A NOTE ON STARLIKENESS
OF A CERTAIN INTEGRAL

SHIGEYOSHI OWA

Let $A$ be the class of functions $f(z)$ which are analytic in the open unit disk $U$ with the normalizations $f(0) = 0$ and $f'(0) = 1$. Denoting by $R(\alpha)$ the subclass of $A$ consisting of functions $f(z)$ which satisfy $\Re\{f'(z)\} > \alpha$ for some $\alpha (\alpha < 1)$ and for all $z \in U$, the starlikeness of an integral $g(z) = \int_0^z \{f(t)/t\}dt$ is shown.

1. Introduction

Let $A$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

(1.1)

which are analytic in the open unit disk $U = \{z : |z| < 1\}$. A function $f(z)$ belonging to $A$ is said to be a member of the class $R(\alpha)$ if it satisfies

$$\Re\{f'(z)\} > \alpha \quad (z \in U)$$

(1.2)

for some $\alpha (\alpha < 1)$. Further, a function $f(z) \in A$ is said to be in the class $S^*(\beta)$ if it satisfies

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \beta \quad (z \in U)$$

(1.3)

for some $\beta (\beta < 1)$.

For $f(z)$ belonging to $A$, we define the function $g(z)$ defined by the following integral

$$g(z) = \int_0^z \frac{f(t)}{t}dt.$$ 

(1.4)

For such an integral, Singh and Singh [6] have shown

1990 Mathematics Subject Classification. Primary 30C45.
Key words and phrases. Analytic, Class $R(\alpha)$, Class $S^*(\beta)$, starlikeness.
THEOREM A. If \( f(z) \in R(0) \), then \( g(z) \in S^*(0) \).

In the present paper, we improve the above theorem by Singh and Singh [6]. Furthermore, Bulboaca [1, p. 162] has given

**Problem.** If \( f(z) \in R(\alpha) \), find the best \( Q(\alpha) \) for which \( g(z) \in S^*(Q(\alpha)) \); or for a given \( \alpha \), find the best \( \Psi(\alpha) \) for which \( f(z) \in R(\Psi(\alpha)) \) implies \( g(z) \in S^*(\alpha) \).

2. Starlikeness of the integral

We begin with the statement of the following lemma due to Owa, Ma and Liu [4, Corollary 1].

**Lemma 1.** If \( f(z) \in R(\alpha) \), then

\[
\text{Re}\left\{ \frac{f(z)}{z} \right\} > 2\alpha - 1 + 2(1 - \alpha)\log 2 \quad (z \in U). \tag{2.1}
\]

The result is sharp.

Further, we have to recall here the following lemma by Jack [2] (also, by Miller and Mocanu [3]).

**Lemma 2.** Let \( w(z) \) be regular in \( U \), with \( w(0) = 0 \). If \( |w(z)| \) attains its maximum value in the circle \( |z| = r < 1 \) at a point \( z_0 \in U \), then

\[
z_0w'(z_0) = kw(z_0), \tag{2.2}
\]

where \( k \) is real and \( k \geq 1 \).

An application of the above lemmas derives

**Theorem 1.** If \( f(z) \in R(\alpha) \) with \( \gamma \leq \alpha < 1 \), then \( g(z) \in S^*(\beta) \), where \( 0 \leq \beta \leq \frac{1}{2} \), \( t = 2\beta^2 + \beta - 1 \), and

\[
\gamma = \frac{8t \log 2 - 4t(\log 2)^2 - 3t}{8t \log 2 - 4t(\log 2)^2 - 4t + 2}. \tag{2.3}
\]

**Proof.** Since

\[
\text{Re}\{f'(z)\} = \text{Re}\{g'(z) + zg''(z)\} > \alpha, \tag{2.4}
\]
Lemma 1 gives that
\[ \text{Re}\left\{ \frac{f(z)}{z} \right\} = \text{Re}\{g'(z)\} > 2\alpha - 1 + 2(1 - \alpha) \log 2, \] (2.5)
so that,
\[ \text{Re}\left\{ \frac{g(z)}{z} \right\} > 4\alpha - 3 + 8(1 - \alpha) \log 2 - 4(1 - \alpha)(\log 2)^2. \] (2.6)
Define the function \( w(z) \) by
\[ \frac{zg'(z)}{g(z)} = \beta + (1 - \beta) \frac{1 + w(z)}{1 - w(z)} \quad (w(z) \neq 1). \] (2.7)
Then \( w(z) \) is regular in \( U \) and \( w(0) = 0 \). It is easy to see that
\[ \text{Re}\{f'(z)\} = \text{Re}\{g'(z) + zg''(z)\} \]
\[ = \text{Re}\left\{ \frac{g(z)}{z} \left( (\beta + (1 - \beta) \frac{1 + w(z)}{1 - w(z)})^2 + (1 - \beta) \frac{2zw'(z)}{(1 - w(z))^2} \right) \right\} \]
If we suppose that there exists a point \( z_0 \in U \) such that
\[ \max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1 \quad (w(z_0) \neq 1), \]
then we can write \( w(z_0) = e^{i\theta} \) \((0 \leq \theta < 2\pi)\). Therefore, applying Lemma 2, we have
\[ \text{Re}\{f'(z_0)\} \]
\[ = \text{Re}\left\{ \frac{g(z_0)}{z_0} \left( \beta + (1 - \beta) \frac{1 + w(z_0)}{1 - w(z_0)} + (1 - \beta) \frac{2kw(z_0)}{(1 - w(z_0))^2} \right) \right\} \]
\[ = \left( \beta^2 + (1 - \beta)^2 \cos \theta + 1 + \frac{k(1 - \beta)}{\cos \theta - 1} \right) \text{Re}\left\{ \frac{g(z_0)}{z_0} \right\} \]
\[ \leq \left( \beta^2 - \frac{k(1 - \beta)}{2} \right) \text{Re}\left\{ \frac{g(z_0)}{z_0} \right\} \]
\[ \leq \left( \beta^2 - \frac{(1 - \beta)}{2} \right) \text{Re}\left\{ \frac{g(z_0)}{z_0} \right\} \]
\[ \leq (2\beta^2 + \beta - 1) \left\{ 2\alpha - \frac{3}{2} + 4(1 - \alpha) \log 2 - 2(1 - \alpha)(\log 2)^2 \right\}, \]
because
\[ \Re \left\{ \frac{g(z_0)}{z_0} \right\} > 4\alpha - 3 + 8(1 - \alpha) \log 2 - 4(1 - \alpha)(\log 2)^2 > 0 \]

for \( \gamma \leq \alpha < 1 \), where \( \gamma \) is the root of the equation
\[ (2\beta^2 + \beta - 1) \left\{ 2\gamma - \frac{3}{2} + 4(1 - \gamma) \log 2 - 2(1 - \gamma)(\log 2)^2 \right\} = \gamma. \]

Further, noting that
\[
\frac{2\beta^2 + \beta - 1}{2} < (2\beta^2 + \beta - 1) \left\{ 2\alpha - \frac{3}{2} + 4(1 - \alpha) \log 2 - 2(1 - \alpha)(\log 2)^2 \right\} \leq \gamma
\]

we know that (2.9) contradicts our condition of the theorem. Thus we conclude that \( |w(z)| < 1 \) for all \( z \in U \), that is, that
\[ \Re \left\{ \frac{zg'(z)}{g(z)} \right\} > \beta \quad (z \in U). \tag{2.10} \]

This completes the assertion of the theorem.

Letting \( \beta = 0 \) in Theorem 1, we have

**COROLLARY 1.** If \( f(z) \in R(-0.26228 \cdots) \), then \( g(z) \in S^\ast(0) \), and if \( f(z) \in R(0) \), then \( g(z) \in S^\ast(1/2) \).

**REMARK.** Corollary 1 is the improvement of Theorem A by Singh and Singh [6]. The first half of Corollary 1 was given by Owa [5].

**ACKNOWLEDGMENT.** The author would like to thank the referee for his encouragement and advice for the paper.

**References**


Department of Mathematics  
Kinki University  
Higashi-Osaka, Osaka 577  
Japan