SOME ANALYTIC CLASSIFICATION OF PLANE CURVE SINGULARITIES TOPOLOGICALLY EQUIVALENT TO THE EQUATION $z^n + y^k = 0$ WITH $gcd(n, k) = 1$

CHUNGHYUK KANG AND CHANGHO KEEM

1. Introduction

We know [6] that the analytic classification of complex hypersurfaces with isolated singularity at the origin is the same as the algebraic classification of their corresponding moduli algebra over the complex field. In fact, even an algebraic classification of irreducible plane curve singularities at the origin is a very delicate and complicated problem. For example, consider the family of analytic irreducible plane curve singularities $f_\alpha$ at the origin parametrically defined by $y = t^4$ and $z = t^9 + t^{10} + \alpha t^{11}$ where $\alpha$ is a number. Then for any $\alpha f_\alpha$ is clearly topologically equivalent to the equation $z^4 + y^9 = 0$ at the origin, but for any two numbers $\alpha \neq \beta f_\alpha$ and $f_\beta$ are analytically different at the origin [2].

Let $V = \{f(z, y) = 0\}$ and $W = \{z^n + y^k = 0\}$ with $gcd(n, k) = 1$ be analytic irreducible plane curves with isolated singularities at the origin. Assume that $V$ and $W$ are topologically equivalent at the origin. Then denote this relation by $f \sim z^n + y^k$ for brevity. So by a nonsingular linear change of coordinates $f$ can be written as $u(z^n + a_2 y^\alpha z^{n-2} + \cdots + a_{n-1}y^\alpha z + y^k)$ where $u$ is a unit and the $a_i = a_i(y)$ are units in $2 O$, the ring of germs of holomorphic functions at the origin in $\mathbb{C}^2$ and $\frac{a_i}{i} > \frac{k}{n}$ for $i = 2, \ldots, n - 1$ [5]. If $V$ and $W$ are analytically equivalent at the origin, then denote this relation by $f \approx g$. If not, we write $f \not\approx g$.


Supported by Ministry of Education-university research institute support program. Also supported in part by the GARC-KOSEF.
Then we are going to prove the following cases:

(1) If \( f = z^n + uy\alpha z^\beta + y^k \) where \( u \) is a unit in \( \mathcal{O}_2 \), then \( f \approx z^n + y\alpha z^\beta + y^k \).

(2) If \( f = z^n + uy\alpha z^\beta + y^k \) and \( g = z^n + v\gamma z^\delta + y^k \) where \( u, v \) are units in \( \mathcal{O}_2 \) and \( 1 \leq \alpha, \gamma \leq k - 2 \) and \( 1 \leq \beta, \delta \leq n - 2 \), then \( f \approx g \) if and only if \( \alpha = \gamma \) and \( \beta = \delta \).

(3) In the case (2), if \( 1 \leq \alpha \leq k - 1 \) and \( 1 \leq \beta \leq n - 1 \), then \( f \approx g \) does not imply that \( \alpha = \gamma \) or \( \beta = \delta \).

(4) Suppose that \( f \sim z^n + y^k \) and \( g \sim z^n + y^k \). By [1], \( f \approx z^n + y^k + \sum c_i P_i \) and \( g \approx z^n + y^k + \sum d_j Q_j \) where each \( c_i \) and \( d_j \) are nonzero numbers if exist and \( P_i = y^{\alpha_i} z^{\beta_i}, Q_j = y^{\gamma_i} z^{\delta_j} \); \( 1 \leq \alpha_i, \gamma_j \leq k - 2 \); \( 1 \leq \beta_i, \delta_j \leq n - 2 \) satisfying that \( n\alpha_i + k\beta_i > nk \) and \( n\gamma_j + k\delta_j > nk \). Let \( m(f) = \min\{\alpha_i + \beta_i; c_i \neq 0\} \) and \( m(g) = \min\{\gamma_j + \delta_j; d_j \neq 0\} \). If \( f \approx g \), then \( \{(\alpha_i, \beta_i) : \gamma_j + \delta_j = m(g)\} \) as sets.

(5) Let \( f = z^n + y^k + \Sigma c_i P_i \) where each \( c_i \) is a nonzero number if exists and \( P_i = y^{\alpha_i} z^{\beta_i} \) with \( n\alpha_i + k\beta_i > nk \) and \( 1 \leq \alpha_i \leq k - 2 \), \( 1 \leq \beta_i \leq n - 2 \). Then \( f \approx z^n + y^k \) if and only if all \( c_i \) are zero.

2. Known preliminaries

**Definition 2.1.** Let \( V = \{z \in \mathbb{C}^n : f(z) = 0\} \) and \( W = \{z \in \mathbb{C}^n : g(z) = 0\} \) be germs of complex analytic hypersurfaces with isolated singular points at the origin. (i) \( V \) and \( W \) are said to be topologically equivalent at the origin if there is a germ at the origin of homeomorphisms \( \phi : (U_1, 0) \rightarrow (U_2, 0) \) such that \( \phi(V) = W \) and \( \phi(0) = 0 \) where \( U_1 \) and \( U_2 \) are open subset containing the origin in \( \mathbb{C}^n \). In this case denote this relation by \( f \sim g \). (ii) \( V \) and \( W \) are said to be analytically equivalent at the origin if there is a germ at the origin of biholomorphisms \( \psi : (U_1, 0) \rightarrow (U_2, 0) \) such that \( \psi(V) = W \) and \( \psi(0) = 0 \) where \( U_1 \) and \( U_2 \) are open subsets containing the origin in \( \mathbb{C}^n \). Then denote this relation by \( f \approx g \). If not, we write \( f \not\approx g \). Let \( \mathcal{O} \) denote the ring of germs of holomorphic functions at the origin in \( \mathbb{C}^n \).

**Theorem 2.2** [5]. Let \( f(z, y) = a_0 z^n + a_1 y^{\alpha_1} z^{n-1} + \cdots + a_n y^{\alpha_n} \) be irreducible in \( \mathcal{O}_2 \) where each \( a_i \) is a unit in \( \mathcal{O}_2 \) if exists and the \( \alpha_i \)
are positive integers. Then \( \frac{\alpha_i}{i} \geq \frac{\alpha_n}{n} \) for all \( i \). Moreover, if \( \alpha_n = nk \) for some integer \( k \), then \( \frac{\alpha_n}{n} = \frac{\alpha_i}{i} \) for all \( i = 1, \ldots, n - 1 \).

**Corollary 2.3.** Let \( f(z, y) = z^n + a_1 y^{\alpha_1} z^{n-1} + \cdots + a_{n-1} y^{\alpha_{n-1}} z + y^k \) with \( (n, k) = 1 \) where \( a_i = a_i(y) \) is a unit in \( \mathcal{O} \) if exists and the \( \alpha_i \) are positive integers. Then \( f \) is irreducible in \( \mathcal{O} \) if and only if
\[
\frac{k}{n} < \frac{\alpha_i}{i}
\]
for all \( i \neq n \). Moreover, in this case \( f \sim z^n + y^k \) in \( \mathcal{O} \).

**Definition 2.4.** The polynomial \( f(z_1, \ldots, z_n) \) is called weighted homogeneous of type \((\frac{1}{a_1}, \ldots, \frac{1}{a_n})\) if there is some positive rational numbers \( a_1, \ldots, a_n \) such that
\[
f(t^{a_1} z_1, \ldots, t^{a_n} z_n) = tf(z_1, \ldots, z_n).
\]

**Theorem 2.5 (Mather-Yau [6]).** Suppose that \( V = \{f(z_1, \ldots, z_n) = 0\} \) and \( W = \{g(z_1, \ldots, z_n) = 0\} \) have the isolated singular point at the origin. Then the following conditions are equivalent:

(i) \( f \approx g \).

(ii) \( A(f) \) is isomorphic to \( A(g) \) as a \( \mathbb{C} \)-algebra where \( A(f) = n\mathcal{O}/(f, \Delta(f)) \), \( A(g) = n\mathcal{O}/(g, \Delta(g)) \) and \( (f, \Delta(f)) \) is the ideal in \( n\mathcal{O} \) generated by \( f, \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n} \).

(iii) \( B(f) \) is isomorphic to \( B(g) \) as a \( \mathbb{C} \)-algebra where \( B(f) = n\mathcal{O}/(f, m\Delta(f)) \), \( B(g) = n\mathcal{O}/(g, m\Delta(g)) \) and \( (f, m\Delta(f)) \) is the ideal in \( n\mathcal{O} \) generated by \( f \) and \( z_i \frac{\partial f}{\partial z_j} \) for all \( i, j = 1, \ldots, n \).

**Theorem 2.6 (Arnold [1]).** Assume that \( n < k \), \( (n, k) = 1 \) and that \( g = z^n + a_1 y^{\alpha_1} z^{n-1} + \cdots + a_{n-1} y^{\alpha_{n-1}} z + y^k \sim z^n + y^k \) at the origin in \( \mathbb{C}^2 \) where each \( a_i = a_i(y) \) is a unit in \( \mathcal{O} \) if exists and the \( \alpha_i \) are positive integers. Then \( g \approx z^n + y^k + \Sigma c_i P_i \) where each \( c_i \) is a nonzero number if exists and \( P_i = y^{\alpha_i} z^{\beta_i} \) with \( 1 \leq \beta_i \leq n - 2 \) and \( 1 \leq \alpha_i \leq k - 2 \) with \( n\alpha_i + k\beta_i > nk \).
3. Some analytic classification of irreducible plane curve
singularities

THEOREM 3.1. Let \( f = z^n + uy^\alpha z^\beta + y^k \) with \( n\alpha + k\beta > nk \) where
\( n < k, (n,k) = 1 \) and \( u = u(z,y) \) is a unit in \( \mathcal{O} \). Then \( f \sim z^n + y^k \)
and \( f \approx z^n + y^\alpha z^\beta + y^k \).

Proof. By Corollary 2.3, clearly \( f \sim z^n + y^k \). First note that for
any number \( c \neq 0 \) \( f_c(z,y) = z^n + cy^\alpha z^\beta + y^k \approx z^n + y^\alpha z^\beta + y^k \)
because \( f_c(t^k z, t^n y) = t^{nk}(z^n + c t^{\alpha + k\beta - nk} y^\alpha z^\beta + y^k) \).

(i) To show that \( f \approx z^n + y^\alpha z^\beta + y^k \), first assume that \( u(z,y) = u(z,0) \) is a unit in \( \mathcal{O} \). Then by a nonsingular linear change of coordinates at the origin, \( f = z^n + u(z,0) y^\alpha z^\beta + y^k \approx v(z) z^n + y^\alpha z^\beta + y^k = h \)
where \( v = v(z) \) is a unit in \( \mathcal{O} \). Now it is enough to show that
\( h \approx v(0) z^n + y^\alpha z^\beta + y^k = g \). To use Theorem 2.5 compute the ideal
\( I = (h, m\Delta(h)) \) in \( \mathcal{O} \) generated by \( h, zh_z, yh_y, yh_z, zh_y \) as follows:

\[
\begin{align*}
h &= v z^n + y^\alpha z^\beta + y^k \\
zh_z &= (zv' +nv) z^n + \beta y^\alpha z^\beta \\
yh_y &= \alpha y^\alpha z^\beta + ky^k \\
yh_z &= (zv' +nv) y z^{n-1} + \beta y^{\alpha+1} z^{\beta-1} \\
zh_y &= \alpha y^{\alpha-1} z^{\beta+1} + ky^{k-1} z.
\end{align*}
\]

Then solve the equation \( h \equiv zh_z \equiv yh_y \equiv 0 \) (mod \( I \)) with respect to
\( z^n, y^\alpha z^\beta \) and \( y^k \) as below:

\[
\begin{vmatrix}
v & 1 & 1 \\
zv' + nv & \beta & 0 \\
0 & \alpha & k
\end{vmatrix} = v(n\alpha + k\beta - nk) + zv'(k - \alpha) \neq 0
\]

at the origin. Thus \( z^n, y^\alpha z^\beta, y^k \) belong to \( I \). Considering \( yh_z \) and \( zh_y \),
then \( I = (z^n, y^\alpha z^\beta, y^k, nv(0) z^{n-1} + \beta y^{\alpha+1} z^{\beta-1}, \alpha y^{\alpha-1} z^{\beta+1} + ky^{k-1} z) \).
So \( I = (g, m\Delta(g)) \). By Theorem 2.5 \( f \approx v(0) z^n + y^\alpha z^\beta + y^k \approx z^n + y^\alpha z^\beta + y^k \).

(ii) Now let \( f = z^n + u(z,y) y^\alpha z^\beta + y^k \). Also by a nonsingular linear
change of coordinates, it is enough to consider \( \ell(z,y) = z^n + y^\alpha z^\beta + \)
Some analytic classification of plane curve singularities

v(z, y)y^k where v = v(z, y) is a unit in \( \mathcal{O} \). Then compute the ideal \( J = (\ell, m\Delta(\ell)) \) as follows:

\[
\ell = z^n + y^\alpha z^\beta + vy^k \\
z\ell_z = nz^n + \beta y^\alpha z^\beta + v_zy^kz \\
y\ell_y = \alpha y^\alpha z^\beta + (yv_y + kv)y^k \\
y\ell_z = nyz^{n-1} + \beta y^{\alpha+1}z^{\beta-1} + vz^{k+1} \\
z\ell_y = \alpha y^{\alpha-1}z^{\beta+1} + (yv_y + kv)y^{k-1}z.
\]

Similarly as the previous case, solve the equation \( \ell \equiv z\ell_z \equiv y\ell_y \equiv 0 \pmod{J} \) relative to \( z^n, y^\alpha z^\beta \) and \( y^k \). Then we prove easily that \( z^n, y^\alpha z^\beta \) and \( y^k \) belong to \( J \). Considering \( y\ell_z \) and \( z\ell_y \), then \( J = (z^n, y^\alpha z^\beta, y^k, nyz^{n-1} + \beta y^{\alpha+1}z^{\beta-1}, \alpha y^{\alpha-1}z^{\beta+1} + kv(z, 0)y^{k-1}z) \) where \( v(z, 0) \) is a unit in \( \mathcal{O} \). Therefore \( \ell \approx z^n + y^\alpha z^\beta + v(z, 0)y^k \) by Theorem 2.5. By another nonsingular linear change of coordinates at the origin, \( z^n + y^\alpha z^\beta + v(z, 0)y^k \approx z^n + w(z)y^\alpha z^\beta + y^k \) where \( w(z) \) is a unit in \( \mathcal{O} \). By (i), we proved the theorem.

**Theorem 3.2.** Let \( f(z, y) = z^n + y^k + uy^\alpha z^\beta \) and \( g = z^n + y^k + vy^\gamma z^\delta \) where \( n < k \), \( (n, k) = 1 \) and \( u = u(z, y) \), \( v = v(z, y) \) are units in \( \mathcal{O} \) and \( 1 \leq \beta, \delta \leq n-2; 1 \leq \alpha, \gamma \leq k-2 \) with \( na + k\beta > nk \) and \( n\gamma + k\delta > nk \). Then \( f \approx g \) if and only if \( \alpha = \gamma \) and \( \beta = \delta \).

**Proof.** By Theorem 3.1, if \( \alpha = \gamma \) and \( \beta = \delta \) then \( f \approx g \). Now suppose that \( f \approx g \). Then by Theorem 3.1, we may put \( f = z^n + y^k + y^\alpha z^\beta \) and \( g = z^n + y^k + y^\gamma z^\delta \). Let us prove the condition that \( \alpha = \gamma \) and \( \beta = \delta \). Assume the contrary. So it is enough to consider the following cases: (I) \( \alpha + \beta < \gamma + \delta \) and (II) \( \alpha + \beta = \gamma + \delta \) with \( \alpha \neq \gamma \).

By definition, if \( f \approx g \) then there is a biholomorphic mapping \( \phi : (U_1, 0) \to (U_2, 0) \) such that \( uf = g \circ \phi \), where \( U_1 \) and \( U_2 \) are open subsets containing the origin and \( u \) is a unit in \( \mathcal{O} \). Write \( \phi(z, y) = (H, L) \) as follows:

\[
H = H(z, y) = az + by + H_2 + H_3 + \cdots \\
L = L(z, y) = cz + dy + L_2 + L_3 + \cdots
\]
where $H_n$ and $L_n$ are homogeneous polynomials of degree $n$ with $H_n = H_n(z, y) = a_{n,0}z^n + a_{n-1,1}z^{n-1}y + \cdots + a_{0,n}y^n$ and $L_n = L_n(z, y) = b_{n,0}z^n + b_{n-1,1}z^{n-1}y + \cdots + b_{0,n}y^n$.

Note that $ad - bc \neq 0$. Then $g \circ \phi(z, y) = (az + by + H_2 + H_3 + \cdots)^n + (cz + dy + L_2 + L_3 + \cdots)^\delta (az + by + H_2 + H_3 + \cdots)^\delta + (cz + dy + L_2 + L_3 + \cdots)^k = u(z^n + y^\alpha z^\beta + y^k)$ where $u$ is a unit in $\mathcal{O}$. We know that $b = 0$ because $n < k$, $\alpha + \beta > n$ and $\gamma + \delta > n$. Let us prove the first case (I).

(I) We are going to separate this fact into the following three cases:

(i) $\alpha + \beta < k$: Observe that $H_2, H_3, \cdots, H_{\alpha + \beta - n}$ can be analytically divisible by $z$ in the expansion of $H_n$ in $g \circ \phi(z, y)$, considering $\mathcal{O}$ as a unique factorization ring up to a unit [3]. Therefore in the expansion of $H_n = (az \cdot \text{unit} + H_{\alpha + \beta - n+1} + \cdots)^n$, we cannot find a nonzero monomial $y^\alpha z^\beta$ with $\beta \leq n - 2$ because $\beta + (n - \beta)(\alpha + \beta - n + 1) > \alpha + \beta$ if and only if $(\alpha + \beta - n)(n - \beta - 1) > 0$.

(ii) $\alpha + \beta = k$: Observe also that $H_2, H_3, \cdots, H_{k-n}$ can be analytically divisible by $z$ in the expansion of $H_n$ in $g \circ \phi(z, y)$. To find the set of nonzero monomials $y^\ell z^m$ with $\ell + m = k$ and $m \leq n - 1$ in the expansion of $g \circ \phi(z, y)$, it is enough to consider the following:

$$n(az)^{n-1}H_{k-n+1} + (cz + dy)^k.$$ Since the coefficient of monomial $y^{k-1}z$ must be zero, $ad - bc \neq 0$ implies that $c = 0$. Therefore we cannot find a nonzero monomial $y^\alpha z^\beta$ in the expansion of $g \circ \phi(z, y)$ because $\beta \leq n - 2$.

(iii) $\alpha + \beta > k$: Note that $H_2, H_3, \cdots, H_{k-n}$ can be analytically divisible by $z$ in the expansion of $H_n$ in $g \circ \phi(z, y)$. Since there is no nonzero monomial $y^{k-1}z$ in $uf$, $\alpha + \beta > k$ implies that $c = 0$ and also $H_{k-n+1}$ can be divided by $z$ analytically in $\mathcal{O}$. So in the expansion of $g \circ \phi(z, y)$ to find a nonzero monomial $y^\alpha z^\beta$, it is sufficient to consider the following:

$$(az \cdot \text{unit} + H_{k-n+2} + H_{k-n+3} + \cdots)^n + (dy + L_2 + L_3 + \cdots)^k.$$ Now we are going to show that (iii a) there is no nonzero term $y^\alpha z^\beta$ in the expansion of $H_n$ and (iii b) there is no nonzero term $y^\alpha z^\beta$ in the expansion of $L^k$. Consider the case (iii a). If $H_j$ can be analytically
Some analytic classification of plane curve singularities

divisible by \( z \) for all \( j \), there is nothing to prove. If not, let \( m \) be the smallest positive integer such that \( H_m \) cannot be analytically divisible by \( z \). Then in the expansion of \( H^n, z^{n-1}H_m \) contains a nonzero monomial \( z^{n-1}y^m \). If \( n-1+m \geq \alpha + \beta \), there is no nonzero term \( y^\alpha z^\beta \) in the expansion of \( H^n \). If \( n-1+m < \alpha + \beta \) and there is no nonzero monomial \( y^m z^{n-1} \) in the expansion of \( L_k \), then we cannot find a nonzero term \( y^m z^{n-1} \) in \( u_f \), because if exists then \( \alpha \leq m \) and \( \beta \leq n-2 \) would imply that \( \alpha + \beta \leq m + n - 2 < \alpha + \beta \). It is a contradiction. So it is enough to prove that if \( n-1+m < \alpha + \beta \) and there is a nonzero term \( y^m z^{n-1} \) in \( L_k \), we could find a contradiction. Then let \( r \) be the smallest positive integer such that \( L_r \) cannot be analytically divisible by \( y \). Thus we would get some inequality as follows:

\[
\begin{align*}
n - 1 + m & \geq m + r(k - m). \\
\end{align*}
\]

Claim that \( n-1+m = m+r(k-m) \). If \( n-1+m > m+r(k-m) \), then there is neither a nonzero term \( y^m z^{r(k-m)} \) in \( u_f \) nor in \( H^n \) because \( k+n-4 \geq \alpha + \beta > n-1+m > m+r(k-m) \) implies that \( k-3 > m \) and \( r(k-m) < n-1 \). Since there exists a nonzero term \( y^m z^{r(k-m)} \) in \( g \circ \phi(z,y) \), it would be a contradiction. Therefore we get the equation:

\[
\begin{align*}
(A) \quad n - 1 + m &= m + r(k - m). \\
\end{align*}
\]

Now consider a nonzero term \( y^{k-1}z^r \) in \( L_k \). Note that \( k-1+r < \alpha + \beta \) because \( \alpha + \beta > n-1+m = m+r(k-m) \) and \( m+r(k-m)-(k-1+r) = (k-m-1)(r-1) > 0 \). Since there is no nonzero monomial \( y^{k-1}z^r \) in \( u_f \), if there is no nonzero term \( y^{k-1}z^r \) in \( H^n \) then there is nothing to prove. If there is a nonzero term \( y^{k-1}z^r \) in \( H^n \), by the similar method just as before, we get another equation: \( \begin{align*}
(B) \quad k - 1 + r &= r + m(n - r). \\
\end{align*} \)

From two equations (A) and (B), \( k(1+r) = n(m+1) \), i.e., \( \frac{k}{n} = \frac{m+1}{r+1} \).

Note that \( (n,k) = 1 \). So the equation \( k(1+r) = n(m+1) \) does not hold because \( k-1+r < \alpha + \beta \leq n+k-4 \) implies \( r < n - 3 \). Thus we proved the case (\( \text{iii}_a \)).

Next, using the similar technique as in the case (\( \text{iii}_a \)), we can prove the case (\( \text{iii}_b \)). Thus we proved the theorem in the case (I).

Similarly, we can prove the case (II).

In the assumption of Theorem 3.2, if \( f = z^n + y^k + y^\alpha z^\beta \sim z^n + y^k \) with \( 1 \leq \alpha \leq k-1 \) and \( 1 \leq \beta \leq n-1 \), then we can prove that the result of Theorem 3.2 may not be true by the following examples:
(1) Let \( f = z^4 + y^8 z + y^9 \) and \( g = z^4 + y^7 z^2 + y^9 \). Then \( f \approx g \) but \( f \not\approx z^4 + y^9 \) by Theorem 2.5.

(2) Let \( f = z^4 + y^4 z^3 + y^9 \) and \( g = z^4 + y^8 z^2 + y^9 \). Then \( f \approx g \approx z^4 + y^9 \) by Theorem 2.5 and Theorem 3.1.

**Theorem 3.3.** Let \( f = z^n + y^k + \sum c_i P_i \) and \( g = z^n + y^k + \sum d_j Q_j \) where \( n < k \), \( (n, k) = 1 \) and each \( c_i \) and \( d_j \) are nonzero numbers if exist and \( P_i = y^{\alpha_i} z^{\beta_i}, Q_j = y^{\gamma_j} z^{\delta_j} \) with \( 1 \leq \alpha_i, \gamma_j \leq k - 2 \) and \( 1 \leq \beta_i, \delta_j \leq n - 2 \) satisfying that \( n\alpha_i + k\beta_i > nk \) and \( n\gamma_j + k\delta_j > nk \). Let \( m(f) = \text{Min}\{\alpha_i + \beta_i : c_i \neq 0\} \) and \( m(g) = \text{Min}\{\gamma_j + \delta_j : d_j \neq 0\} \). If \( f \approx g \), then \( \{(\alpha_i, \beta_i) : \alpha_i + \beta_i = m(f)\} = \{\{\gamma_j, \delta_j) : \gamma_j + \delta_j = m(g)\} \) as sets.

**Proof.** By the similar method as in the proof of Theorem 3.2, we can prove it.

**Theorem 3.4.** Let \( f = z^n + y^k + \sum c_i P_i \) where \( n < k \), \( (n, k) = 1 \) and each \( c_i \) is a number if exist and \( P_i = y^{\alpha_i} z^{\beta_i} \) with \( n\alpha_i + k\beta_i > nk \) and \( 1 \leq \alpha_i \leq k - 2, 1 \leq \beta_i \leq n - 2 \). Then \( f \approx z^n + y^k \) if and only if all \( c_i \) are zero.

**Proof.** See [4] or use the similar technique as in the proof of Theorem 3.2.

**References**

4. C. Kang, *On the type of plane curve singularities analytically equivalent to the equation* \( z^n + y^k = 0 \), *with gcd(n, k) = 1*, J. KMS. 29 no. 2 (1992), 281-295.
Some analytic classification of plane curve singularities

Department of Mathematics
Seoul National University
Seoul 151-742, KOREA