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EXTENSIONS OF t-MODULES

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1. Introduction

An elliptic module is an analogue of an elliptic curve over a function field [D]. The dual of an elliptic curve E is represented by $\mathrm{Ext}(E, \mathbb{G}_m)$ and the Cartier dual of an affine group scheme G is represented by $\mathrm{Hom}(G, \mathbb{G}_m)$. In the category of elliptic modules the Carlitz module C plays the role of $\mathbb{G}_m$. Taguchi [T] showed that a notion of duality of a finite t-module can be represented by $\mathrm{Hom}(G, C)$ in a suitable category. Our computation shows that the Ext-group as it stands is rather too "big" to represent a dual of an elliptic module.

2. Elliptic modules and t-modules

Throughout this paper we fix the following notations: p is a fixed prime, A is the polynomial ring $\mathbb{F}_p[t]$, K is a perfect field containing A and θ is the image of t in K. As usual $\mathbb{G}_{a,K}$ denotes the additive group scheme over K. It is well known that the ring of endomorphisms $\mathrm{End}_K(\mathbb{G}_a)$ is a noncommutative polynomial ring $K[\tau]$ with a commutation relation,

$$\tau x = x^p \tau \quad \text{for} \quad x \in K.$$

DEFINITION 1. An *elliptic module* or a *Drinfeld module* E of rank r is the additive group scheme $\mathbb{G}_a$ together with an A-action

$$\psi : A \to \mathrm{End}_K(\mathbb{G}_a) = K[\tau]$$

such that

(i) degree of ψ_a in τ is the same as $\deg(a) r$,

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(ii) the constant term of ψ_a is the same as the image of a in K.

If (E_1, ψ_1) and (E_2, ψ_2) are elliptic modules then a morphism from E_1 to E_2 is defined to be an endomorphism u of $\mathbb{G}_a$ such that $u \circ \psi_1 = \psi_2 \circ u$.

Anderson [A] gave a definition of higher dimensional analogue of elliptic modules.

DEFINITION 2. An *abelian t-module* over K is the A-module valued functor E such that

(i) as a group valued functor E is isomorphic to $\mathbb{G}_a^n$ for some n,

(ii) $(t-\theta)^N \mathrm{Lie}(E) = 0$ for some positive integer N,

(iii) there is a finite dimensional subspace V of the group $\mathrm{Hom}(E, \mathbb{G}_a)$ of the morphisms of K-algebraic groups such that

$$\mathrm{Hom}(E, \mathbb{G}_a) = \sum_{j=0}^{\infty} V \circ t^j.$$

A morphism between t-modules is simply a natural transformation of the functors.

Let $K[t, \tau]$ be the noncommutative ring generated by t and τ over K with the relations; $t\tau = \tau t$, $xt = tx$, $\tau x = x^p \tau$ for $x \in K$.

DEFINITION 3. A *t-motive* M is a left $K[t, \tau]$-module with the following properties,

(i) M is free of finite rank over $K[t]$,

(ii) $(t-\theta)^N (N/\tau M) = 0$ for some positive integer N,

(iii) M is finitely generated over $K[\tau]$.

A morphism between t-motives is simply a $K[t, \tau]$-linear map.

Anderson [A] showed that the category of t-modules is anti-equivalent to the category of t-motives. To state his theorems let E be a t-module and let $M(E)$ be the set of all morphisms $E \to \mathbb{G}_a$ of K-algebraic groups equipped with $K[t, \tau]$-module structure,

$$\begin{cases} (xm)(e) = x(m(e)), \\ \tau(m)(e) = m(e)^p, \\ tm(e) = m(t(e)), \end{cases}$$

for $e \in E$.

THEOREM 1. *The functor sending E to $M(E)$ is an anti-equivalence of categories between t-modules and t-motives.*

We recall another result of Anderson [A] for future use. Let E be a t-module. Let

$$H_*(E) = \text{ the kernel of exp } : \text{Lie}(E) \to E(K).$$

THEOREM 2. *Let K be the algebraic closure of $\mathbb{F}_p((1/t))$. The following are equivalent:*
(i) $\text{rank}_A(H_*(E)) = \text{rank}(E)$.
(ii) $\exp : \text{Lie}(E) \to E(K)$ *is surjective.*
A t-module E satisfying any one of the conditions will be said to be uniformizable.

3. Extensions of t-modules

In this section we make explicit computation of Ext groups and study their related properties.

PROPOSITION 1. *Let M_1 and M_2 be t-motives. If*

$$0 \to M_1 \to M \to M_2 \to 0$$

is an exact sequence of $K[t,\tau]$-modules, then M is again a t-motive. In particular, if $0 \to E_1 \to E \to E_2 \to 0$ is an extension of t-modules, then E is isomorphic to $\mathbb{G}_a^n$ for some n.

If E_1 and E_2 are uniformizable, then so is E.

Proof. For the first statement we need to check three conditions of Definition 3. First M is free of finite rank, since so are M_1 and M_2. Second we need to check $(t-\theta)^N(M/\tau M) = 0$ for some N. Let $(t-\theta)^r(M_1/\tau M_1) = 0$, $(t-\theta)^s(M_2/\tau M_2) = 0$. Since we have an exact sequence

$$M_1/\tau M_1 \to M/\tau M \to M_2/\tau M_2 \to 0,$$

we see that $(t-\theta)^{r+s}(M/\tau M) = 0$. The third condition is immediate.

For the last statement, consider the following commutative diagram;

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Lie}(E_1) & \longrightarrow & \mathrm{Lie}(E) & \longrightarrow & \mathrm{Lie}(E_2) & \longrightarrow & 0 \\ & & \downarrow \exp & & \downarrow \exp & & \downarrow \exp & & \\ 0 & \longrightarrow & E_1(K) & \longrightarrow & E(K) & \longrightarrow & E_2(K) & \longrightarrow & 0. \end{array}$$

Our assertion follows since exponential maps are surjective on $E_1(K)$ and $E_2(K)$.

PROPOSITION 2. *Let E_1, E_2 be t-modules then, we have an isomorphism*

$$Ext_{t-\mathrm{mod}}(E_1, E_2) \cong Ext_{K[t,\tau]}(M(E_1), M(E_2)).$$

Proof. Immediate from Theorem 1 and Proposition 1.

THEOREM 3. *Let E be an elliptic module of rank r and C be the Carlitz module. Then $Ext^1(E, C)$ is isomorphic to K^r as an abelian group.*

Proof. Write $R = K[t, \tau]$. To compute Ext-group we use a free resolution,

$$0 \to R \xrightarrow{d_1 = t - \psi_t^C} R \xrightarrow{\pi} M(C) \to 0$$

where $d_1(r) = r(t - \psi_t^C)$ and π sends t to ψ_t^C and τ to τ. Now apply $\mathrm{Hom}_R(-, M(E))$ to the above resolution,

$$\mathrm{Hom}_R(M(C), M(E)) \to \mathrm{Hom}_R(R, M(E)) \xrightarrow{d_1^*} \mathrm{Hom}_R(R, M(E)) \xrightarrow{d_2^*} 0.$$

Here we identify $\alpha \in \mathrm{Hom}(R, M(E))$ with $\alpha(1) \in M(E) = K[\tau]$ and $d_1^*(\alpha) = \alpha\psi_t^E - \psi_t^C\alpha$. Hence $\mathrm{Ext}^1(E, C)$ is isomorphic to $K[\tau]/\mathcal{B}$ where

$$\mathcal{B} = \{\alpha\psi_t^E - \psi_t^C\alpha \mid \alpha \in K[\tau]\}.$$

Here we note that $\mathcal{B}$ is not a K-submodule of $K[\tau]$.

To prove our assertion, we claim that for a given f there is a unique α such that the degree of $(f - (\alpha\psi_t^E - \psi_t^C\alpha))$ is less than r which is the rank of E. To prove this we use induction on the degree of f. If the degree of f is less than r, then we can choose α to be 0. Now suppose that $\deg(f) = n+1$. Since we can write $f = b_{n+1}\tau^{n+1} + f_n$ where f_n is a polynomial in τ of degree less than or equal to n and since we are assuming our assertion for f_n, we only need to prove our assertion for $b_{n+1}\tau^{n+1}$. First assume $(n+1) < 2r$. Then by Euclidean algorithm [A] in $K[\tau]$ we see that there are unique α and γ' in $K[\tau]$ such that

$$b_{n+1}\tau^{n+1} = \alpha\psi_t^E + \gamma' \quad \text{and} \quad \deg(\gamma') < r,$$

where $\deg(\gamma') < r$ and $\deg(\alpha) < r$ since $(n+1) < 2r$. Therefore

$$b_{n+1}\tau^{n+1} = \alpha\psi_t^E - \psi_t^C\alpha + \gamma$$

where $\gamma = \gamma' + \psi_t^C\alpha$. Now proceed in the same way to get rid of our extra assumption that $(n+1) < 2r$.

Now the map

$$\mathrm{Ext}^1(E, C) = K[\tau]/\mathcal{B} \to K^r$$

sending f to the coefficients of $(f - (\alpha\psi_t^E - \psi_t^C\alpha))$ is obviously an isomorphism.

REMARK. The group $\mathrm{Ext}^1(E, C)$ is not a K-vector space.

PROPOSITION 3. *The t-action on $\mathrm{Ext}^1(E, C) = K[\tau]/\mathcal{B}$ is given by right multiplication by ψ_t^E or which is the same as left multiplication by ψ_t^C.*

Proof. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R & \overset{d_1 = t - \psi_t^C}{\longrightarrow} & R & \overset{\pi}{\longrightarrow} & M(C) & \longrightarrow & 0 \\ & & \downarrow t & & \downarrow t & & \downarrow t = \psi_t^C & & \\ 0 & \longrightarrow & R & \overset{d_1 = t - \psi_t^C}{\longrightarrow} & R & \overset{\pi}{\longrightarrow} & M(C) & \longrightarrow & 0. \end{array}$$

Apply the functor $\mathrm{Hom}_R(-, M(E))$ and carefully chase the diagram.

Let E be an elliptic module and C be the Carlitz module. Let

$$0 \to C \to \mathcal{E} \to E \to 0$$

be an extension of algebraic groups. Then by Proposition 1 we see that $\mathcal{E}$ is isomorphic to $\mathbb{G}_a^2$ as an algebraic group. So the extension $\mathcal{E}$ depends only on the t-module structure on $\mathbb{G}_a^2$. Given f in $\mathrm{Ext}^1(E, C) = K[\tau]/\mathcal{B}$ we will describe the corresponding extension, namely the t-action on $\mathbb{G}_a^2$.

THEOREM 4. *Let $f \in Ext^1(E,C)$ and*

$$0 \to M(E) \to M \to M(C) \to 0$$

be the corresponding extension of t-motives. Then $M \cong K[\tau] \oplus K[\tau]$ with t-action given by

$$\begin{bmatrix} \psi_t^E, & f \\ 0, & \psi_t^C \end{bmatrix}.$$

Proof. The extension $\mathcal{E}$ corresponding to $f \in K[\tau]/\mathcal{B}$ is given by

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R & \overset{d_1 = t - \psi_t^E}{\longrightarrow} & R & \overset{\pi}{\longrightarrow} & M(C) & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow & & \downarrow id & & \\ 0 & \longrightarrow & M(E) & \longrightarrow & M(E) \oplus R/\mathcal{R} & \longrightarrow & M(C) & \longrightarrow & 0 \end{array}$$

where $\mathcal{R} = \{(-rf, d_1(r)) \mid r \in R\}$. We define a map

$$\phi : M(E) \oplus R/\mathcal{R} \to K[\tau] \oplus K[\tau]$$

by sending (m,t) to $m + (\psi_t^C + f)$ and (m, τ) to $((m+\tau), \tau)$. For brevity we write $M = K[\tau] \oplus K[\tau]$. Now one checks that this is an isomorphism. The inverse of ϕ sends (a,b) to $(a-b, b)$.

To get the t-action on M we transport the t-action on $M(E) \oplus R/\mathcal{R}$ via ϕ and send it back to M: To find t-action on (a,b) in M we lift it to $(a-b, b)$ in $M(E) \oplus R/\mathcal{R}$. Hence in $M(E) \oplus R/\mathcal{R}$ we have,

$$t(a-b, b) = ((a-b)\psi_t^E,\ tb).$$

Now send this to M via ϕ to get the t-action on M

$$t(a,b) = ((a-b)\psi_t^E + b(\psi_t^C + f), b\psi_t^C).$$

Now our assertion follows since $(\psi_t^E - \psi_t^C) \in \mathcal{B}$.

Let ϕ be an isogeny of an elliptic module. Then the kernel of ϕ is a finite t-module in the sense of Taguchi [T]. By the standard results of homological algebra [S], we have an exact sequence,

$$0 \to \mathrm{Hom}(G, C) \xrightarrow{\delta} \mathrm{Ext}^1(E, C) \xrightarrow{\phi^*} \mathrm{Ext}^1(E, C).$$

We have zero on the left because there is no morphisms between the elliptic modules of different rank. (We are assuming the rank of E is bigger than 1.) We want to compute the map δ.

THEOREM 4. *$\delta(f)$ is given by $F \in K[\tau]/\mathcal{B}$ such that*

$$F\phi = f\psi_t^E - \psi_t^C f.$$

Proof. Consider the following diagram,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R & \stackrel{d_1=t-\psi_t^E}{\longrightarrow} & R & \stackrel{\pi}{\longrightarrow} & M(C) & \longrightarrow & 0 \\
 & & \downarrow F & & \downarrow (f,1) & & \downarrow id & & \\
0 & \longrightarrow & M(E) & \stackrel{(\phi,0)}{\longrightarrow} & f^*M(E) & \longrightarrow & M(C) & \longrightarrow & 0 \\
 & & \downarrow id & & \downarrow & & \downarrow f & & \\
0 & \longrightarrow & M(E) & \longrightarrow & M(E) & \longrightarrow & M(G) & \longrightarrow & 0.
\end{array}$$

Here $f^*M(E)$ is the fiber product $M(E)x_{M(G)}M(C)$. First lift f to $M(E)$ which we still call f. Now chase the upper left square to get F which satisfies the property $F\phi = f\psi_t^E - \psi_t^C f$.

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