THE EULER CHARACTERISTIC OF
A MANIFOLD WITH BOUNDARY

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1. Introduction

The Morse theory says that for an admissible Morse function $f$ on
a manifold without boundary if $f^{-1}[a, b]$ contains critical points of
index $k$, the manifold is obtained from $f^{-1}[a]$ by attaching $k$-cells up to
homotopy equivalence. Hence the indices of critical points of a Morse
function determine the topology of the manifold. In [1], we used Morse
theory on the distance function to characterize compact $n$-manifold in
$n$-space. In this paper, we study the topology of compact manifolds
with boundary by using results from [1]. It turns out that the homology
types of the manifold do not change at the outward critical points, and
changes at the inward critical points (cf. Lemma 2). From this we
derive a formula for the Euler characteristic of a compact $n$-manifold
$W$ in $n$-space with $C^2$-boundary in terms of the directional type of
critical points of the distance function (cf. Theorem 1).

2. Preliminaries

The tangent space of a $C^2$-manifold $M$ at a point $p$ will be denoted
by $T_pM$, and if $g : M \rightarrow N$ is a $C^2$-map with $g(p) = q$, then the induced
linear map of tangent space will be denoted by $g_* : T_pM \rightarrow T_qN$.

**DEFINITION 1.** Let $f$ be a $C^2$-real valued function on a manifold
$M$. A point $p \in M$ is called a critical point of $f$ if the induced map
$f_* : T_pM \rightarrow T_{f(p)}\mathbb{R}$ is zero, and a critical point $p$ of $f$ is called nonde-
generate if the Hessian matrix $H = \left( \frac{\partial^2 f}{\partial x_i \partial x_j}(p) \right)$ is nonsingular.

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If we choose a local coordinate system \((x_1, \ldots, x_n)\) in a neighborhood \(U\) of a critical point \(p\), we have

\[
\nabla f(p) = \left( \frac{\partial f}{\partial x_1}(p), \ldots, \frac{\partial f}{\partial x_n}(p) \right) = 0.
\]

The real number \(f(p)\) is called a critical value of \(f\).

**Definition 2.** Let \(p \in M\) be a nondegenerate critical point of \(f : M \rightarrow \mathbb{R}\). The index of \(p\) is the index of the matrix \(H\), which is the number of negative eigenvalues of \(H\), counting multiplicities.

This index gives us valuable information about the local behavior of \(f\) near \(p\) by the Morse lemma (see [3], page 6).

**Definition 3.** The number of critical points of index \(k\), \(0 \leq k \leq n\), of a Morse function \(f : M \rightarrow \mathbb{R}\) is called the \(k\)-th type number of \(f\). We denote it by \(C_k(f)\), and \(f\) is said to be of the type \((C_0, \ldots, C_n)\).

Let \(M\) be a manifold (or a hypersurface) in \(\mathbb{R}^n\). For a fixed \(a \in \mathbb{R}^n\), the distance function \(d_a : M \rightarrow \mathbb{R}\) is defined by

\[
d_a(p) = \|p - a\|^2,
\]

for \(p \in M\). It is known that for all \(a \in \mathbb{R}^n - \{\text{focal points of } M\}\), the function \(d_a\) is a Morse function, that is, it has only nondegenerate critical points.

Throughout this paper, we assume that \(W\) is a compact \(n\)-manifold in \(\mathbb{R}^n\) with \(C^2\)-boundary \(\partial W\). For a fixed point \(a \in \mathbb{R}^n\), we denote by \(d|_{\partial W}\) the restriction of the distance function \(d_a\) on \(\partial W\). Note that a critical point of \(d_a\) on \(\partial W\) means that of \(d|_{\partial W}\).

Note that, since

\[
\frac{\partial d_a}{\partial x_i} = 2(p - a) \cdot \frac{\partial p}{\partial x_i},
\]

\(d_a\) has a critical point at \(p\) if and only if \(p - a\) is normal to \(M\) at \(p\).

**Definition 4.** A critical point \(p \neq a\) of the distance function \(d_a\) is called an inward critical point, if the outward normal vector at \(p\) has the opposite direction to the vector \(p - a\). Otherwise, \(p\) is called an outward critical point.
From the observation above, for \( a \notin \partial W \cup \{ \text{focal points of } \partial W \} \), each critical point of the Morse function \( d|_{\partial W} : \partial W \to \mathbb{R} \) is of either inward or outward one. Hence we divide the type numbers of \( d|_{\partial W} \) according to the directional types of the critical points: Let \( I_k \) and \( O_k \) denote, respectively, the numbers of inward and outward critical points of index \( k \), \( 0 \leq k \leq n - 1 \).

**Definition 5.** Let \( I_0, \ldots, I_{n-1} \) denote the inner type numbers and \( O_0, \ldots, O_{n-1} \) the outer type numbers of \( d|_{\partial W} \). The function \( d|_{\partial W} \) is said to be of the type \((I_0, \ldots, I_{n-1}; O_0, \ldots, O_{n-1})\).

### 3. Computations of the Euler characteristic

**Lemma 1.** Let \( s < t \). Suppose that \( d|_{\partial W}^{-1}(s, t] \) contains no critical point, and \( d|_{\partial W} \) is a Morse function. Then

\[
H_i(d|_{\partial W}^{-1}[0, s]) \cong H_i(d|_{\partial W}^{-1}[0, t]),
\]

for any \( i \geq 0 \).

**Proof.** See [1].

Note that, for a manifold without boundary, we can have

\[
H_i(d|_{\partial W}^{-1}[0, r_1]) \cong H_i(d|_{\partial W}^{-1}[0, r_1 + \delta]),
\]

and the torus with the height function shows that we can not have

\[
H_i(d|_{\partial W}^{-1}[0, r_1 - \delta]) \cong H_i(d|_{\partial W}^{-1}[0, r_1]).
\]

However, for a manifold with boundary, if a critical value \( r_1 \) has only outward critical points, then it seems that we could have

\[
H_i(d|_{\partial W}^{-1}[0, r_1 - \delta]) \cong H_i(d|_{\partial W}^{-1}[0, r_1]) \cong H_i(d|_{\partial W}^{-1}[0, r_1 + \delta]).
\]

For instance, consider the solid torus and the height function: There are four critical points: the first one is inward, the second outward, the third inward and the fourth outward. At the second outward critical point, we have

\[
H_i(d|_{\partial W}^{-1}[0, r_1 - \delta]) \cong H_i(d|_{\partial W}^{-1}[0, r_1]) \cong H_i(d|_{\partial W}^{-1}[0, r_1 + \delta]),
\]

since they are all the same homotopy type. The following Lemma says this is true in general. Note that at the third inward critical point, we do not have

\[
H_i(d|_{\partial W}^{-1}[0, r_1 - \delta]) \cong H_i(d|_{\partial W}^{-1}[0, r_1]).
\]
Lemma 2. If \( r_1 \) is a positive critical value of Morse function \( d|_{\partial W} \) such that all the critical points at the level \( r_1 \) are outward, then there is an \( \epsilon > 0 \) such that for every \( i \geq 0 \) and \( \delta \) with \( 0 \leq \delta \leq \epsilon \),

\[
H_i(d^{-1}[0,r_1]) \cong H_i(d^{-1}[0,r_1-\delta]).
\]

**Proof.** See [1].

This shows that for an \( n \)-dimensional manifold with smooth boundary in \( \mathbb{R}^n \), the homology types do not change at an outward critical point. This is the key point in the following main theorem.

**Theorem.** Let \( W \) and \( d \) be as in the above definition. Suppose \( a \notin W \). If \( \beta_k \) denotes the rank of \( H_k(W ; \mathbb{R}) \), then the alternating sum of inner type numbers equals the Euler characteristic of \( W \):

\[
\sum_{k=0}^{n-1} (-1)^k I_k(d|_{\partial W}) = \sum_{k=0}^{n-1} (-1)^k \beta_k = \sum_{k=0}^{n-1} (-1)^k O_{n-k}(d|_{\partial W}).
\]

**Proof.** The last equality comes from the following observations:

\[
I_k(-d|_{\partial W}) = O_{n-k}(d|_{\partial W}),
\]

\[
\sum_{k=0}^{n-1} (-1)^k \beta_k = \sum_{k=0}^{n-1} (-1)^k I_k(-d|_{\partial W}).
\]

Let \( r_1, \ldots, r_m \) be critical values of \( d|_{\partial W} \) with \( 0 < r_1 < r_2 < \cdots < r_m \) such that, for each \( i \) with \( 1 \leq i \leq m \), \( d|_{\partial W}^{-1}(r_i) \) contains at least one inward critical point. For all outward critical points \( p_{i_1}, \ldots, p_{i_k} \) belonging to \( r_i \), we now construct a new space \( W_* \) from \( W \) such that \( \partial W_* \) is smooth, \( d|_{\partial W_*}^{-1}[r_1-\epsilon,r_1] \) contains no critical points of \( d|_{\partial W} \) for sufficiently small \( \epsilon \), and \( d|_{\partial W_*} \) is a Morse function on \( \partial W_* \). Since \( d|_{\partial W} \) is a Morse function, the critical points \( p_{i_1}, \ldots, p_{i_k} \) are nondegenerate critical points. By the Morse lemma (cf. [3], page 6), there are disjoint neighborhoods \( U_{i_1}, \ldots, U_{i_k} \) in \( \partial W \) of \( p_{i_1}, \ldots, p_{i_k} \), respectively. For each \( p_{ij} \), we can find a smooth function \( \lambda_{ij} : \partial W \to [0,1] \) with support in \( U_{ij} \) and with its value 1 on some neighborhood \( V_{ij} \subset U_{ij} \) of \( p_{ij} \). Without lose of generality, we may assume \( a = 0 \) in \( \mathbb{R}^n \).
We define $W_*$ as the union:

$$W \cup \left\{ y = x + \sum_{j=1}^{k} t\epsilon_{ij} \lambda_{ij}(x) \left\| \frac{x}{\|x\|} \right\| \in \mathbb{R}^n \mid x \in \partial W, 0 \leq t \leq 1, \epsilon_{ij} > 0 \right\}.$$

Note that $W_* \to W$ as $\max \epsilon_{ij} \to 0$, and $W_*$ is a manifold with $C^2$ boundary $\partial W_*$. Note also that $d|_{\partial W_*}$ is a smooth function and has an expression of the following form: for $y = x + \sum_{j=1}^{k} \epsilon_{ij} \lambda_{ij}(x) \| \frac{x}{\|x\|} \| \in \partial W_*$ (i.e., $t = 1$),

$$d|_{\partial W_*}(y) = d|_{\partial W}(x) + 2 \sum_{j=1}^{k} \epsilon_{ij} \lambda_{ij}(x) \sqrt{d|_{\partial W}(x)} + \sum_{j=1}^{k} (\epsilon_{ij} \lambda_{ij}(x))^2.$$

We now consider a function $f : \partial W \to \partial W_*$ defined as: for $x \in \partial W$,

$$f(x) = x + \sum_{j=1}^{k} \epsilon_{ij} \lambda_{ij}(x) \left\| \frac{x}{\|x\|} \right\|.$$

Then

$$d|_{\partial W_*} \circ f(x) = d|_{\partial W}(x) + 2 \sum_{j=1}^{k} \epsilon_{ij} \lambda_{ij}(x) \sqrt{d|_{\partial W}(x)} + \sum_{j=1}^{k} (\epsilon_{ij} \lambda_{ij}(x))^2.$$

Therefore,

$$\nabla(d|_{\partial W_*} \circ f)(x) = \nabla(d|_{\partial W_*(f(x))}) \cdot \nabla f(x)$$

$$= \nabla d|_{\partial W}(x) + 2 \sum_{j=1}^{k} \epsilon_{ij} \sqrt{d|_{\partial W}(x)} \nabla \lambda_{ij}(x)$$

$$+ 2 \sum_{j=1}^{k} \epsilon_{ij} \lambda_{ij}(x) \nabla (\sqrt{d|_{\partial W}(x)})$$

$$+ 2 \sum_{j=1}^{k} (\epsilon_{ij})^2 \lambda_{ij}(x) \nabla \lambda_{ij}(x).$$
Thus, if \( x_0 \in \bigcup_{j=1}^{k} U_{ij} \) is not a critical point of \( d|_{\partial W} \), then \( f(x_0) \) is not a critical point of \( d|_{\partial W_*} \). On the other hand, if \( x_0 \notin \bigcup_{j=1}^{k} U_{ij} \) is not a critical point, then obviously \( f(x_0) = x_0 \) is not a critical point of \( d|_{\partial W_*} \). Since \( r_1 \) is a critical value of \( d|_{\partial W} \), there is a small \( \epsilon > 0 \) so that \( d|_{\partial W}^{-1}[r_1 - \epsilon, r_1) \) contains no outward critical points. Thus \( d|_{\partial W_*}^{-1}[r_1 - \epsilon, r_1] \) contains only inward critical points. Therefore, by replacing \( W \) by \( W_* \), if necessary, we may assume that each \( d|_{\partial W}^{-1}(r_i) \) contains only inward critical points.

We now set, for sufficiently small \( \epsilon > 0 \),
\[
X_0 = d^{-1}[0, r_1 - \epsilon] = \phi,
\]
\[
X_i = d^{-1}[0, r_i], \quad \tilde{X}_i = d^{-1}[0, r_{i+1} - \epsilon],
\]
\[
X_m = d^{-1}[0, r_m],
\]
where \( i = 1, \ldots, m - 1 \).

If there is a critical value \( \tilde{r}_i \) with \( r_i < \tilde{r}_i < r_{i+1} \), then \( d^{-1}(\tilde{r}_i) \) contains only outward critical points by the choice of \( r_i \)'s. Hence, by the Lemmas 1 and 2 and the remark above the Lemma 2, for each \( k \) and \( i = 1, \ldots, m - 1 \), we have \( H_k(X_i) \cong H_k(\tilde{X}_i) \).

Let \( \tilde{d} \) be the distance function on the closure of \( \mathbb{R}^n - W \) with the origin \( a \). By the Mayer-Vietories exact homology sequence, we have
\[
H_k(d|_{\partial W}^{-1}[0, r_i - \epsilon]) \cong H_k(X_{i-1}) \oplus H_k(\tilde{d}^{-1}[0, r_i - \epsilon])
\]
\[
H_k(d|_{\partial W}^{-1}[0, r_i]) \cong H_k(X_i) \oplus H_k(\tilde{d}^{-1}[0, r_i]),
\]
for sufficiently small \( \epsilon > 0 \).

Note that the closure of \( \mathbb{R}^n - W \) is a manifold with \( C^2 \)-boundary \( \partial W \), and the critical points of \( d \) belonging to \( r_i \)'s are all outward critical points of \( \tilde{d} \) on the closure of \( \mathbb{R}^n - W \). Thus by the Lemma 2 again, we get
\[
rkH_k(X_i) - rkH_k(X_{i-1}) = rkH_k(d|_{\partial W}^{-1}[0, r_i]) - rkH_k(d|_{\partial W}^{-1}[0, r_i - \epsilon]).
\]

Since \( d|_{\partial W}^{-1}[0, r_i] \) is obtained from the empty set by adding cells of dimension \( \leq n - 1 \), \( H_k(d|_{\partial W}^{-1}[0, r_i]) = 0 \) for \( k > n - 1 \). Hence, for each \( i \), we have the following exact sequence:
\[
0 \rightarrow H_{n-1}(\tilde{Y}_{i-1}) \rightarrow H_{n-1}(Y_i) \rightarrow H_{n-1}(Y_i, \tilde{Y}_{i-1})
\]
\[
\rightarrow H_{n-2}(\tilde{Y}_{i-1}) \rightarrow \cdots \rightarrow H_0(Y_i, \tilde{Y}_{i-1}) \rightarrow 0,
\]
where \( Y_i = d|_{\partial W}^{-1}[0, r_i] \) and \( \check{Y}_{i-1} = d|_{\partial W}^{-1}[0, r_i - \epsilon] \).

Exactness implies the vanishing of the corresponding alternating sum of dimensions of the vector spaces in the sequence. Observe that \( H_i(X_m) = H_i(W) \), for all \( i \). Let

\[
\beta_i = rk H_i(X_m), \quad \beta(i, j) = rk H_i(X_j),
\]
\[
\gamma(i, j) = rk H_i(Y_j), \quad \alpha(i, j) = rk H_i(Y_j, \check{Y}_{j-1}).
\]

Then

\[
\sum_{k=0}^{n-1} (-1)^k \beta_k = \sum_{k=0}^{n-1} (-1)^k (\beta_k - \beta(k, 0))
\]
\[
= \sum_{k=0}^{n-1} (-1)^k \sum_{j=1}^{m} (\beta(k, j) - \beta(k, j - 1))
\]
\[
= \sum_{k=0}^{n-1} (-1)^k \sum_{j=1}^{m} (\gamma(k, j) - \gamma(k, j - 1))
\]
\[
= \sum_{j=1}^{m} \sum_{k=0}^{n-1} (-1)^k (\beta(k, j) - \beta(k, j - 1))
\]
\[
= \sum_{k=0}^{n-1} (-1)^k \sum_{j=1}^{m} \alpha(k, j) = \sum_{k=0}^{n-1} (-1)^k I_k.
\]

**Corollary 1.** If \( W \) is a compact \( n \)-manifold with \( C^2 \)-boundary \( \partial W \), then

\[
\chi(\partial W) = \begin{cases} 
2 \cdot \chi(W), & \text{if } n \text{ is odd,} \\
0, & \text{if } n \text{ is even,}
\end{cases}
\]

where \( \chi \) denotes the Euler characteristic.

**Proof.** This follows immediately from the Theorem above and the fact that

\[
\chi(\partial W) = \sum_{k=0}^{n-1} (-1)^k (I_k(d|_{\partial W}) + O_k(d|_{\partial W})).
\]
References


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