NIJENHUIS TENSOR FUNCTIONAL ON A SUBSPACE OF METRICS

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1. Introduction

The study of the integral of the scalar curvature,

$$A(g) = \int_M R dV_g$$

as a functional on the set $\mathcal{M}$ of all Riemannian metrics of the same total volume on a compact orientable manifold M is now classical, dating back to Hilbert [6] (see also Nagano [8]). Riemannian metric g is a critical point of $A(g)$ if and only if g is an Einstein metric.

Since there are so many Riemannian metrics on a manifold, one can regard, philosophically, the finding of critical metrics as an approach to searching for the best metric for the given manifold. Other functions of the curvature have been taken as intergrands as well, most notably

$$B(g) = \int_M R^2 dV_g, \quad C(g) = \int_M |Ric|^2 dV_g$$

and

$$D(g) = \int_M |Riem|^2 dV_g,$$

where Ric denotes the Ricci tensor and $Riem$ denotes the Riemannian curvature tensor; the critical point conditions it is easy to see that Einstein metrices are critical for $B(g)$ and $C(g)$ but not necessarily converserly (e.g. see Theorem 5.1 of [9] for a non-Einstein critical metric of $C(g)$). Similarly metrices of constant

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curvature and Kähler metrics of constant holomorphic curvature are critical for $D(g)$ [7];

Our study in this paper is primarily motivated by two kinds of questions. 1. Given an integral functional restricted to a smaller set of metrics, what is the critical point condition; one would expect a weaker one. The smaller sets of metrics we have in mind are the sets of metrics associated to a symplectic. 2. Given these sets of metrics are there other natural integrands depending on the structure as well as the curvature ?

To set the stage for our study let us first introduce the classical notations. Let M be a compact orientable manifold and $\mathcal{M}$ the set of all Riemannian metrics normalized by the condition of having the same total volume, usually taken to be 1, but we don't insist on the particular value in a given problem.

Now the approach to these critical point problems is to differentiate the functional in question along a path of metrics. So let $g(t)$ be a path of metrics in $\mathcal{M}$ and

$$D_{ij} = \frac{\partial g_{ij}}{\partial t}|_{t=0}$$

its tangent vector at $g = g(0)$. We define two other tensor fields by

$$D_{ji}{}^{h} = \frac{1}{2}(\nabla_j D_i{}^h + \nabla_i D_j{}^h - \nabla^h D_{ji}), \quad D_{kji}{}^{h} = \nabla_k D_{ji} \quad h - \nabla_j D_{ki}{}^{h}$$

where ∇ denotes the Riemannian connection of $g(0)$ and we note that

$$D_{kji}{}^{h} = \frac{\partial \Gamma_{ji}{}^{h}}{\partial t}|_{t=0}, \quad D_{kji}{}^{h} = \frac{\partial R_{kji}{}^{h}}{\partial t}|_{t=0}$$

where $\Gamma_{ji}{}^{h}$ denote the Christoffel symbols and curvature tensor of $g(t)$.

2. Preliminaries

As mentioned in the previous section our primary interest will be in metrics associated to a symplectic and we begin by reviewing these notions.

By a *symplectic manifold* we mean a C^∞ manifold M^{2n} together with a closed 2-form Ω such that $\Omega^n \neq 0$. Kähler manifolds and cotangent bundles are well known examples.

The notion of an associated metric can be introduced in two ways; first the easy way, namely that a Riemannian metric g is an *associated metric* for a symplectic form Ω, if there exists an almost complex structure J on M such that

$$\Omega(X,Y) = g(X, JY).$$

There is also a constructive approach to these metrics. For a symplectic manifold M let k be any Riemannian metric and $X_1, \cdots, X_{2n}$ be a local k-orthonormal basis. Consider the $2n \times 2n$ matrix $\Omega(X_i, X_j)$; it is non-singular and hence may be written as the product GF of a positive definite symmetric matrix G and an orthogonal matrix F. G then defines a new metric g and F defines an almost complex structure J; checking the overlaps of local charts, it follows from the uniqueness of the polar decomposition that g and J are created simultaneously by polarization.

A metric g created in this way is called an *associated metric* and the set of these metrics will be denoted by $\mathcal{A}$. In particular $\mathcal{A}$ is the set of all almost Kähler metrics on M which have Ω as their fundamental 2-form. This approach suggests that there are indeed many associated metrics and it will be evident from Lemma 3.2 below that the space of all such metrics is infinite dimensional. We note also that all associated metrics have the same volume element $dV = \frac{(-1)^n}{2^n n!}\Omega^n$.

3. Associated metrics for a symplectic form

In this section we will consider a number of integral functionals defined on the set of metrics associated to a symplectic structure. To begin our study we need to see how the set $\mathcal{A}$ of associated metrics sits in the set $\mathcal{M}$ of all Riemannian metrics with the same total volume; for a more detailed treatment see [1].

Let M be a symplectic manifold and $g_t = g + tD + O(t^2)$ be a path of metrics in $\mathcal{A}$. We will use the same letter D to denote D as a tensor field of type (1, 1) and of type (0, 2), $D^i{}_j = g^{ik}D_{kj}$. Now

$$g(X, JY) = \Omega(X,Y) = g_t(X, J_tY) = g(X, J_tY) + tg(X, DJ_t) + O(t^2)$$

from which

$$J = J_t + tDJ_t + O(t^2).$$

Applying J_t on the right and J of the left we have

$$J_t = J + tJD + O(t^2)$$

Squaring this yields $JDJ - D = 0$ and hence $JD + DJ = 0$. Conversely if D is a symmetric tensor field which anti-commutes with J, then $g_t = ge^{tD}$ is a path of associated metrics. We summarize this as follows [1, 2].

Lemma 3.1. *Let M be a symplectic and $g \in \mathcal{A}$. A symmetric tensor field D is tangent to a path in $\mathcal{A}$ at g if and only if*

$$DJ + JD = 0 \tag{3.1}$$

We end this section with the following lemma [2,3,4] for critical point problems on $\mathcal{A}$.

Lemma 3.2. *Let T be a second order symmetric tensor field on M. Then $\int_M T^{ij} D_{ij}\, dV_g = 0$ for all symmetric tensor fields D satisfying (3.1) if and only if $TJ = JT$.*

4. Square of the norm of the Nijenhuis tensor as a symplectic invariant

The $*$-*Ricci tensor* and $*$-*scalar curvature* of an almost Hermitian manifold are defined by

$$R^*_{ji} = R_{iklt} J^{kl} J_j{}^t, \quad R^* = R^{*i}_i.$$

On a Kähler manifold

$$R^*_{ij} = R_{ij}. \tag{4.1}$$

The most important property of R^* on a sympletic manifold is that

$$R - R^* = -\frac{1}{2}|\nabla J|^2 \tag{4.2}$$

and hence $R - R^* \leq 0$ with equality holding if and only if the metric is Kähler. Thus for M compact, Kähler metrics are maxima of the functional

$$K(g) = \int_M (R - R^*) dV_g$$

on $\mathcal{A}$ and hence it is natural to ask for the critical point condition in general. In [3] S. Ianus and Blair computed the critical point condition for $K(g)$ and obtained the following lemma by using Lemmas 3.1 and 3.2:

Lemma 4.1. *Let M be a compact sympletic manifold and $\mathcal{A}$ the set of metrics associated to the symplectic form. Then $g \in \mathcal{A}$ is a critical point of $K(g)$ if and only if $QJ = JQ$, where Q denotes the Ricci operator of g.*

From now on we consider the integral functional

$$N(g) = \int_M |N|^2 dV_g$$

defined on A, where N denotes the Nijenhuis tensor formed with J.

By the way on an almost Kähler manifold the followings hold good:

$$\nabla_i J_j^i = 0 \tag{4.3}$$

$$(\nabla_k J_{ih}) J_j^k = (\nabla_h J_{ij}) J_k^h \tag{4.4}$$

By using (4. 3) and (4. 4), we can easily see that

$$|N|^2 = 32\{|\nabla J|^2 + R - R^*\},$$

which together with (4. 2) implies

$$|N|^2 = 16|\nabla J|^2. \tag{4.5}$$

Thus for M compact, Kähler metrics are minima of the functional $N(g)$. Moreover, combining with (4. 2), (4. 5) and Lemma 4. 1, we have

Theorem. *Let M be a compact sympletic manifold and $\mathcal{A}$ the set of metrics associated to the symplectic form. Then $g \in \mathcal{A}$ is a critical point of $N(g)$ if and only if $QJ = JQ$.*

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