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INTEGRAL OPERATORS FOR OPERATOR VALUED MEASURES

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1. Introduction

Let $\mathcal{P}_0$ be a δ-ring (a ring closed with respect to the forming of countable intersections) of subsets of a nonempty set Ω. Let X and Y be Banach spaces and $L(X,Y)$ the Banach space of all bounded linear operators from X to Y.

A set function $m : \mathcal{P}_0 \to L(X,Y)$ is called an *operator valued measure countably additive in the strong operator topology* if for every $x \in X$ the set function $E \to m(E)x$ is a countably additive vector measure.

From now on, m will denote an operator valued measure countably additive in the strong operator topology.

We denote by $\mathfrak{S}(\mathcal{P}_0)$ the smallest σ-ring containing $\mathcal{P}_0$. By a $\mathcal{P}_0$-*simple* function on Ω with values in X we mean a function of the form

$$f = \sum_{i=1}^{r} x_i \chi_{E_i}$$

where $x_i \in X$, $E_i \in \mathcal{P}_0$ and $E_i \cap E_j = \emptyset$ for $i \neq j$, $i,j = 1,2,\ldots,r$. Its integral is defined in the standard way.

For a function $f : \Omega \to X$ and a set $A \subset \Omega$, put $\|f\|_A = \sup_{t\in A}|f(t)|$, where $|f(t)|$ denotes the norm of $f(t)$. By $\mathfrak{B}(\Omega, X)$ we mean the Banach space of all bounded function $f : \Omega \to X$ with the supremum norm.

For each $E \in \mathfrak{S}(\mathcal{P}_0)$, the *semivariation* $\hat{m}(E)$ of the measure m is defined by

$$\hat{m}(E) = \sup |\sum_{i=1}^{n} m(E_i)x_i|$$

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where the supremum is taken over all the finite and measurable partitions of $E \in \mathfrak{S}(\mathcal{P}_0)$ and all the finite families $\{x_i\}_{i=1}^n \subset X$ with $\|x_i\| \le 1$ for $i = 1, 2, \ldots, n$. From the definition, we note that $\hat{m}$ is monotone and countably subadditive.

For a δ-ring $\mathcal{P}_0$, $\mathcal{P}_1$ will denote the class of those sets from $\mathfrak{S}(\mathcal{P}_0)$ which have finite semivariation. Put $\mathcal{P} = \mathcal{P}_0 \cap \mathcal{P}_1$.

Elements of $\mathcal{P}$ will be called *integrable* sets. A $\mathcal{P}$-simple integrable function on Ω with values in X will be called a *simple integrable* function. The set of all simple integrable functions will be denoted by $\mathfrak{T}_s$.

A function $f : \Omega \to X$ is called *measurable* if there is a sequence of simple integrable functions (f_n) such that $\lim_{n\to\infty} f_n(t) = f(t)$ for each $t \in \Omega$. A measurable function $f : \Omega \to X$ is called *integrable* if there exists a sequence of simple integrable functions (f_n) converging almost everywhere m to f for which the integrals $\int_{\cdot} f_n dm$, $n = 1, 2, \ldots$ are uniformly countably additive on $\mathfrak{S}(\mathcal{P})$. In that case, the integral of the function f on set $A \in \mathfrak{S}(\mathcal{P})$ is defined by $\int_A f dm = \lim_{n\to\infty} \int_A f_n dm$.

It is shown in [2, Theorem 16] that if there exists a sequence of the integrable functions (f_n) which converges almost everywhere m to f and there exists the limit $\lim_{n\to\infty} \int_A f_n dm \in Y$ for each $A \in \mathfrak{S}(\mathcal{P})$, then f is integrable and $\int_A f dm = \lim_{n\to\infty} \int_A f_n dm$. This integral, called the Dobrakov integral, was introduced by I. Dobrakov in [2].

For a measurable function g and $E \in \mathfrak{S}(\mathcal{P})$, the L_1-norm $\hat{m}(g, E)$ of g on E is a nonnegative not necessarily finite number defined by

$$\hat{m}(g, E) = \sup\{|\int_E f dm| : f \in \mathfrak{T}_s, |f(t)| \le |g(t)| \text{ for each } t \in E\}.$$

The L_1-norm of the function g is defined by $\hat{m}(g, \Omega) = \sup_{E \in \mathfrak{S}(\mathcal{P})} \hat{m}(g, E)$.

All terms not defined in this paper can be found in [2], [3] and [4].

In this paper, we prove the bounded convergence theorems for the Dobrakov integral, and we study the operators on $\mathfrak{B}(\Omega)$ represented by the Dobrakov integral, where $\mathfrak{B}(\Omega)$ is the space of all bounded measurable scalar valued functions with the usual supremum norm on Ω.

2. The Bounded Convergence Theorem

We start with an analogue of the Bartle's Bounded Convergence Theorem [1, Theorem II.4.1].

THEOREM 2.1. *Let* (f_n) *be a bounded sequence of integrable functions in* $\mathfrak{B}(\Omega, X)$ *which converges almost everywhere* m *to a measurable function* f. *Let* $F = \bigcup_{n=0}^{\infty}\{t \in \Omega : |f_n(t)| > 0\}$, *where* $f_0 = f$. *If for each* $\varepsilon > 0$ *there exists a set* $E \in \mathcal{P}$ *with* $\hat{m}(F - E) < \varepsilon$ *such that* (f_n) *converges uniformly to* f *on* E, *then* f *is integrable and* $\int_A f dm = \lim_{n\to\infty} \int_A f_n dm$ *for each* $A \in \mathfrak{S}(\mathcal{P})$.

Proof. Suppose $\|f_n\|_\Omega \leq K$ for all n. Let $\varepsilon > 0$ be given. Then there exists a set $E \in \mathcal{P}$ with $\hat{m}(F-E) < \varepsilon$ such that (f_n) converges uniformly to f on E.

For each $A \in \mathfrak{S}(\mathcal{P})$, we have

$$\begin{aligned}
&\overline{\lim_{n,p}} \mid \int_A f_n dm - \int_A f_p dm \mid = \overline{\lim_{n,p}} \mid \int_{A\cap F} (f_n - f_p) dm \mid \\
&\leq \overline{\lim_{n,p}} \{ \mid \int_{A\cap(F-E)} (f_n - f_p) dm \mid \\
&\qquad + \mid \int_{A\cap F\cap E} (f_n - f) dm \mid + \mid \int_{A\cap F \cap E} (f - f_p) dm \mid \} \\
&\leq 2K\hat{m}(A \cap (F-E)) + \overline{\lim_n} \|f_n - f\|_E \, \hat{m}(E) + \overline{\lim_p} \|f - f_p\|_E \, \hat{m}(E) \\
&\leq 2K\hat{m}(F-E) \\
&< 2K\varepsilon.
\end{aligned}$$

Thus there exists the limit $\lim_{n\to\infty} \int_A f_n dm \in Y$. By [2, Theorem 6], f is integrable and $\int_A f dm = \lim_{n\to\infty} \int_A f_n dm$ for each $A \in \mathfrak{S}(\mathcal{P})$.

COROLLARY 2.2. *Let* (f_n) *be a bounded sequence of integrable functions in* $\mathfrak{B}(\Omega, X)$ *which converges almost everywhere* m *to a measurable function* f. Let $F = \bigcup_{n=0}^{\infty}\{t \in \Omega : |f_n(t)| > 0\}$, where $f_0 = f$. If for *each* $\varepsilon > 0$ *there exists a set* $E \in \mathfrak{S}(\mathcal{P})$ *with* $\hat{m}(F - E) < \varepsilon$ *such that* (f_n) *is a Cauchy sequence in* L_1*-norm on* E, *then* f *is integrable and* $\int_A f dm = \lim_{n\to\infty} \int_A f_n dm$ *for each* $A \in \mathfrak{S}(\mathcal{P})$.

Proof. Let $\varepsilon > 0$ be given. Then there exists a set $E \in \mathfrak{S}(\mathcal{P})$ with $\hat{m}(F-E) < \varepsilon$ such that (f_n) is a Cauchy sequence in L_1-norm on E. Suppose $\|f_n\|_\Omega \leq K$ for all n. Then the desired result follows immediately from the next relation;

$$\begin{aligned}
&\overline{\lim_{n,p}} \left| \int_A f_n dm - \int_A f_p dm \right| \\
&\leq \overline{\lim_{n,p}} \left| \int_{A\cap(F-E)} (f_n - f_p) dm \right| + \overline{\lim_{n,p}} \left| \int_{A\cap F\cap E} (f_n - f_p) dm \right| \\
&\leq 2K\ \hat{m}(A\cap(F-E)) + \overline{\lim_{n,p}}\ \hat{m}(f_n - f_p, A\cap F\cap E) \\
&\leq 2K\ \hat{m}(F-E) + \overline{\lim_{n,p}}\ \hat{m}(f_n - f_p, E) \\
&< 2K\varepsilon
\end{aligned}$$

for each $A \in \mathfrak{S}(\mathcal{P})$.

COROLLARY 2.3. *Let (f_n) be a bounded sequence of integrable functions in $\mathfrak{B}(\Omega, X)$ which converges almost everywhere m to a measurable function f.*

If $\hat{m}$ is continuous on $\mathfrak{S}(\mathcal{P})$ (i.e., if $E_n \in \mathfrak{S}(\mathcal{P})$, $E_n \searrow \emptyset$, $n = 1, 2, \ldots$, then $\lim_{n\to\infty} \hat{m}(E_n) = 0$), then f is integrable and $\int_A f dm = \lim_{n\to\infty} \int_A f_n dm$ for each $A \in \mathfrak{S}(\mathcal{P})$.

Proof. Let $F = \bigcup_{n=0}^{\infty} \{t \in \Omega : |f_n(t)| > 0\}$, where $f_0 = f$. Then $F \in \mathfrak{S}(\mathcal{P})$. By Egoroff-Lusin's theorem [2, Theorem 1], there is a set $N \in \mathfrak{S}(\mathcal{P})$ and a nondecreasing sequence of sets $F_k \in \mathcal{P}, k = 1, 2, \ldots,$ with $\bigcup_{n=0}^{\infty} F_k = F - N$ such that N is a m-zero set and on each F_k the sequence (f_n) converges uniformly to the function f.

Since $\hat{m}$ is continuous on $\mathfrak{S}(\mathcal{P})$, for each $\varepsilon > 0$ we can select F_k such that $\hat{m}(F - F_k) < \varepsilon$. The desired result now follows immediately from the Theorem 2.1.

3. Operators on $\mathfrak{B}(\Omega)$

By $\mathfrak{L}_1\mathfrak{M}(m)$ or $\mathfrak{L}_1\mathfrak{T}(m)$, we denote the set of all measurable or integrable functions with $\hat{m}(g, \Omega) < \infty$. By $\mathfrak{L}_1\mathfrak{T}_s(m)$, we denote the closure in the L_1-norm of the set of all simple integrable functions $\mathfrak{T}_s$ in

$\mathfrak{L}_1\mathfrak{M}(m)$. By $\mathfrak{L}_1(m)$ we denote the set of all functions $g \in \mathfrak{L}_1\mathfrak{M}(m)$ whose L_1-norms $\hat{m}(g,\cdot)$ are continuous on $\mathfrak{S}(\mathcal{P})$. It is well-known in [3, Theorem 4] that

$$\mathfrak{L}_1(m) \subset \mathfrak{L}_1\mathfrak{T}_s(m) \subset \mathfrak{L}_1\mathfrak{T}(m) \subset \mathfrak{L}_1\mathfrak{M}(m).$$

If $f \in \mathfrak{B}(\Omega)$ and $g \in \mathfrak{L}_1\mathfrak{T}(m)$, then fg is integrable [2, Theorem 4]. For $g \in \mathfrak{L}_1\mathfrak{T}(m)$, we consider the operator $T : \mathfrak{B}(\Omega) \to Y$ defined by $Tf = \int fgdm$. It is easy to show that the operator T is bounded and $\|T\| \leq \hat{m}(g,\Omega)$.

THEOREM 3.1. *Let $g \in \mathfrak{L}_1\mathfrak{T}(m)$ and $F = \{t \in \Omega : |g(t)| > 0\}$. Define $T : \mathfrak{B}(\Omega) \to Y$ by $Tf = \int fgdm$. Then T is compact if and only if for each $\varepsilon > 0$ there exists $E_\varepsilon \in \mathfrak{S}(P)$ with $\hat{m}(g, F - E_\varepsilon) < \varepsilon$ such that the operator T_ε defined by $T_\varepsilon f = \int_{E_\varepsilon} fgdm$ is compact.*

Proof. Suppose that T is compact. Since g is measurable, $F \in \mathfrak{S}(\mathcal{P})$. By taking $E_\varepsilon = F$ for each $\varepsilon > 0$, it follows that $T_\varepsilon = T$ and T_ε is compact.

To prove the converse, let $\varepsilon > 0$. Then there exists $E_\varepsilon \in \mathfrak{S}(\mathcal{P})$ with $\hat{m}(g, F - E_\varepsilon) < \varepsilon$ such that T_ε is compact.

Let U be the unit ball of $\mathfrak{B}(\Omega)$. Then $\{\int_{E_\varepsilon} fgdm : f \in U\}$ is relatively compact and hence totally bounded by ε-balls. For $f \in U$, we have

$$\begin{aligned} | \int_{\Omega - E_\varepsilon} fgdm \; | &= | \int_{F - E_\varepsilon} fgdm \; | \\ &\leq \hat{m}(fg, F - E_\varepsilon) \leq \hat{m}(g, F - E_\varepsilon) < \varepsilon. \end{aligned}$$

It follows easily that

$$\{Tf : f \in U\} = \{\int_{E_\varepsilon} fgdm + \int_{\Omega - E_\varepsilon} fgdm : f \in U\}$$

is totally bounded by 2ε-balls. Hence T is compact.

In paticular, if $g \in \mathfrak{L}_1\mathfrak{T}_s(m)$, then we can prove that the operator T in Theorem 3.1 is compact.

THEOREM 3.2. *Let $g \in \mathfrak{L}_1\mathfrak{T}_s(m)$ and $T : \mathfrak{B}(\Omega) \to Y$ be the linear operator defined by $Tf = \int fgdm$. Then T is compact.*

Proof. Since $g \in \mathfrak{L}_1\mathfrak{T}_s(m)$, there exists a sequence (g_n) of simple integrable functions such that (g_n) converges to g in L_1-norm in $\mathfrak{L}_1\mathfrak{M}(m)$.
Define the operator $T_n : \mathfrak{B}(\Omega) \to Y$ by $T_n f = \int fg_n dm$. Since each g_n has a finite range, T_n is a finite rank continuous linear operator.
For $f \in \mathfrak{B}(\Omega)$, we have

$$\begin{aligned} |(T - T_n)f| &= |\int f(g - g_n)dm| \\ &\leq \hat{m}(f(g - g_n), \Omega) \leq \|f\|_\Omega \, \hat{m}(g - g_n, \Omega). \end{aligned}$$

Since (g_n) converges to g in L_1-norm and each T_n is compact, T is compact.

Now proceeding like in the proof of Theorem 3.2, we get the following corollary.

COROLLARY 3.3. *Let $g, g_n \in \mathfrak{L}_1\mathfrak{T}(m)$ $(n = 1, 2, \ldots)$. Let $T, T_n : \mathfrak{B}(\Omega) \to Y$ be operators defined by $Tf = \int fgdm$ and $T_n f = \int fg_n dm$, respectively. If each T_n is compact and g_n converges to g in L_1-norm, then T is compact.*

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