DIRICHLET PROBLEM ON THE UPPER
HALF PLANE – A HEURISTIC ARGUMENT

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The Dirichlet problem (DP) on the upper half plane \( \{z = x + iy : y > 0\} \) is to find a real-valued harmonic function \( u(x,y) \) satisfying \( u(x,0) = g(x) \) almost everywhere for some reasonably nice function \( g \) defined on the real line, which is called the data on the boundary for (DP). To find such a function we use the formula

\[
    u(x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{g(\xi)y}{(x-\xi)^2 + y^2} d\xi \quad \text{for } y > 0.
\]

In most references it is derived using Cauchy’s integral formula. In this short article we derive the formula using elementary ideas. First we need

**LEMMA.** Let \( g \) be a real valued function defined on the real line such that \( g(x) = 1 \) for \( a \leq x < b \) and \( g(x) = 0 \) elsewhere, i.e., \( g(x) = \chi_{[a,b]} \). Choose a branch for \( \log z \) so that it is single-valued and analytic on the upper half plane. For example, put \( \log z = \log |z| + \text{Arg}(z), \frac{\pi}{2} < \text{Arg}(z) < \frac{3\pi}{2} \). Then the solution of (DP) is given by

\[
    u_{ab}(x,y) = \frac{1}{\pi} \text{Im} \left[ \log \frac{z-b}{z-a} \right].
\]

**Proof.** Since \( \text{Arg}(z), z \neq 0 \) is the imaginary part of the analytic function \( \log z \), it is harmonic. Hence \( \text{Arg}(z-b) - \text{Arg}(z-a) \) is harmonic on the upper half plane, which is equal to the imaginary part of \( \log(z-b) - \log(z-a) = \log \frac{z-b}{z-a} \). It is easy to see that the function satisfies the boundary condition except at \( z = a, b \). For the details, see [1, p. 377].

Note that (DP) is linear with respect to the data \( g \) on the boundary in the sense that if \( u_1, u_2 \) are solutions for (DP) with boundary data

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\( g_1, g_2, \) respectively, then \( u = c_1u_1 + c_2u_2 \) is the solution for \( (DP) \) with boundary data \( g = c_1g_1 + c_2g_2 \) where \( c_1, c_2 \) are arbitrary constants.

For a general Dirichlet Problem we consider the case when \( g \) is piecewise continuous and integrable along the real line. We will generalize the concept of linearity of \( (DP) \) up to an infinite sum of \( g_i \)'s and decompose the given data \( g \) into an infinite linear combination of characteristic functions of infinitesimally short intervals.

We partition the real axis into very short intervals \( I_k = [x_k, x_{k+1}), \) \( -\infty < k < \infty, \) and consider the \( (DP) \) for \( g_k(x) \equiv g(x_k) \cdot \chi_{I_k}(x) \) and find the corresponding solution

\[
\begin{align*}
    u_k(z) &\equiv g(x_k) \cdot \frac{1}{\pi} \text{Im} \left[ \log \frac{z - x_{k+1}}{z - x_k} \right].
\end{align*}
\]

Note that \( g \) is approximately the sum of all \( g_k \) since \( \Delta x_k \equiv x_{k+1} - x_k \) is very small and \( g \) is continuous.

Since \( \log \frac{z - x_{k+1}}{z - x_k} = \log(1 - \frac{\Delta x_k}{z - x_k}) \) is approximately equal to \( \frac{\Delta x_k}{x_k - z} \) by the first order approximation, \( u_k(z) \) is approximately equal to \( g(x_k) \cdot \frac{1}{\pi} \text{Im} \left[ \frac{\Delta x_k}{x_k - z} \right], \) hence the solution \( u(z) \) of the original \( (DP) \) with the boundary data \( g(x) \) is approximately equal to

\[
\sum_{k=-\infty}^{\infty} g(x_k) \cdot \frac{1}{\pi} \text{Im} \left[ \frac{1}{x_k - z} \right] \Delta x_k.
\]

As the partition of the real line becomes finer and finer, i.e., the lengths \( \Delta k \) get shorter indefinitely, we obtain

\[
u(z) = \int_{-\infty}^{\infty} g(\xi) \cdot \frac{1}{\pi} \text{Im} \left[ \frac{1}{\xi - z} \right] d\xi.
\]

Now we use \( \text{Im} \left[ \frac{1}{\xi - z} \right] = \text{Im} \left[ \frac{1}{\xi - x - iy} \right] = \frac{y}{(\xi - x)^2 + y^2}, \) which completes the proof.
REFERENCES


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