1. Introduction

Let $M$ be a compact orientable Riemannian manifold and let $\mu(M)$ be the space of $C^\infty$ Riemannian metrics $G$ on $M$ satisfying $\int_M dV_G = 1$, where $dV_G$ is the volume element measured by $G$. For an element $G$ in $\mu(M)$, we assume that $f(\kappa)$ is a scalar field on $M$ determined by $G$ as the contraction of a tensor product of the curvature tensor. Then $H_M[G] = \int_M f(\kappa) dV_G$ defines a mapping $H_M : \mu(M) \rightarrow R$. From now on, we denote $H_M$ by $H$. In this case, a critical point of $H$ is called a critical Riemannian metric with respect to the field $f(\kappa)$ and denoted by $G_H$ (cf. [1.6]).

Following M. Berger [1], we have four kinds of critical Riemannian metrics $G_A$, $G_B$, $G_C$ and $G_D$ as the most prominent ones. The corresponding integrals are

$$A_M[G] = \int_M K dV_G, \quad B_M[G] = \int_M K^2 dV_G,$$

$$C_M[G] = \int_M S^2 dV_G, \quad D_M[G] = \int_M R^2 dV_G,$$

where $R$, $S$ and $K$ are the Riemannian curvature tensor, Ricci curvature tensor and scalar curvature respectively. The equations of the critical Riemannian metrics obtained by M. Berger can be written in the following form in tensor notations

$$A_{ji} = c_A G_{ji}, \quad B_{ji} = c_B G_{ji}, \quad (1.1)$$

$$C_{ji} = c_C G_{ji}, \quad D_{ji} = c_D G_{ji},$$


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where $c_A$, $c_B$, $c_C$ and $c_D$ are undetermined constants and $A_{ji}$, $B_{ji}$, $C_{ji}$ and $D_{ji}$ are given by

$$A_{ji} = -S_{ji} + \frac{1}{2} K G_{ji}, \quad (1.2)$$

$$B_{ji} = 2 \nabla_j \nabla_i K - 2(\nabla_k \nabla^k K) G_{ji} - 2K S_{ji} + \frac{1}{2} K^2 G_{ji}, \quad (1.3)$$

$$C_{ji} = \nabla_j \nabla_i K - \nabla_k \nabla^k S_{ji} - \frac{1}{2}(\nabla_k \nabla^k K) G_{ji} - 2R_{jkh} S_{ki} + \frac{1}{2} S_{kh} S^{kh} G_{ji}, \quad (1.4)$$

$$D_{ji} = 2 \nabla_j \nabla_i K - 4 \nabla_k \nabla^k S_{ji} + 4S_{jk} S_{ki} - 4R_{jkh} S^{kh} - 2R_{jkl} R_i^{kh} + \frac{1}{2} R_{khlm} R^{khlm} G_{ji}, \quad (1.5)$$

where $\nabla$ is the Riemannian connection with respect to $G$ on $M$.

The purpose of this paper is to study the fibred Sasakian space forms with critical Riemannian metric and we get, in this case, $M$ is a space of constant curvature, the base space is a complex space form and each fibre is 1-dimension and totally geodesic. Essential examlpes for this result can be found in [4].

2. Fibred Riemannian spaces

Let $\{M, N, G, \pi\}$ be a fibred Riemannian space, that is, $\{M, G\}$ is an $m$-dimensional total space with Riemannian metric $G$, $N$ an $n$-dimensional base space, and $\pi : M \rightarrow N$ the projection with maximum rank $n$. The fibre passing through a point $P \in M$ is denoted by $M(P)$ or generally $\tilde{M}$, and the metric tensor $G$ is projectable. Throughout this paper, the range of indices are as follows;

$$h, i, j, k, l = 1, 2, \ldots, m,$$
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\[ a, b, c, d, e = 1, 2, \ldots, n, \]
\[ x, y, z, u, v = n + 1, \ldots, n + p = m. \]

One of the present author [5] proved that if the Sasakian structure \((\phi, \xi, \eta)\) on \(M\) is of constant \(\phi\)-holomorphic sectional curvature \(k\), that is, the curvature tensor \(R\) of \(M\) is given by

\[
R_{kji}^h = \frac{k + 3}{4}(G_{ji} \delta_k^h - G_{ki} \delta_j^h) \tag{2.1}
\]
\[
- \frac{k - 1}{4}(\eta_j \eta_i \delta_k^h - \eta_k \eta_i \delta_j^h + G_{ji} \eta_k \xi^h
- G_{ki} \eta_j \xi^h - \phi_{ji} \phi_k^h + \phi_{ki} \phi_j^h + 2\phi_{kj} \phi_i^h),
\]

then the base space is a complex space form, each fibre is minimal and

\[
\overline{R}_{uzy}^x = \frac{k + 3}{4}(\overline{g}_{zy} \delta_u^x - \overline{g}_{uy} \delta_z^x) \tag{2.2}
\]
\[
- \frac{k - 1}{4}(\overline{\eta}_z \overline{\eta}_y \delta_u^x - \overline{\eta}_u \overline{\eta}_y \delta_z^x + \overline{g}_{zy} \overline{\eta}_u \overline{\xi}^x
- \overline{g}_{uy} \overline{\eta}_z \overline{\xi}^x - \overline{\phi}_{zy} \overline{\phi}_u^x + \overline{\phi}_{uy} \overline{\phi}_z^x + 2\overline{\phi}_{uz} \overline{\phi}_y^x)
+ h_{zye} \overline{h}_u^x \overline{h}_z^e,
\]
\[
S_{zy} = \overline{S}_{zy} + \nabla^* \phi_{zy} h_{zy}^e + \eta_{zy} \eta_y, \tag{2.3}
\]
\[
L_{cb}^x = J_{cb} \overline{\xi}^x, \tag{2.4}
\]

where we have put

\[
\nabla^* \phi_{h_{xy}^a} = \partial_d h_{xy}^a + \Gamma^a_d \phi_{h_{xy}^e} - Q_{dx}^z h_{zy}^a - Q_{dy}^z h_{xz}^a, \tag{2.5}
\]
\[
Q_{dx}^y = P_{dx}^y - h_{x}^y d, \tag{2.6}
\]

\(P_{dx}^y\) are local functions related to \(L_{C_x} C^y = P_{dx}^y E^d, \ \{E^a, C^x\}\) are dual to the local frame \(\{E_b, C_y\}\), \(J\) is the complex structure induced on \(N, (\phi, \xi, \eta, \bar{\eta}, \bar{g})\) is the Sasakian structure induced on \(M\), \(S_{zy} = S(C_x, C_y)\), \(\overline{R}\) and \(\overline{S}\) are the curvature tensor and Ricci curvature tensor on \(M\) respectively, \(h = (h_{xy}^a)\) and \(L = (L_{cb}^x)\) are the components of the second fundamental tensor and normal connection of each fibre \(M\) respectively.
3. Main results

First we show that, if the Riemannian metric $G$ on the Sasakian space form $M$ is a critical Riemannian metric $G_A$, then each fibre is totally geodesic and the dimension of $M$ is one.

From (2.1), we get

$$S_{ji} = \frac{(m+1)k + 3m - 5}{4} G_{ji} - \frac{(m+1)(k-1)}{4} \eta_j \eta_i$$  \hspace{1cm} (3.1)

and since $M$ is an Einstein space, it is easily seen that $k = 1$ by use of the fact that $m \neq 1$. Therefore $M$ becomes a space of constant curvature and

$$\| h_{xy}^a \|^2 = -n(p - 1)$$

by use of (2.2), (2.3) and (3.1). Since $\| h_{xy}^a \|^2$ is non-negative, we have $p = 1$ and that $h$ vanishes identically. Thus we have

**Theorem 3.1.** Let $M$ be the fibred Sasakian space with constant $\phi$-holomorphic sectional curvature $k$. If $G$ is a critical Riemannian metric $G_A$, then $M$ has 1-dimensional totally geodesic fibres, $M$ is a space of constant curvature and the base space $N$ is a complex space form and the Riemannian metric $g$ on $N$ is a critical Riemannian metric $G_A$.

By the same argument of Theorem 3.1, we get

**Theorem 3.2.** Let $M$ be the fibred Sasakian space with constant $\phi$-holomorphic sectional curvature $k$. If $G$ is a critical Riemannian metric $G_B$, then we have the same result of Theorem 3.1 except that $g$ on $N$ is a critical Riemannian metric $G_B$.

Assume that $G$ is a critical Riemannian metric $G_C$, then

$$\nabla_k \nabla^k S_{ji} = -\frac{(m+1)(k-1)}{2} (G_{ji} - \eta_j \eta_i),$$  \hspace{1cm} (3.2)

$$R_{jkh}^k S_{kh} = \{(m-1)\alpha \gamma + 2\beta \gamma - \alpha \delta + \beta \delta\} G_{ji}$$

$$+ \{\alpha \delta - \beta \delta - (m+1)\beta \gamma\} \eta_j \eta_i$$  \hspace{1cm} (3.3)
where we have put
\[
\alpha = \frac{k + 3}{4}, \quad \beta = \frac{k - 1}{4},
\]
\[
\gamma = \frac{(m + 1)k + 3m - 5}{4}, \quad \delta = \frac{(m + 1)(k - 1)}{4}.
\]
From (1.1), (1.4) and (3.2)~(3.4), we see that the coefficient of \(\eta_j\eta_i\) in \(C_{ji} - cCG_{ji}\) is \(\frac{1}{8}(m + 1)(k - 1)((m + 1)k - m - 9)\). Hence \(k = 1\) or \(k = \frac{m + 9}{m + 1}\).

(I) if \(k = 1\), then \(M\) is a space of constant curvature and \(\|h_{xy}^a\|^2 = -n(p - 1)\) by use of (2.2) and (2.3), so \(p = 1\) and \(h = 0\).

(II) if \(k = \frac{m + 9}{m + 1}\), then
\[
\|h_{xy}^a\|^2 = \frac{p - 1}{m + 1} \{1 - 1\} = \frac{p - 1}{m + 1} \{(m + 3)p + 2\} - \{(m + 2)(p - 1)\},
\]
that is,
\[
(m + 1)\|h_{xy}^a\|^2 = (1 - p)(m + 3)(m - p).
\]
Since \(m + 1)\|h_{xy}^a\|^2 \geq 0\), we get \(p \leq 1\). Hence we have \(p = 1\) and that \(h = 0\). Thus we obtain

**Theorem 3.3.** If the fibred Sasakian space \(M\) with constant \(\phi\)-holomorphic sectional curvature \(k\) has a critical Riemannian metric \(G_c\), then

(1) \(M\) is a space of constant curvature,

(2) \(M\) has 1-dimensional totally geodesic fibres,

(3) the base space \(N\) is a complex space form and \(g\) on \(N\) is a critical Riemannian metric \(G_c\).

Finally, if the total space \(M\) has a critical Riemannian metric \(G_D\), then
\[
S_{ji}S^{ji} = \gamma^2G_{ji} + \delta(\delta - 2\gamma)\eta_j\eta_i,
\]
\[
R_{jikl}R^{ijkl} = \left\{\frac{1}{8}(k + 3)(m - 1) + (k + 3)(k - 1)\right\}G_{ji}
\]
\[
- \frac{1}{2}(k + 3)(k - 1)(m + 1)\eta_j\eta_i.
\]
Since the coefficient of $\eta_j\eta_i$ in $D_{ji} - c_D G_{ji}$ vanishes identically, we have

$$(k - 1)\{k(m + 1) - (6m^2 + 3m - 11)\} = 0,$$  \hspace{1cm} (3.8)$$

that is, $k = 1$ or $k = \frac{6m^2 + 3m - 11}{m + 1}$.

For the case of $k = 1$, we have same result of (I) by the similar argument.

If $k = \frac{6m^2 + 3m - 11}{m + 1}$, then we obtain

$$S_{ji} = \frac{3m^2 + 3m - 8}{2} G_{ji} - \frac{3m^2 + m - 6}{2} \eta_j\eta_i,$$  \hspace{1cm} (3.9)$$

and that

$$K = \frac{3(m - 1)^2(m + 2)}{2}$$  \hspace{1cm} (3.10)$$

and that

$$2(m + 1)||h_{xy}^a||^2 = (1 - p)(m - p)(3m^2 + 3m - 4)$$  \hspace{1cm} (3.11)$$

by use of (2.2), (2.3), (3.9) and (3.10).

Since $3m^2 + 3m - 4 > 0$, we get $p = 1$ and that $h = 0$. Thus we have

**Theorem 3.4.** If the fibred Sasakian space $M$ with constant $\phi$-holomorphic sectional curvature $k$ has a critical Riemannian metric $G_D$, then

1. $M$ is a space of constant curvature,
2. $M$ has 1-dimensional totally geodesic fibres,
3. the base space $N$ is a complex space form and $g$ on $N$ is a critical Riemannian metric $G_D$.

**References**


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