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ON DISCRETE GROUPS

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The concept of a continuous module is a generalization of that of an injective module, and conditions (C_1), (C_2) and (C_3) are given for this concept in [4]. In this paper, we study modules with properties that are dual to continuity. These will be called discrete and we discuss discrete abelian groups.

Throughout R is a ring with identity, M is a module over R, G is an abelian group of finite rank, E is the ring of endomorphisms of G and S is the center of E.

Dual to the notion of essential submodules, we define small submodules of a module M over R.

DEFINITION. A submodule X of a module M is called small in M (notation $X \ll M$) if $X + Y \neq M$ for any proper submodule Y of M. A module H is called hollow if every proper submodule of H is small.

By proposition 9.13 in [1], the Jacobson radical of M denoted by $\operatorname{Rad} M = \cap\{K \leq M \mid K \text{ is maximal in } M\} = \sum\{S \leq M \mid S \text{ is small in } M\}$. Note that there are two types of hollow modules H:

(i) $H \neq \operatorname{Rad} H$; in this case, $H = Rx$ for every $x \notin \operatorname{Rad} H$.
(ii) $H = \operatorname{Rad} H$; in this case, H is not finitely generated.

The following conditions (D_i) are dual to (C_i), respectively:

(D_1) For every submodule X of M, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq X$ and $X \cap M_2 \ll M$;
(D_2) If $X \leq M$ such that M/X is isomorphic to a direct summand of M, then X is a direct summand of M.
(D_3) If M_1 and M_2 are direct summands of M with $M_1 + M_2 = M$, then $M_1 \cap M_2$ is a direct summand of M.

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The following immediate consequences about (D_i) are used freely.
(1) If a module M has (D_2), then it has (D_3).
(2) Any direct summand of M with (D_i) also satisfies (D_i).
(3) An indecomposable module M has (D_1) if and only if M is hollow.

DEFINITION. A module M is called discrete if it has (D_1) and (D_2); M is called quasi-discrete if it has (D_1) and (D_3).

It is clear that every hollow module is quasi-discrete. Also, Discrete modules are quasi-discrete.

The next theorem characterizes quasi-discrete modules.

THEOREM [4, THEOREM 4.15]. *Any quasi-discrete module M has a decomposition $M = \oplus_{i\in I} H_i$ where each H_i is hollow. Moreover, such a decomposition complements summands, and hence is unique up to isomorphism.*

Recall the following:
Injective $\Rightarrow$ quasi-injective $\Rightarrow$ continuous $\Rightarrow$ quasi-continuous.
But we have:
Projective $\Rightarrow$ quasi-projective $\not\Rightarrow$ discrete $\Rightarrow$ quasi-discrete.

Now consider an abelian group G as a Z-module. $Q, Z(p^\infty)$, Q/Z are examples for continuous (in fact, injective) groups. But Q is indecomposable but not hollow and $Z(p^\infty)$ does not have (D_2). Therefore above groups are not discrete. Also, projective group Z does not have (D_1) and hence is not discrete.

LEMMA 1. *If G contains an element of infinite order, then G is not discrete.*

Proof. Suppose G is discrete. Then

$$G = (\oplus_{i\in I} H_i) \oplus (\oplus_{j\in J} K_j)$$

where each H_i is hollow and $\operatorname{Rad} H_i \neq H_i$, and each K_j is hollow and $\operatorname{Rad} K_j = K_j$ by Theorem 4.15 and Corollary 4.18 in [4]. Since $H_i = Zx_i$ for every $x_i \notin \operatorname{Rad} H_i$, H_i is isomorphic to Z for the x_i of infinite order. But H_i is discrete and Z is not discrete. Therefore K_j contains an element of infinite order for some $j \in J$. Since K_j is hollow and $\operatorname{Rad} K_j = K_j$, we have an irredundant sum $K_j = \sum S_n$ where S_n is indecomposable small subgroup of K_j. By Lemma 4.22 in [4], $K_j = \oplus S_n$. But K_j is hollow. Hence $K_j \ll K_j$ and $K_j = 0$. This contradicts the fact that K_j contains an element of infinite order. Thus G is not discrete.

Therefore G is discrete only if G is torsion. But there is a torsion free abelian group which is discrete over E. Consider Q. Then $E \simeq Q$ and Q is discrete as an E-module.

LEMMA 2. (1) *Z_{p^n} is discrete for a prime p and $n \in \mathbb{N}$.*
(2) *For a prime p and $n_i \in N$, $\oplus_{i \in I} Z_{p^{n_i}}$ is discrete if and only if $n_i = n_j$ for $i, j \in I$.*

Proof. (1) Since Z_{p^n} is indecomposable and every proper subgroup of Z_{p^n} is small, Z_{p^n} has (D_1). The only direct summands of Z_{p^n} are 0 and Z_{p^n}. Therefore, a subgroup X such that Z_{p^n}/X is isomorphic to a direct summand of Z_{p^n} is 0 or Z_{p^n} since Z_{p^n} is a finite group. Hence Z_{p^n} has (D_2). Thus Z_{p^n} is discrete.
(2) Note that if $n_i < n_j$, then $Z_{p^{n_i}}$ is not $Z_{p^{n_j}}$-projective. By theorem 4.48 in [4], $n_i = n_j$ for $i, j \in I$. For the converse, apply above (1), corollary 4.50 and theorem 5.2 in [4].

Here I is a finite set since we assume that rank of G is finite.

PROPOSITION 1. *G is discrete if and only if $G \simeq \oplus_{i \in I} Z_{p_i^{n_i}}$ where each p_i is a prime number, $n_i \in \mathbb{N}$ and $n_i = n_j$ if $p_i = p_j$.*

Proof. the necessity follows from lemma 2, corollary 4.50 and theorem 5.2 in [4]. Conversely assume that G is discrete. Then $G = (\oplus_{i \in I} H_i) \oplus (\oplus_{j \in J} K_j)$ as in the proof of lemma 1. For each H_i, $H_i = Zx_i$ for $x_i \notin \operatorname{Rad} H_i$. Since G is torsion and H_i is hollow, $H_i \simeq Z_{p_i^{n_i}}$ for some prime p_i and $n_i \in \mathbb{N}$. For $j \in J$, $K_j = 0$ by the proof of lemma 1. Hence $G = \oplus_{i \in I} H_i \simeq \oplus_{i \in I} Z_{p_i^{n_i}}$. And $n_i = n_j$ for $p_i = p_j$ by lemma 2.

LEMMA 3. *Let p be a prime, $A_p = \oplus_I Z_{p^n}$ and E_p be the endomorphism ring of A_p. Then A_p is discrete over E_p.*

Proof. Let B_p be a submodule of A_p. Since B_p is a fully invariant subgroup of A_p, $B_p = \oplus_I (Z_{p^n} \cap B_p)$ and $Z_{p^n} \cap B_p = p^k Z_{p^n}$ for some k $(0 \leq k \leq n)$. Here $p^k Z_{p^n} \ll Z_{p^n}$ for $k > 0$. Thus every proper submodule of A_p is small and A_p is hollow as an E_p-module. Hence

A_p is indecomposable over E_p and has (D_1). From above, a submodule B_p of A_p is of the form $B_p = \oplus_I p^k Z_{p^n}$ $(0 \leq k \leq n)$. Let B_p be a submodule of A_p such that A_p/B_p is isomorphic to a direct summand of A_p. Since $A_p/B_p \simeq \oplus_I p^{n-k} Z_{p^n}$, we have $\oplus_I p^{n-k} Z_{p^n} \simeq 0$ or $\oplus_I p^{n-k} Z_{p^n} \simeq \oplus_I Z_{p^n}$, and hence $k = 0$ or $k = n$. Therefore $B_p = A_p$ or $B_p = 0$ i.e., B_p is a direct summand of A_p. Thus A_p has (D_2).

PROPOSITION 2. *If ${}_ZG$ is discrete, then ${}_EG$ is discrete.*

Proof. Since ${}_ZG$ is discrete, $G = \oplus_{p \in \Pi} A_p$ where Π is a set of prime numbers and $A_p \simeq \oplus Z_{p^n}$ for some $n \in \mathbb{N}$ by proposition 1. It is clear that A_p is a fully invariant subgroup of G since $\mathrm{Hom}(Z_p, Z_q) = 0$ for distinct primes p and q. Hence each A_p is a submodule of ${}_EG$ and ${}_EA_p = {}_{E_p}A_p$ where E_p is the ring of endomorphisms of A_p. By lemma 3, A_p is discrete over E and hollow as an E-module. Therefore G is discrete over E by theorem 4.48 and 5.2 in [4].

Recall that there exists a group which is not discrete but discrete as an E-module. By corollary 2.1 in [2] and theorem 5.4 in [4], we have the following results.

COROLLARY. (1) *If G is regular over E, then G is discrete over E.*
(2) *If S is regular, then G is discrete over E.*

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