A NOTE OF PI–RINGS WITH RESTRICTED DESCENDING

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In this paper, some properties for a PI-ring satisfying the descending chain condition on essential left ideals are studied: Let R be a ring with a polynomial identity satisfying the descending chain condition on essential ideals. Then all minimal prime ideals in R are maximal ideals. Moreover, if R has only finitely many minimal prime ideals, then R is left and right Artinian. Consequently, if every primeideal of R is finitely generated as a left ideal, then R is left and right Artinian. A finitely generated PI-algebra over a commutative Noetherian ring satisfying the descending chain condition on essential left ideals is a finite module over its center.

All rings R considered here are associative with identity and all modules are unitary. The Jacobson radical of a ring R will be denoted by $J(R)$ and the socle of a left R-module M by $Soc(M)$. Also, for any subset X of a ring R, $l(X)$ represents the left annihilator of X.

Let Z be the ring of all integers and $Z < x_1, x_2, ... >$ be a free associative algebra over Z in countably many indeterminates. A ring R satisfies a polynomial identity if there is a nonzero polynomial $f(x_1, x_2, ..., x_n)$ (one of the monomials of f of the highest total degree has coefficient 1 or -1) in $Z < x_1, x_2, ... >$ which vanishes when evaluated on any $r_1, r_2, ..., r_n \in R$. A ring satisfying a polynomial identity is called a PI-ring.

We refer to [4] for a ring with a polynomial identity.

A left ideal of R is said to be essential if it has nonzero intersection with each nonzero left ideal of R.

Some properties for rings satisfying the descending chain condition and the ascending chain condition on essential left ideals are studied in [1] and [2].

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Lemma 1 [1]. *The following conditions are equivalent:*

(*i*) *A ring R satisfies the descending chain condition on essential left ideals.*

(*ii*) $R/Soc(R)$ *is left Artinian.*

Proof. Since $Soc(R)$ is the intersection of essential left ideals of R, if $R/Soc(R)$ is left Artinian, R satisfies the descending chain condition on essential left ideals. Conversely, suppose that R satisfies the descending chain condition on essential left ideals. Then $Soc(R)$ is a finite intersection of essential left ideals, and is itself an essential left ideal of R. Therefore, any left ideal of R containing $Soc(R)$ is essential and so $R/Soc(R)$ is left Artinian.

The homomorphic image of a ring satisfying the descending chain condition on essential left ideals also satisfies the descending chain condition on essential left ideals.

If a ring R satisfies the descending chain condition on essential left ideals, then $Soc(R)$ is an essential left ideal of R. But, there is a non left artinian ring whose left socle is left essential.

Here, we note that any maximal ideal in a ring R is a prime ideal. Moreover, if $P_1 \supseteq P_2 \supseteq \cdots$ is a chain of prime ideals of R, then $\cap P_i$ is a prime ideal of R. Thus, by Zorn's Lemma, any prime ideal in a ring R contains a minimal prime ideal.

Lemma 2 (Wedderburn-Artin). *The following conditions on a left Artinian ring R are equivalent.*

(*i*) *R is simple.*

(*ii*) *R is primitive.*

(*iii*) *For some positive integer n, R is isomorphic to the ring of all n x n matrices over a division ring.*

Proof. See [5].

We can show that a prime ideal in a left Artinian ring is a maximal ideal using Lemma 2.

Lemma 3 (Kaplansky). *Let R be a primitive ring satisfying a polynomial identity of degree d. Then R is a central simple algebra of dimension n^2 over its center $Z(R)$, with $2n \leq d$.*

Proof. See [4].

Theorem 4. *Let R be a PI-ring satisfying the descending chain condition on essential left ideals. Then any minimal prime ideal P of R is a maximal ideal.*

Proof. If $Soc(R) = 0$, then R is left Artinian. Thus, P is a maximal ideal. Suppose that $Soc(R)$ is nonzero. Let L be a minimal left ideal of R. Then $PL \neq 0$ or $PL = 0$.

(i) If $PL \neq 0$ for any minimal left ideal L in R, then $0 \neq PL \subseteq L \cap P \subseteq L$. Thus $L \cap P = L$ and so $L \subseteq P$. Therefore, $Soc(R) \subseteq P$ and so, R/P is left Artinian. Moreover, since R/P is a prime ring, R/P is a primitive ring. By Lemma 2, R/P is a simple ring. Thus P is a maximal ideal.

(ii) Suppose that $PL = 0$ for some minimal left ideal $L \subsetneq P$. Then $P = l(L)$. Consider a ring $R/P = R/l(L)$. Then L is a faithful simple left $R/P(= R/l(L))$-module. Therefore, R/P is a primitive ring. By Lemma 3 , R/P is a simple ring. Thus, P is a maximal ideal.

There is a PI-ring satisfying the descending chain condition on essential left ideals which is not left Artinian.

Example 5. Let F be a field and $V = \oplus_{\aleph_0} F_i$, $F_i = F$, vector space over F with dimension $\aleph_0$. Let $R = \begin{pmatrix} F & V \\ 0 & F \end{pmatrix}$. Then R is neither left nor right Artinian but R satisfies the descending chain condition on essential left ideals.

In Example 5, R is a PI-ring satisfying the descending chain condition on essential left ideals, but R is not von Neumann regular. However R is strongly π-regular.

A ring R is called π-regular[strongly π-regular] if for every $a \in R$ there exists a positive integer n, depending on a, and $b \in R$ such that $a^n b a^n = a^n [a^{n+1} b = a^n]$.

Lemma 6 [3]. *For a PI-ring R, the following are equivalent.*

(a) *R is π-regular.*

(b) *Every prime ideal of R is primitive.*

(c) *Every prime ideal of R is maximal.*

(*d*) *R is strongly* π*-regular.*

(*e*) *Every prime factor ring of R is von Neumann regular.*

Proof. See [3].

Proposition 7. *Let R be a PI-ring satisfying the descending chain condition on essential left ideals. Then R is strongly* π*-regular.*

Proof. It follows from Theorem 4 and Lemma 6.

We note that a subring of a prime PI-ring has a finite number of minimal prime ideals.

Theorem 8 . *Let R be a PI-ring satisfying the descending chain condition on essential left ideals. If R has only finitely many minimal prime ideals, then R is left and right Artinian.*

Proof. Let $P_1, P_2, \ldots P_m$ be all minimal prime ideals of R, and N be the lower nil radical of R. Then $N = P_1 \cap P_2 \cap \cdots \cap P_m$ and each minimal prime ideal P_i is a maximal ideal by Theorem 4. Since each R/P_i satisfies the descending chain condition on essential left ideals, each R/P_i is left Artinian. Thus $R/N \cong R/P_1 \oplus R/P_2 \oplus \ldots \oplus R/P_m$ is left Artinian. On the other hand, since R satisfies the descending chain condition on essential left ideals, the Jacobson radical $J(R)$ is nilpotent. In fact, $(J(R)+Soc(R))/Soc(R) \subseteq J(R/Soc(R))$. Since R/Soc(R) is left Artinian, $J(R/Soc(R))$ is nilpotent. Thus, $(J(R)+Soc(R))/Soc(R)$ is nilpotent and so $(J(R))^t \subseteq Soc(R)$ for some integer $t \geq 1$. But, $J(R)$ annihilates $Soc(R)$. Thus, $(J(R))^{t+1} = 0$. Since $J(R)$ is nilpotent, N = J(R), i.e., R/N is semiprimitive left Artinian. $P_m P_{m-1} \cdots P_1 \subseteq P_1 \cap P_2 \cap \cdots \cap P_m = J(R)$ and so $(P_m P_{m-1} \cdots P_1)^k = 0$ for some integer $k \geq 1$. Therefore, there are maximal ideals $P_1, P_2, \ldots, P_s$ such that $P_s P_{s-1} \ldots P_1 = 0$. Now, let $M_i = P_i P_{i-1} \cdots P_1$, $1 \leq i \leq s$ and $M_0 = R$. Then each M_{i-1}/M_i can be viewed naturally as a finitely generated left and right R/P_i-module. But, the dimension of R/P_i over its center $Z(R/P_i)$ is finite by Lemma 3. Thus, M_i/M_{i-1} is a finite dimensional vector space over the field $Z(R/P_i)$. Therefore, $R = M_0 \supseteq M_1 \supseteq M_2 \subseteq \cdots \subseteq M_s = 0$ can be refined to a composition series. Hence, R is left and right Artinian.

Lemma 9. *Every prime ideal of a ring R is finitely generated as a left ideal. Then for every ideal* $I \neq R$*, there exist finitely many prime ideals* $P_1, P_2, \ldots, P_n$ *such that* $I \subseteqq P_i$ *for* $1 \leq i \leq n$*, and* $P_1 P_2 \cdots P_n \subseteqq I$

Proof. Suppose that it is false. Let $T = \{A | A$ is an ideal of R and does not contain a finite product of prime ideals P_i such that $P_i \supseteq A\}$. Then $T \neq \emptyset$. Consider $A_1 \subset A_2 \subset \cdots$, where $A_i \in T$. Then $B = \cup A_i \in T$. If it is not, there exist prime ideals $C_1, C_2 \ldots, C_k$ of R such that $B \subseteq C_i$, $1 \leq i \leq k$ and $C_1 C_2 \cdots C_k \subseteq B = \cup A_i$. Since $C_1, C_2, \ldots, C_k$ are finitely generated, $C_1 C_2 \cdots C_k$ is a finitely generated left ideal of R. But $C_1 C_2 \cdots C_k \subseteq A_i$ for some i. Thus $A_i \notin T$. This is a contradiction. By Zorn's Lemma, there is a maximal element D of T. D is clearly not prime. Therefore, there are ideals E, $F \supsetneqq D$ but $EF \subseteqq D$. Since E, $F \notin T$, there exist prime ideals $E_1, \ldots, E_l, F_1, \ldots, F_t$ of R such that $E \subseteq E_i$, $1 \leq i \leq l$, $F \subseteq F_i$, $1 \leq i \leq t$, $E_1 \cdots E_l \subseteq E(\subseteq \cap_{i=1}^{i} E_i)$, and $F_1 \cdots F_t \subseteq F(\subseteq \cap_{i=1}^{t} F_i)$. Thus $E_i \cdots E_l F_1 \cdots F_t \subseteq EF \subseteq D(\subseteq (\cap_{i=1}^{l} E_i)(\cap_{i=1}^{t} F_i))$ and $D \notin T$. This is a contradiction.

Proposition 10. *Let R be a PI-ring satisfying the descending chain condition on essential left ideals. If every prime ideal of R is finitely generated as a left ideal, then R is left and right Artinian.*

Proof. There exist finitely many maximal ideals $P_1, P_2, \ldots, P_s$ such that $P_s P_{s-1} \cdots P_1 = 0$ by Theorem 4 and Lemma 9. Therefore, by the same method of Theorem 8, R is left and right Artinian.

We can apply these results to a finitely generated PI-algebra over a commutative Noetherian ring.

Lemma 11. *Let R be a finitely generated PI-algebra over a commutative Noetherian ring. Then R has only finitely many minimal prime ideals.*

Proof. See [6].

Corollary 12. *Let R be a finitely generated PI-algebra over a commutative Noetherian ring and satisfy the descending chain condition on essential left ideals. Then R is left and right Artinian.*

Proof. It follows from Theorem 8 and Lemma 11.

Lemma 13. *Let K be a field and R a finitely generated PI-algebra over K. Then R is finite dimensional over K.*

Proof. See [7].

Lemma 14. *Let R be a finitely generated PI-algebra over a commutative ring K such that R is not a finite module over K. Then there exists a prime ideal I of R such that I is maximal with respect to R/I not finite over K.*

Proof. See [7].

Note that if R is a finitely generated PI-algebra over a commutative Noetherian ring and satisfies the descending chain condition on essential left ideals, then $J(R)$ is nilpotent and $R/J(R)$ is Artinian, i.e., R is semiprimary.

Corollary 15. *Let R be a finitely generated PI-algebra over a commutative Noetherian ring and satisfy the descending chain condition on essential left ideals. Then R is a finte module over its center.*

Proof. Suppose not. There is a prime ideal P such that R/P is not finite over its center $Z(R/P)$ by Lemma 14. R/P is finitely generated over $Z(R)/(P \cap Z(R))$, where $Z(R)$ is the center of R. Now, $Z(R)/(P \cap Z(R))$ is a field by Theorem 4. This is a contradiction to Lemma 13.

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