EXTENSIONS OF $t$-MODULES

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1. Introduction

An elliptic module is an analogue of an elliptic curve over a function field [D]. The dual of an elliptic curve $E$ is represented by $\text{Ext}(E, \mathbb{G}_m)$ and the Cartier dual of an affine group scheme $G$ is represented by $\text{Hom}(G, \mathbb{G}_m)$. In the category of elliptic modules the Carlitz module $C$ plays the role of $\mathbb{G}_m$. Taguchi [T] showed that a notion of duality of a finite $t$-module can be represented by $\text{Hom}(G, C)$ in a suitable category. Our computation shows that the $\text{Ext}$-group as it stands is rather too “big” to represent a dual of an elliptic module.

2. Elliptic modules and $t$-modules

Throughout this paper we fix the following notations: $p$ is a fixed prime, $A$ is the polynomial ring $\mathbb{F}_p[t]$, $K$ is a perfect field containing $A$ and $\theta$ is the image of $t$ in $K$. As usual $\mathbb{G}_{a,K}$ denotes the additive group scheme over $K$. It is well known that the ring of endomorphisms $\text{End}_K(\mathbb{G}_a)$ is a noncommutative polynomial ring $K[\tau]$ with a commutation relation,

$$\tau x = x^p \tau \quad \text{for} \quad x \in K.$$

**DEFINITION 1.** An elliptic module or a Drinfeld module $E$ of rank $r$ is the additive group scheme $\mathbb{G}_a$ together with an $A$-action

$$\psi : A \to \text{End}_K(\mathbb{G}_a) = K[\tau]$$

such that

(i) degree of $\psi_a$ in $\tau$ is the same as $\text{deg}(a)r$,
(ii) the constant term of $\psi_\alpha$ is the same as the image of $\alpha$ in $K$.

If $(E_1, \psi_1)$ and $(E_2, \psi_2)$ are elliptic modules then a morphism from $E_1$ to $E_2$ is defined to be an endomorphism $u$ of $G_a$ such that $u \circ \psi_1 = \psi_2 \circ u$.

Andersen [A] gave a definition of higher dimensional analogue of elliptic modules.

**Definition 2.** An abelian $t$-module over $K$ is the $A$-module valued functor $E$ such that

(i) as a group valued functor $E$ is isomorphic to $G_a^n$ for some $n$,
(ii) $(t - \theta)^N \text{Lie}(E) = 0$ for some positive integer $N$,
(iii) there is a finite dimensional subspace $V$ of the group $\text{Hom}(E, G_a)$ of the morphisms of $K$-algebraic groups such that

$$\text{Hom}(E, G_a) = \sum_{j=0}^{\infty} V \circ t^j.$$  

A morphism between $t$-modules is simply a natural transformation of the functors.

Let $K[t, \tau]$ be the noncommutative ring generated by $t$ and $\tau$ over $K$ with the relations; $t\tau = \tau t$, $xt = tx$, $\tau x = x^p \tau$ for $x \in K$.

**Definition 3.** A $t$-motive $M$ is a left $K[t, \tau]$-module with the following properties,

(i) $M$ is free of finite rank over $K[t]$,
(ii) $(t - \theta)^N (N/\tau M) = 0$ for some positive integer $N$,
(iii) $M$ is finitely generated over $K[\tau]$.

A morphism between $t$-motives is simply a $K[t, \tau]$-linear map.

Andersen [A] showed that the category of $t$-modules is anti-equivalent to the category of $t$-motives. To state his theorems let $E$ be a $t$-module and let $M(E)$ be the set of all morphisms $E \to G_a$ of $K$-algebraic groups equipped with $K[t, \tau]$-module structure,

$$\begin{cases}
(xm)(e) = x(m(e)), \\
\tau(m)(e) = m(e)^p, \\
tm(e) = m(t(e)),
\end{cases}$$

for $e \in E$.  

Theorem 1. The functor sending $E$ to $M(E)$ is an anti-equivalence of categories between $t$-modules and $t$-motives.

We recall another result of Anderson [A] for future use. Let $E$ be a $t$-module. Let

$$H_*(E) = \text{the kernel of exp : Lie}(E) \to E(K).$$

Theorem 2. Let $K$ be the algebraic closure of $\mathbb{F}_p((1/t))$. The following are equivalent:

(i) $\text{rank}_A(H_*(E)) = \text{rank}(E)$.

(ii) $\exp : \text{Lie}(E) \to E(K)$ is surjective.

A $t$-module $E$ satisfying any one of the conditions will be said to be uniformizable.

3. Extensions of $t$-modules

In this section we make explicit computation of Ext groups and study their related properties.

Proposition 1. Let $M_1$ and $M_2$ be $t$-motives. If

$$0 \to M_1 \to M \to M_2 \to 0$$

is an exact sequence of $K[t, t^\tau]$-modules, then $M$ is again a $t$-motive. In particular, if $0 \to E_1 \to E \to E_2 \to 0$ is an extension of $t$-modules, then $E$ is isomorphic to $G^n_\alpha$ for some $n$.

If $E_1$ and $E_2$ are uniformizable, then so is $E$.

Proof. For the first statement we need to check three conditions of Definition 3. First $M$ is free of finite rank, since so are $M_1$ and $M_2$. Second we need to check $(t - \theta)^N(M/\tau M) = 0$ for some $N$. Let $(t - \theta)^r(M_1/\tau M_1) = 0$, $(t - \theta)^s(M_2/\tau M_2) = 0$. Since we have an exact sequence

$$M_1/\tau M_1 \to M/\tau M \to M_2/\tau M_2 \to 0,$$

we see that $(t - \theta)^{r+s}(M/\tau M) = 0$. The third condition is immediate.
For the last statement, consider the following commutative diagram;

\[
\begin{array}{cccc}
0 & \rightarrow & \text{Lie}(E_1) & \rightarrow & \text{Lie}(E) & \rightarrow & \text{Lie}(E_2) & \rightarrow & 0 \\
\downarrow \text{exp} & & \downarrow \text{exp} & & \downarrow \text{exp} & & \\
0 & \rightarrow & E_1(K) & \rightarrow & E(K) & \rightarrow & E_2(K) & \rightarrow & 0.
\end{array}
\]

Our assertion follows since exponential maps are surjective on \(E_1(K)\) and \(E_2(K)\).

**Proposition 2.** Let \(E_1, E_2\) be \(t\)-modules then, we have an isomorphism

\[
\text{Ext}_{t-\text{mod}}(E_1, E_2) \cong \text{Ext}_{K[t, \tau]}(M(E_1), M(E_2)).
\]

**Proof.** Immediate from Theorem 1 and Proposition 1.

**Theorem 3.** Let \(E\) be an elliptic module of rank \(r\) and \(C\) be the Carlitz module. Then \(\text{Ext}^1(E, C)\) is isomorphic to \(K^r\) as an abelian group.

**Proof.** Write \(R = K[t, \tau]\). To compute Ext-group we use a free resolution,

\[
0 \rightarrow R \xrightarrow{d_1 = t - \psi_t^C} R \xrightarrow{\pi} M(C) \rightarrow 0
\]

where \(d_1(r) = r(t - \psi_t^C)\) and \(\pi\) sends \(t\) to \(\psi_t^C\) and \(\tau\) to \(\tau\). Now apply \(\text{Hom}_R(\cdot, M(E))\) to the above resolution,

\[
\text{Hom}_R(M(C), M(E)) \rightarrow \text{Hom}_R(R, M(E)) \xrightarrow{d_1^*} \text{Hom}_R(R, M(E)) \xrightarrow{d_2^*} 0.
\]

Here we identify \(\alpha \in \text{Hom}(R, M(E))\) with \(\alpha(1) \in M(E) = K[\tau]\) and \(d_1^*(\alpha) = \alpha \psi_t^E - \psi_t^C \alpha\). Hence \(\text{Ext}^1(E, C)\) is isomorphic to \(K[\tau]/\mathcal{B}\) where

\[
\mathcal{B} = \{\alpha \psi_t^E - \psi_t^C \alpha \mid \alpha \in K[\tau]\}.
\]

Here we note that \(\mathcal{B}\) is not a \(K\)-submodule of \(K[\tau]\).

To prove our assertion, we claim that for a given \(f\) there is a unique \(\alpha\) such that the degree of \((f - (\alpha \psi_t^E - \psi_t^C \alpha))\) is less than \(r\) which is the rank of \(E\). To prove this we use induction on the degree of \(f\). If the
degree of $f$ is less than $r$, then we can choose $\alpha$ to be 0. Now suppose that $\deg(f) = n + 1$. Since we can write $f = b_{n+1} \tau^{n+1} + f_n$ where $f_n$ is a polynomial in $\tau$ of degree less than or equal to $n$ and since we are assuming our assertion for $f_n$, we only need to prove our assertion for $b_{n+1} \tau^{n+1}$. First assume $(n + 1) < 2r$. Then by Euclidean algorithm [A] in $K[\tau]$ we see that there are unique $\alpha$ and $\gamma'$ in $K[\tau]$ such that

$$b_{n+1} \tau^{n+1} = \alpha \psi^E_t + \gamma'$$

and $\deg(\gamma') < r$, where $\deg(\gamma') < r$ and $\deg(\alpha) < r$ since $(n + 1) < 2r$. Therefore

$$b_{n+1} \tau^{n+1} = \alpha \psi^E_t - \psi^C_t \alpha + \gamma$$

where $\gamma = \gamma' + \psi^C_t \alpha$. Now proceed in the same way to get rid of our extra assumption that $(n + 1) < 2r$.

Now the map

$$\text{Ext}^1(E, C) = K[\tau]/\mathcal{B} \rightarrow K^r$$

sending $f$ to the coefficients of $(f - (\alpha \psi^E_t - \psi^C_t \alpha))$ is obviously an isomorphism.

**Remark.** The group $\text{Ext}^1(E, C)$ is not a $K$-vector space.

**Proposition 3.** The $t$-action on $\text{Ext}^1(E, C) = K[\tau]/\mathcal{B}$ is given by right multiplication by $\psi^E_t$ or which is the same as left multiplication by $\psi^C_t$.

**Proof.** Consider the commutative diagram

$$
\begin{array}{cccccc}
0 & \longrightarrow & R & \overset{d_1=t-\psi^C_t}{\longrightarrow} & R & \overset{\pi}{\longrightarrow} & M(C) & \longrightarrow & 0 \\
\downarrow t & & \downarrow t & & \downarrow t=\psi^C_t & & \\
0 & \longrightarrow & R & \overset{d_1=t-\psi^C_t}{\longrightarrow} & R & \overset{\pi}{\longrightarrow} & M(C) & \longrightarrow & 0.
\end{array}
$$

Apply the functor $\text{Hom}_R(-, M(E))$ and carefully chase the diagram.

Let $E$ be an elliptic module and $C$ be the Carlitz module. Let

$$0 \rightarrow C \rightarrow E \rightarrow E \rightarrow 0$$

be an extension of algebraic groups. Then by Proposition 1 we see that $E$ is isomorphic to $G^2_a$ as an algebraic group. So the extension $E$ depends only on the $t$-module structure on $G^2_a$. Given $f$ in $\text{Ext}^1(E, C) = K[\tau]/\mathcal{B}$ we will describe the corresponding extension, namely the $t$-action on $G^2_a$. 
Theorem 4. Let \( f \in \text{Ext}^1(E, C) \) and
\[
0 \to M(E) \to M \to M(C) \to 0
\]
be the corresponding extension of \( t \)-motives. Then \( M \cong K[\tau] \oplus K[\tau] \) with \( t \)-action given by
\[
\begin{bmatrix}
\psi^E_t & f \\
0 & \psi^C_t
\end{bmatrix}.
\]

Proof. The extension \( \mathcal{E} \) corresponding to \( f \in K[\tau]/B \) is given by
\[
\begin{array}{cccccc}
0 & \to & R & \overset{d_1 = t - \psi^E_t}{\to} & R & \overset{\pi}{\to} & M(C) & \to & 0 \\
\downarrow f & & \downarrow & & \downarrow & & \downarrow \text{id}
\end{array}
\]
where \( \mathcal{R} = \{(-rf, d_1(r)) | r \in R \} \). We define a map
\[
\phi : M(E) \oplus R/\mathcal{R} \to K[\tau] \oplus K[\tau]
\]
by sending \((m, t)\) to \( m + (\psi^C_t + f) \) and \((m, \tau)\) to \(((m + \tau), \tau)\). For brevity we write \( M = K[\tau] \oplus K[\tau] \). Now one checks that this is an isomorphism. The inverse of \( \phi \) sends \((a, b)\) to \((a - b, b)\).

To get the \( t \)-action on \( M \) we transport the \( t \)-action on \( M(E) \oplus R/\mathcal{R} \) via \( \phi \) and send it back to \( M \): To find \( t \)-action on \((a, b)\) in \( M \) we lift it to \((a - b, b)\) in \( M(E) \oplus R/\mathcal{R} \). Hence in \( M(E) \oplus R/\mathcal{R} \) we have,
\[
t(a - b, b) = ((a - b)\psi^E_t, tb).
\]
Now send this to \( M \) via \( \phi \) to get the \( t \)-action on \( M \)
\[
t(a, b) = ((a - b)\psi^E_t + b(\psi^C_t + f), b\psi^C_t).
\]
Now our assertion follows since \((\psi^E_t - \psi^C_t) \in \mathcal{B}\).

Let \( \phi \) be an isogeny of an elliptic module. Then the kernel of \( \phi \) is a finite \( t \)-module in the sense of Taguchi [T]. By the standard results of homological algebra [S], we have an exact sequence,
\[
0 \to \text{Hom}(G, C) \overset{\delta}{\to} \text{Ext}^1(E, C) \overset{\phi^*}{\to} \text{Ext}^1(E, C).
\]
We have zero on the left because there is no morphisms between the elliptic modules of different rank. (We are assuming the rank of \( E \) is bigger than 1.) We want to compute the map \( \delta \).
Theorem 4. $\delta(f)$ is given by $F \in K[\tau]/B$ such that

$$F \phi = f \psi^E_t - \psi^C_t f.$$

Proof. Consider the following diagram,

$$
\begin{array}{ccccccccc}
0 & \rightarrow & R & \xrightarrow{d_1=t-\psi^E_t} & R & \xrightarrow{\pi} & M(C) & \rightarrow & 0 \\
\downarrow F & & \downarrow (f,1) & & \downarrow id \\
0 & \rightarrow & M(E) & \xrightarrow{(\phi,0)} & f^*M(E) & \rightarrow & M(C) & \rightarrow & 0 \\
\downarrow id & & \downarrow & & \downarrow f \\
0 & \rightarrow & M(E) & \rightarrow & M(E) & \rightarrow & M(G) & \rightarrow & 0.
\end{array}
$$

Here $f^*M(E)$ is the fiber product $M(E) \times_{M(G)} M(C)$. First lift $f$ to $M(E)$ which we still call $f$. Now chase the upper left square to get $F$ which satisfies the property $F \phi = f \psi^E_t - \psi^C_t f$.

References


[T] Taguchi, A duality for finite $t$-modules, (a circulating note).

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