xi} = \tau_{ik,t}\ \tau_{jk}$$

, where τ_{ij} are the elements of **T**.

The curvatures and angular velocities are given as

$$\kappa_1 = \theta_{1,s} + \theta_{2,s}\ \sin\theta_3$$
$$\kappa_2 = \theta_{3,s}\ \sin\theta_1 + \theta_{2,s}\ \cos\theta_1\ \cos\theta_3$$
$$\kappa_3 = \theta_{3,s}\ \cos\theta_1 - \theta_{2,s}\ \sin\theta_1\ \cos\theta_3$$

$$\omega_1 = \theta_{1,t} + \theta_{2,t}\ \sin\theta_3$$
$$\omega_2 = \theta_{3,t}\ \sin\theta_1 + \theta_{2,t}\ \cos\theta_1\ \cos\theta_3$$
$$\omega_3 = \theta_{3,t}\ \cos\theta_1 - \theta_{2,t}\ \sin\theta_1\ \cos\theta_3$$