PROPERTIES OF FINITE GROUPS WHOSE IRREDUCIBLE CHARACTER DEGREES ARE PRIMES

JOONG-SANG SHIN

1. Introduction

Let $G$ be a finite group and let $\text{Irr}(G)$ be the set of irreducible complex characters of $G$. Let $\text{c.d.}(G)$ be the set of degrees of all irreducible characters in $\text{Irr}(G)$.

I.M. Issacs and D.S. Passman have been obtained a characterization of groups with the property that every nonlinear irreducible character has a prime degree [4]:

**Theorem 1.** Let $G$ be a finite group with the property that every nonlinear irreducible character has prime degree. Suppose further that at least two distinct primes occur. Then there exist primes $p$ and $q$, $p \neq q$ such that $G$ has one of the following normal series

(I) $G > N > Z(G) = Z(N)$

with $G/Z(G)$ and $N$ both nonabelian.

(II) $G > N > A = Z(N) \times R$

with both $G/A$ and $N$ nonabelian and $Z(G) = Z(N)$. Here $R$ is elementary abelian of order $r^m$ for some prime $r$ and $N/A$ acts irreducibly on it. Also $r^m - 1 = q(r^m - 1)$.

Conversely if group $G$ has either of the above structure then $\text{c.d.}(G) = \{1, p, q\}$.

By the above theorem, a group $G$ with the property that every nonlinear irreducible character has a prime degree is precisely a group with $\text{c.d.}(G) = \{1, p, q\}$, $p \neq q$ primes.

It is not hard to show that the above Theorem can be restated as follow (cf. [6, chapter II]).

Received August 24, 1992.
THEOREM 2. Let $G$ be a finite group with $\text{c.d.}(G) = \{1, p, q\}$, where $p$ and $q$ are distinct primes. Let $K$ be a maximal element in the set 

$$\{\ker \chi | \chi \in \text{Irr}(G), \chi(1) \neq 1\}$$

and let $N/K = (G/K)^I$. Then $G/K$ is a Frobenius group with kernel $N/K$ and $Z(G) = Z(N)$ and one of the following holds.

(I) $|G : N| = p$, $N/K$ is an elementary abelian $q$-group of order $q^2$, $K = Z(G)$, $c.d.(N) = \{1, q\}$ and $c.d.(G/K) = \{1, p\}$

(II) $|G : N| = p$, $N/K$ is an elementary abelian $q$-group of order $q$, $K = Z(G) \times R$ where $R$ is an elementary abelian $q$-group of order $r^m$, $c.d.(N) = \{1, q\}$ and $c.d.(G/K) = \{1, p\}$

REMARK. (i) In (I) of Theorem 2, if $p > q$ then it must be $p = 3$ and $q = 2$. (ii) The case when $p > q$ does not occur in (II) of Theorem 2.

Throughout this paper we fix the notation in Theorem 2 and we say that $G$ is of type (I) and of type (II) if $c.d.(G) = \{1, p, q\}$ and $G$ satisfies the conclusion (I) and (II) in Theorem 2 respectively.

Let $G$ be a finite group and let $p$ and $q$ be primes with $p \neq q$. We denote the matrix of degree type of $G$ by

$$d.t.(G) = \begin{bmatrix} 1 & p & q \\ x & y & z \end{bmatrix}$$

if the following two conditions hold:

(i) $c.d.(G) = \{1, p, q\}$

(ii) $G$ has exactly $x$ linear characters, $y$ irreducible characters of degree $p$ and $z$ irreducible characters of degree $q$.

Our main result is the following.

THEOREM. Let $G$ be a finite group with $c.d.(G) = \{1, p, q\}$, $p \neq q$ primes. Let $s$ be the order of $Z(G)$. Then the following hold.

(i) If $G$ is of type (I) or of type (II) with $r = q$, then the commutator subgroup $G'$ is of order $q^3$ and

$$d.t.(G) = \begin{bmatrix} 1 \\ p \\ q \end{bmatrix} \begin{bmatrix} p & q \\ (q^2-1)p & (q-1)pq \\ q & q \end{bmatrix}$$
(ii) If $G$ is of type (II) and $r \neq q$, then $G' = G$ is of order $qr^m$ and
\[
d.t.(G) = \begin{bmatrix}
    1 & p & (q-1)s \\
    p & (q-1)s & (r^m-1)p^s \\
    1 & p & (r^m-1)p^s
\end{bmatrix}
\]

Remark. It can be shown that the commutator subgroup $G'$ is isomorphic to the extra special group $M(q)$ of order $q^3$ if $G$ is of type (I) or of type (II) with $r = q$ and $G'$ is the semidirect product of an elementary abelian group of order $r^m$ and a group of order $q$.

For notations and terminologies one confer [3].

2. Properties of a finite group $G$ with $c.d.(G) = \{1, p, q\}$

For convenience we describe the following well known theorems without proof (cf. [3])

Theorem (Clifford). Let $N < G$ and let $\chi \in \text{Irr}(G)$. Let $\phi$ be an irreducible constituent of $\chi_N$ and suppose that $\phi = \phi_1, \cdots, \phi_t$ are the distinct conjugates of $\phi$ in $G$. Then $\chi_N = e(\phi_1 + \cdots + \phi_t)$, where $e = [\chi_N, \phi]$ and $t = |G : I_G(\phi)|$.

Theorem (Ito). Let $A < G$ be abelian. Then $\chi(1)$ devides $|G : A|$ for all $\chi \in \text{Irr}(G)$.

Theorem (Gallagher). Let $N < G$ and let $\chi \in \text{Irr}(G)$ be such that $\chi_N = \xi \in \text{Irr}(N)$. Then the characters $\beta \chi$ for $\beta \in \text{Irr}(G/N)$ are irreducible, distinct for distinct $\beta$ and are all of the irreducible constituents of $\xi^G$.

Now we investigate some properties of a finite group $G$ with $c.d.(G) = \{1, p, q\}$.

Proposition 2.1. Suppose that $G$ is of type (I). Let $Q$ be a Sylow $p$-subgroup of $N$. Then $Q' = G'$ and $N' = G''$ is of order $q$.

Proof. Let $P$ be a Sylow $p$-subgroup of $G$. Since $G/Z(G) = G/K$ is of order $pq^2$, the factor group $P/Z(P)$ is a cyclic group and so $P$ is abelian. It follows [3, cor 12. 34] that every Sylow $s$-subgroup of $G$ is abelian normal in $G$ for all primes $s \neq p, q$. Thus we have $|G' \cap Z(G)|$ is a power of $q$ (cf. [3, theorem 5.6]).
Since $N/K = (GK)'$ is of order $q^2$, we have

$$q^2|Z(G)| = |N| = \frac{|G'||Z(G)|}{G' \cap Z(G)}$$

and so $|G'| = q^2 |G' \cap Z(G)|$. Thus $G'$ is a $q$-group.

Since c.d.$(N) = \{1, q\}$, the subgroup $N$ has a normal $q$-complement $L$ (cf. [3, cor 12.2]). Since $|N : Z(N)| = q^2$, we have $L \subset Z(N)$. Now it follows that $Q \triangleleft N$ since $N = QL$. Moreover, $Q$ is also a Sylow $q$-subgroup of $G$ and a characteristic subgroup of $N$. Thus $Q \triangleleft G$ and $G' \subset Q$.

Let $B$ be a normal subgroup of $N$ with $|B : K| = q$. Then $B/Z(B)$ is cyclic and hence $B$ is abelian. Now it follows from [3, lemma 12.12] that

$$|B| = |N'| |Z(N)| = |N'||K|$$

and so $|N'| = q$.

Since $N = G'Z(G) = QZ(G)$, we have $N' = G'' = Q'$. Thus the proposition holds.

**Proposition 2.2.** Suppose that $G$ is of type (II) and $r \neq q$. Then $G'' = N'$ and $N'$ is an elementary abelian group of order $r^m$.

**Proof.** Let $Q$ be a Sylow $q$-subgroup of $N$. Since $K = Z(G) \times R$ for some elementary abelian group $R$ of order $r^m$, it follows that $G/Z(G)$ is of order $pqr^m$ and so $|Q/Z(Q)| \leq q$. Thus $Q$ is abelian.

Since c.d.$(N) = \{1, q\}$, every Sylow subgroup of $N$ is abelian (cf. [3, cor 12.34]). By Theorem 5.6 of [3], we have

$$N' \cap Z(N) = \{1\}.$$

Note that $K$ is abelian and $N/K$ is cyclic of order $q$. Thus we have the following (cf. [3, Lemma 12.12])

$$|K| = |N'||K \cap Z(N)| = |N'||Z(N)|.$$

Since $N'Z(N) \subset K$, we have $K = N' \times Z(N)$.

Now it follows from $K = R \times Z(N)$ that $N' \cong R$ and so $N'$ is an elementary abelian group of order $r^m$.

Note that $N = G'K$ and $N' \subset G'$. Thus we have

$$N = G'K = G'N'Z(N) = G'Z(G)$$

and so $N' = G''$. 


PROPOSITION 2.3. Suppose that $G$ is a finite group with $c.d.(G) = \{1,p,q\}$. Then the set of all irreducible characters of $G$ of degree $q$ is precisely the set of all irreducible constituents of $\phi^G$, where $\phi$ runs over all nonlinear irreducible characters of $G'$.

Proof. Let $\phi$ be an irreducible character of $G'$ of degree $q$. If $\chi$ be an irreducible constituent of $\phi^G$, then Clifford Theorem yields that $\chi(1) = q$.

Now we show that every irreducible character of $G$ of degree $q$ is an irreducible constituent of $\phi^G$ for some $\phi \in \text{Irr}(G')$ of degree $q$.

Let $\chi$ be an irreducible character of $G$ of degree $q$. Assume that $\chi_N$ has a linear constituent $\theta$. Then Gallagher Theorem yield that $\theta^G$ has no linear constituent since $\chi$ is an irreducible constituent of $\theta^G$. Moreover, $\theta^G$ has no irreducible constituent of degree $p$ since $\theta^G(1) = p$ and $\chi$ is an irreducible constituent of $\theta^G$. Thus every irreducible constituent of $\theta^G$ is of degree $q$. Since $p$ and $q$ are distinct primes, this is not the case. Thus $\chi_N$ is an irreducible character of degree $q$ and so $N' \not\subseteq \ker \chi_N$.

Now let $\phi$ be an irreducible constituent of $\chi_{G'}$. Then, by Clifford Theorem, we have

$$\chi_{G'} = e(\phi_1 + \phi_2 + \cdots + \phi_t)$$

where $\phi_1, \cdots, \phi_t$ are the distinct conjugates of $\phi$ in $G$ and $e = [\phi, \chi_{G'}]$. Since

$$\bigcap_{i=1}^t \ker \phi_i = \ker \chi_{G'} = G' \cap \ker \chi_N \not\subseteq N'$$

we have $N' \not\subseteq \ker \phi_i$ for some $i$.

Since $N' = G''$ by Proposition 2.1 and Proposition 2.2, the character $\phi_i$ is not linear. This implies that $\phi(1) = q$ and so $\chi_{G'} = \phi \in \text{Irr}(G')$. Thus $\chi$ is an irreducible constituent of $\phi^G$. This completes the proof.

Let $G$ be a finite group. If $H \triangleleft G$ and $\chi$ is a character of $G$ with $H \subseteq \ker \chi$, then there is a unique character $\tilde{\chi}$ of $G/H$ defined by $\tilde{\chi}(Hg) = \chi(g)$. This formula can also be used to define the character $\chi$ if $\tilde{\chi}$ is given. It is immediate consequence that $\chi$ is irreducible if and only if $\tilde{\chi}$ is. In this paper, we will not distinguish between $\chi$ and $\tilde{\chi}$. 
3. Main Result

In this section we will prove our main Theorem.

**Theorem 3.1.** Suppose that $G$ is of type (I) or of type (II) with $r = q$. Let $s = |Z(G)|$. Then the commutator subgroup $G'$ is of order $q^3$ and

$$d.t.(G) = \left[ \begin{array}{ccc} 1 & \frac{p}{q^3} & \frac{(q^2-1)s}{pq} \\ \frac{ps}{q} & \frac{q}{p} & \frac{(q-1)ps}{q} \end{array} \right]$$

**Proof.** Let $\phi \in \text{Irr}(G)$. If $\phi(1) = 1$, then $\phi_K \in \text{Irr}(K)$ and

$$N' \subset K \cap G' \subset \ker \phi_K.$$

Thus $\phi_K \in \text{Irr}(K/N')$.

Now suppose that $\phi(1) = p$. Since $K = Z(G)$, every irreducible constituent of $\phi_K$ is invariant in $G$. Thus it follows by Clifford Theorem that $\phi_K = p\theta$ for some $\theta \in \text{Irr}(K)$. Since $c.d.(N) = \{1, q\}$ and $|G : N| = p$, we have

$$\phi_N = \chi_1 + \chi_2 + \cdots + \chi_p$$

for some linear characters $\chi_1, \chi_2, \cdots, \chi_p$ of $N$.

Since $N' \subset K \cap \ker \chi_i = \ker(\chi_i)_K$ for all $i = 1, 2, \cdots, p$, we have

$$N' \subset \ker \phi_K.$$

Since $\phi_K = p\theta$, it follows that $\theta \in \text{Irr}(K/N')$ and $\phi$ is an irreducible constituent of $\theta^G$.

In the proof of Proposition 2.3, we showed that if $\phi$ is an irreducible character of $G$ of degree $q$ then $\phi_N \in \text{Irr}(N)$ and $N' \not\subset \ker \phi_N$. In this case we have $N' \not\subset \ker \phi_K$ since $N' \subset K$.

Now we can conclude that $\theta \in \text{Irr}(K/N')$ is extendible to $G$ and every irreducible constituent of $\theta \in \text{Irr}(K) - \text{Irr}(K/N')$ is of degree $q$.

Since $K/N'$ is abelian and $G/K$ has $p$ linear characters and $\frac{(q^2-1)}{p}$ irreducible characters of degree $p$, it follows by Gallagher Theorem that $G$ has $p|K : N'|$ linear characters and $|K : N'|\frac{(q^2-1)}{p}$ irreducible characters of degree $p$. 
Properties of finite groups whose irreducible character degrees are primes

If \( \phi \in \text{Irr}(G) \) is of degree \( q \), then \( \phi_K = q\theta \) for some \( \theta \in \text{Irr}(K) \) - \( \text{Irr}(K/N') \). In this case, \( \theta^G \) has \( p \) irreducible constituent of degree \( q \). Thus \( G \) has \( p(|K| - |K : N'|) \) irreducible characters of degree \( q \).

Note that \( G \) has \( |G : G'| \) linear characters. Thus \( |G : G'| = p|K : N'| \) and then

\[
p|K : K \cap G'| = p|K^G : G'| = |G : G'| = p|K : N'|
\]

Thus \( |K \cap G' : N'| = 1 \). That is, \( K \cap G' = N' \).

Since \( |N : K| = q^2 \) and \( |N'| = q \), we have \( |G' : K \cap G'| = |N : K| = q^2 \) and

\[
|G'| = |G' : K \cap G'| |K \cap G'| = q^2 |N'| = q^3.
\]

**Remark.** If \( G \) is of type (II) with \( r = q \), then \( |N : Z(G)| = q^2 \) (cf. [4]). The proof of this case is similar to type (I).

**Theorem 3.2.** Suppose that \( G \) is of type (I) and \( r \neq q \). Let \( s = |Z(G)| \). Then the commutator subgroup \( G' \) is of order \( qr^m \) and

\[
d.t.(G) = \begin{bmatrix}
1 & p & (r^m q - 1) p s \\
p & (q - 1)s & (r^m - 1) p s
\end{bmatrix}
\]

**Proof.** Let \( \theta \) be an irreducible character of \( Z(G) \). Assume that \( \theta^N \) has \( x \) linear constituents and \( y \) irreducible constituents of degree \( q \). Then since \( \theta \) is invariant in \( G \) and \( \text{c.d.}(N) = \{1, q\} \), we have the equation

\[
qr^m = \theta^N(1) = x + qy^2.
\]

Note that \( q|(r^m - 1) \) (cf. Theorem 1). Thus \( x \neq 0 \) and so \( \theta \) is extendible to \( N \). Since \( (N/Z(G))' = N'Z(G)/Z(G) = K/Z(G) \), \( \theta^N \) has \( \frac{r^m - 1}{q} \) irreducible constituents of degree \( q \) by Gallagher Theorem.

Note that every irreducible character of \( N \) of degree \( q \) is extendible to \( G \) (cf. Proof of Proposition 2.3).

Since \( (\theta^N)^G = \theta^G \), Gallagher Theorem yields that \( \theta^G \) has \( \frac{p(r^m - 1)}{q} \) irreducible constituents of degree \( q \).

Assume that \( \theta \) is not extendible to \( G \). Then since \( \theta \) is invariant in \( G \), we have the equation

\[
pqr^m = \theta^G(1) = xp^2 + qy^2
\]
where $x$ and $y$ are the numbers of irreducible constituents of degree $p$ and of degree $q$ respectively. But since $y = \frac{p(r^m - 1)}{q}$ and $p \nmid q$, it is not the case.

Thus every $\theta \in \text{Irr}(Z(G))$ is extendible to $G$.

Now, by applying Proposition 2.3 to $G/Z(G)$, it follows that $G/Z(G)$ has $p$ linear characters, $\frac{(q-1)}{p}$ irreducible characters of degree $p$ and $\frac{(r^m-1)p}{q}$ irreducible characters of degree $q$. Thus, by Gallagher Theorem, $G$ has $p|Z(G)|$ linear characters, $\frac{(q-1)|Z(G)|}{p}$ irreducible characters of degree $p$ and $\frac{(r^m-1)p|Z(G)|}{q}$ irreducible characters of degree $q$.

Finally we show that $G'$ is of order $qr^m$.

Note that $G$ has $|G : G'|$ linear characters and that

$$
N/G' = G'Z(G)/G' \cong Z(G)/G' \cap Z(G).
$$

Thus we have

$$
p|Z(G)| = |G : G'| = p|Z(G) : G' \cap Z(G)|
$$

and so $G' \cap Z(G) = \{1\}$.

Since $G'Z(G)/Z(G) \cong G'/G' \cap Z(G)$, we have

$$
|G'| = |G' : G' \cap Z(G)||G' \cap Z(G)|
$$

$$
= |G'Z(G) : Z(G)|
$$

$$
= |N : Z(G)|
$$

$$
= qr^m.
$$

We have proved THEOREM which is introduced in section 1.

References


Department of Mathematics  
Kyung-won University  
Sung-nam 461-200, Korea