SOME COMPUTATIONS OF RELATIVE NIELSEN NUMBERS

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1. Introduction

H. Schirmer introduced the relative Nielsen number $N(f; X, A)$ in [8] which is a lower bound for the number of fixed points for all maps in the relative homotopy class of $f$. In [10] the Nielsen number of the boundary $n(f; X, A)$ is a lower bound for the number of fixed points on the boundary of $A$ denoted $\text{Bd } A$ only when any selfmap of $(X, A)$ has a minimal fixed point set and the Nielsen number of the complement $\bar{N}(f; X, A)$, that is, the number of fixed point classes of $f : X \to X$ which do not assume their index in $A$, is a lower bound for the number of fixed points on $\text{Cl}(X - A)$.

In the classical setting, where $A = \phi$, $f_\pi(\pi_1(X)) \subset J(f)$ the trace subgroup of cyclic homotopies and $R(f)$ the Reidemeister number of $f$ introduced in section 3, it follows from $L(f) \neq 0$ that $N(f) = R(f)$ [5, p.33, Theorem 4.2]. It is a purpose of this paper to generalize this fact to maps of pairs of spaces (Theorem 3.1 and 3.2). In section 2, $n(f; X - A)$ will be defined and we will show that maps in the homotopy class of $f$ which have a $N(f; X - A)$ fixed points on $X - A$ must have at least $n(f; X - A)$ fixed points on $\text{Bd } A$, and we will calculate $n(f; X - A)$ in some special cases. In section 3, methods to compute relative Nielsen numbers with relative Lefschetz numbers are given. Throughout this paper, $f : (X, A) \to (X, A)$ will be a selfmap of a pair of compact polyhedra with $X$ connected and we will follow the notations and terminology of [10].

Received September 24, 1992.
Supported by a TGRC-KOSEF grant.
2. Relative Nielsen numbers

Let \( f : (X, A) \rightarrow (X, A) \) be a selfmap of a pair of compact polyhedra. We shall write \( f : A \rightarrow A \) for the restriction of \( f \) to \( A \) and write \( f : X \rightarrow X \) if the condition that \( f(A) \subset A \) is immaterial. Let \( \tilde{A} = \bigcup A_k \) be the disjoint union of all components of \( A \) which are mapped by \( f \) into themselves, and we shall write \( f_k : A_k \rightarrow A_k \) for the restriction of \( f \) to \( A_k \). We write \( \text{Fix} \ f \) for the fixed point set \( \{ x \in X | f(x) = x \} \) and \( F \) for a fixed point class of \( f : X \rightarrow X \). A fixed point class \( F \) of \( f : X \rightarrow X \) is a weakly common fixed point class if and only if it contains a fixed point class of \( f_k : A_k \rightarrow A_k \) for some \( k \) [13]. H. Schirmer defined the fixed point class \( F \) of \( f : X \rightarrow X \) assumes its index in \( A \) if

\[
\text{ind}(X, f, F) = \text{ind}(A, \tilde{f}, F \cap A).
\]

Also she defined the relative Nielsen number \( \bar{n}(f; X, A) \) by the number of common fixed point classes of \( f : X \rightarrow X \) which do not assume their index in \( A \) [10].

**Definition 2.1.** The number of weakly common fixed point classes of \( f : X \rightarrow X \) which do not assume their index in \( A \) is called the relative Nielsen number of the boundary space \( \text{Bd} \ A \), denoted \( n(f; X - A) \).

It is clear \( n(f; X - A) \geq \bar{n}(f; X, A) \) by the definition. In general, \( n(f; X - A) \) is different from \( \bar{n}(f; X, A) \). As a simple example, if we take the identity map \( f : (B^2, S^1) \rightarrow (B^2, S^1) \) of the pair of a 2-dimensional ball and its boundary, then \( \bar{n}(f; X, A) = 0 \), but \( n(f; X - A) = 1 \). If all the fixed point classes of \( \tilde{f} \) are essential, then \( n(f; X - A) = \bar{n}(f; X, A) \).

In [13], \( N(f; X - A) \) is defined by the number of essential fixed point classes of \( f : X \rightarrow X \) which are not weakly common fixed point classes, and \( E(f, \tilde{f}) \) is defined by the number of essential weakly common fixed point classes of \( f \) and \( \tilde{f} \).

**Theorem 2.1.** If \( f : (X, A) \rightarrow (X, A) \) is a map, then \( \tilde{N}(f; X, A) = n(f; X - A) + N(f) - E(f, \tilde{f}) \) and hence \( n(f; X - A) = \tilde{N}(f; X, A) - N(f; X - A) \).

**Proof.** Let \( F \) be a fixed point class of \( f : X \rightarrow X \). A fixed point class \( F \) which is not a weakly common fixed point class of \( f \) and \( \tilde{f} \) is essential if and only if it does not assume its index in \( A \), and so we
get \( n(f; X - A) + [N(f) - E(f, \tilde{f})] \leq \# \{ F|F \text{ is a weakly common fixed point class of } f \text{ and } \tilde{f} \text{ and does not assume its index in } A \} + \# \{ F|F \text{ is not a weakly common fixed point class of } f \text{ and } \tilde{f} \text{ which is essential} \} = \# \{ F|F \text{ is a weakly common fixed point class of } f \text{ and } \tilde{f} \text{ and does not assume its index in } A \} + \# \{ F|F \text{ is not a weakly common fixed point class of } f \text{ and } \tilde{f} \text{ and does not assume its index in } A \} = \tilde{N}(f; X, A).

\[ N(f; X - A) + E(f, \tilde{f}) = N(f) \] shows the second part of the theorem.

The lower bound property of \( n(f; X - A) \) follows immediately from Theorem 2.1.

**Theorem 2.2 (Lower Bound).** Any map \( f : (X, A) \to (X, A) \) which has \( N(f; X - A) \) fixed points on \( X - A \) has at least \( n(f; X - A) \) fixed points on \( \partial A \).

**Example 2.1.** Let \( X = B^{n+1} \) be the unit ball \( \{ x \in \mathbb{R}^{n+1} ||x|| \leq 1 \} \) in \( \mathbb{R}^{n+1} \) for \( n \geq 2 \), \( A = \{ x \in X | 1/2 \leq ||x|| \leq 1 \} \), and \( f : (X, A) \to (X, A) \) be the identity, then

\[
N(\tilde{f}) = \begin{cases} 
0, & \text{if } n \text{ is odd} \\
1, & \text{if } n \text{ is even},
\end{cases}
\]

\[
\tilde{N}(f; X, A) = n(f; X - A) = 1.
\]

By [9, Theorem 4.1], there exists a deformation \( g : (X, A) \to (X, A) \) such that if \( n \) is odd, then \( g \) has \( N(f; X, A) \) fixed point on \( X - A \) and no further fixed point.

### 3. Main results

Pick a base point \( a_k \in A_k \) for each \( A_k \subset \hat{A} \) and a base point \( x_0 \in X \). It is well known that the covering translations of universal covering spaces \( \hat{A}_k \) and \( \hat{X} \) of \( A_k \) and \( X \) form groups \( \mathcal{D}_k = \mathcal{D}_k(A_k, p_k) \) and \( \mathcal{D} = \mathcal{D}(\hat{X}, p) \) which are isomorphic to \( \pi_1(A_k) \) and \( \pi_1(X) \) respectively. Recall that points of \( \hat{A}_k \) and \( \hat{X} \) are respectively in one-to-one correspondence with the path classes in \( A_k \) and \( X \) starting from \( a_k \) and \( x_0 \). Under this identification, let \( \tilde{a}_k = (e_k) \in \hat{A}_k \) and \( \tilde{x}_0 = (e) \in \hat{X} \) be the constant paths. Pick a path \( w_k \) in \( A_k \) from \( a_k \) to \( f_k(a_k) \) for each \( k \) and a path
Let liftings \( \tilde{f}_k \) and \( \tilde{f} \) of \( f_k : A_k \to A_k \) and \( f : X \to X \) such that \( \tilde{f}_k(\langle e_k \rangle) = \langle w_k \rangle \in \tilde{A}_k \) and \( \tilde{f}(\langle e \rangle) = \langle w_0 \rangle \in \tilde{X} \). Let liftings \( \tilde{f}_k \) and \( \tilde{f} \) be chosen as references, then the endomorphism \( \tilde{f}_\pi : \mathcal{D} \to \mathcal{D} \) determined by a lifting \( \tilde{f} \) of \( f \) is defined by

\[
\tilde{f}_\pi(\alpha) \circ \tilde{f} = \tilde{f} \circ \alpha, \alpha \in \mathcal{D},
\]

and the \( \tilde{f}_\pi \)-conjugacy class of \( \gamma \in \pi \) is said to be the coordinate of a fixed point class \( p\text{Fix}(\gamma \circ \tilde{f}) \) [5]. The coordinate of a fixed point class can be obtained geometrically.

**Lemma 3.1.** The coordinate for the class of a fixed point \( x \) of \( f \) is the \( \tilde{f}_\pi \)-conjugacy class of \( \gamma = \langle c(f \circ c)^{-1}w_0^{-1} \rangle \in \pi \), where \( c \) is any path from \( x_0 \) to \( x \). In other words, \( x \in p\text{Fix}(\gamma \circ \tilde{f}) \).

**Proof.** Let \( \tilde{x} = \langle c \rangle \in p^{-1}(x) \). Since \( \tilde{f}(\tilde{x}_0) = \tilde{f}(\langle e \rangle) = \langle w_0 \rangle \), we have \( \tilde{f}(\tilde{x}) = \tilde{f}(\langle c \rangle) = \langle w_0(f \circ c) \rangle \). Hence \( \gamma(f \circ c) = \gamma(w_0(f \circ c)) = \langle c(f \circ c)^{-1}w_0^{-1} \rangle \langle w_0(f \circ c) \rangle = \tilde{x} \).

Let \( f : X \to X \) be a given selfmap. The set of fixed points of \( f \) is denoted by \( \Phi(f) \) instead of \( \text{Fix} f \). Two fixed points \( x, y \in \Phi(f) \) are said to be equivalent if \( x \) and \( y \) belong to the same fixed point class, i.e., if there exists a path \( \lambda : I \to X \) such that \( \lambda(0) = x, \lambda(1) = y \) and \( \lambda \) is homotopic to \( f \circ \lambda \) rel. end points. We denote by \( \Phi(f)/\sim \) the set of equivalence classes of \( \Phi(f) \) by this equivalence relation. Let \( F \in \Phi(f)/\sim \) and \( x \in F \) be given. Define \( \tau(F) \) as the unique class of \( \text{FPC}(f) \) determined by \( x \) where \( \text{FPC}(f) \) is the fixed point class data of \( f \), the weighted set of lifting classes of \( f \), the weight of a class \( [\tilde{f}] \) being \( \text{ind}(X, f, \text{pFix} \tilde{f}) \) [5, Ch. III, Sec. 1]. This correspondence gives a well-defined function \( \tau : \Phi(f)/\sim \to \text{FPC}(f) \). Also we can define \( \mu_k : \Phi(f_k)/\sim \to \Phi(f)/\sim \) by \( \mu_k(F_k) = F \) determined by \( x_k \in F_k \subset F \), and thus we have a commutative diagram

\[
\begin{array}{ccc}
\Phi(f_k)/\sim & \xrightarrow{\tau_k} & \text{FPC}(f_k) \\
\downarrow_{\mu_k} & & \downarrow_{i_{k,\text{FPC}}} \\
\Phi(f)/\sim & \xrightarrow{\tau} & \text{FPC}(f).
\end{array}
\]
Note that we shall fail to distinguish between a path in $X$ and its class in the fundamental groupoid of $X$. In [4], the group homomorphism $f^{w_0} : \pi_1(X, x_0) \to \pi_1(X, x_0)$ defined by $f^{w_0}(\alpha) = w_0f(\alpha)w_0^{-1}$ for every $\alpha \in \pi$ defines an equivalence relation on $\pi$ by setting $\alpha \sim \alpha'$ if there exists a $\beta \in \pi$ such that $\alpha = \beta \alpha' f^{w_0}(\beta^{-1})$. Let $\text{Coker} \ (1 - f^{w_0})$ be the quotient set of $\pi$ by this equivalence relation. The Reidemeister number of $f$ is the number $R(f) = \| \text{Coker} \ (1 - f^{w_0}) \|$. In what follows, $j : \pi \to \text{Coker} \ (1 - f^{w_0})$ denotes the quotient function: if $(\alpha) \in \pi$, then $j((\alpha)) = [(\alpha)] \in \text{Coker} \ (1 - f^{w_0})$.

Pick a path $u_k$ from $x_0$ to $a_k$ and take a lifting $\tilde{i}_k$ of $i_k$ such that $\tilde{i}_k((c_k)) = (u_k)$. Define a function $\nu_{k, \pi} : \pi_1(A_k, a_k) \to \pi_1(X, x_0)$ by

$$\nu_{k, \pi}(\alpha) = (u_k(i_k \circ \alpha)w_k(f \circ u_k)^{-1}w_0^{-1}).$$

**Lemma 3.2.** The function $\nu_{k, \pi}$ induces a transformation

$$\nu_k : \text{Coker} \ (1 - f^{w_k}) \to \text{Coker} \ (1 - f^{w_0})$$

and $\nu_k$ is independent of the choice of the path $u_k$.

**Proof.** See [12, Lemma 1.2].

**Lemma 3.3.** The diagram

$$\Phi(f_k)/ \sim \xrightarrow{p_k} \text{Coker} \ (1 - f^{w_k})$$

$$\downarrow \mu_k \hspace{1cm} \downarrow \nu_k$$

$$\Phi(f)/ \sim \xrightarrow{\rho} \text{Coker} \ (1 - f^{w_0})$$

commutes, where $\rho(F) = [(c(f \circ c)^{-1}w_0^{-1})]$, $c$ is any path in $X$ with $c(0) = x_0, c(1) = x$, for any $x \in F$.

**Proof.** Let $x_k \in F_k \in \Phi(f_k)/ \sim$ and pick a path $c_k$ from $a_k$ to $x_k$ in $A_k$. Since $\rho$ is independent of the choice of the path $c$, pick a path $c$ from $x_0$ to $x_k \in F$ (as $F_k \subset F$) in $X$. By Lemma 3.2, $\nu_k[(\alpha)] = [(u_k(i_k \circ \alpha)w_k(f \circ u_k)^{-1}w_0^{-1})]$, we have

$$\nu_k \rho_k(F_k) = \nu_k[(c_k(f_k \circ c_k)^{-1}w_k^{-1})]$$

$$= [(u_k c_k(f_k \circ c_k)^{-1}w_k^{-1}w_0^{-1}(f \circ u_k)^{-1}w_0^{-1})]$$

$$= [(u_k c_k(f \circ (u_k c_k))^{-1}w_0^{-1})]$$

$$= \rho(F)$$

$$= \rho \mu_k(F_k).$$
We recall two lemmas (see [4, Lemma A.1, A.2]).

**Lemma 3.4.** Let \( f : X \to X, x_0 \in X \) and \( w_0 \) and \( \eta \) be paths in \( X \) connecting \( x_0 \) to \( f(x_0) \). Then, there is an index preserving bijection \( r_{w_0, \eta} : \text{Coker} \ (1 - f^{w_0}) \to \text{Coker} \ (1 - f^{\eta}) \) given by \( r_{w_0, \eta}[(\alpha)] = [(\alpha w_0 \eta^{-1})] \).

**Lemma 3.5.** Let \( f : X \to X, x_0 \in X \) and \( w_0 : I \to X \) be given, with \( w_0(0) = x_0, w_0(1) = f(x_0) \). Let \( a_k \in A_k \subset X \) be another base point and let \( u_k : I \to X \) be a path in \( X \) connecting \( x_0 \) to \( a_k \). Then, \( u_k^{*} = u^{*} : \text{Coker} \ (1 - f^{w_0}) \to \text{Coker} \ (1 - f^{u_k^{*} w_0(\cdot) u_k^{*}}} \) defined by \( u^{*}[(\alpha)] = [(u_k^{-1}\alpha u_k)] \) is an index preserving bijection.

Consider the commutative diagram

\[
\begin{array}{ccc}
\pi_1(A_k, a_k) & \xrightarrow{f^{w_k}} & \pi_1(A_k, a_k) \\
\downarrow{i_k, \pi} & & \downarrow{i_k, \pi} \\
\pi_1(X, a_k) & \xrightarrow{f^{w_k}} & \pi_1(X, a_k).
\end{array}
\]

If \( i_{k, \pi} \) is surjective, then we have an exact sequence

\[
0 \to \ker i_{k, \pi} \to \pi_1(A_k, a_k) \xrightarrow{i_{k, \pi}} \pi_1(X, a_k) \to 0.
\]

**Lemma 3.6.** If \( i_{k, \pi} \) is surjective and the restriction \( f_k^{w_k}|_{\ker i_{k, \pi}} \) of \( f_k^{w_k} \) to \( \ker i_{k, \pi} \) is nilpotent, then

\[
v_k : \text{Coker} \ (1 - f_k^{w_k}) \to \text{Coker} \ (1 - f_k^{w_0})
\]

is bijective.

**Proof.** Applying [3, Proposition 1.11], \( i_{k, \pi} \) induces a bijection

\[
i_k : \text{Coker} \ (1 - f_k^{w_k}) \to \text{Coker} \ (1 - f_k^{w_k})
\]

defined by \( i_k[(\alpha)] = [(i_k \circ \alpha)] \).

With \( u_k \) as above, define \( \eta = u_k^{-1} w_0(f \circ u_k) \). Then, by Lemma 3.4 and 3.5, it suffices to check that the diagram
Some computations of relative Nielsen numbers

\[ \text{Coker } (1 - f_k^{u_k}) \xrightarrow{i_k} \text{Coker } (1 - f_k^{w_k}) \]
\[ \downarrow \nu_k \quad \downarrow \gamma_{w_k, \eta} \]
\[ \text{Coker } (1 - f_{u_0}^w) \xrightarrow{u_k^*} \text{Coker } (1 - f_{u_k^{-1}w_0(f_{0u_k})}) \]

commutes.

Let \( [(\alpha)] \in \text{Coker } (1 - f_k^{w_k}) \), then

\[
\begin{align*}
\gamma_{w_k, \eta} i_k [(\alpha)] &= \gamma_{w_k, \eta} [(i_k \circ \alpha)] \\
&= [(i_k \circ \alpha) w_k \eta^{-1}] \\
&= [(i_k \circ \alpha) w_k (f \circ u_k)^{-1} w_0^{-1} u_k] 
\end{align*}
\]

and

\[
\begin{align*}
u_k u_k [(\alpha)] &= u_k [(u_k (i_k \circ \alpha) w_k (f \circ u_k)^{-1} w_0^{-1})] \\
&= [u_k^{-1} u_k (i_k \circ \alpha) w_k (f \circ u_k)^{-1} w_0^{-1} u_k] \\
&= [(i_k \circ \alpha) w_k (f \circ u_k)^{-1} w_0^{-1} u_k].
\end{align*}
\]

Then we have \( \gamma_{w_k, \eta} i_k [(\alpha)] = u_k \nu_k [(\alpha)] \).

Recall the relative Lefschetz number \( L(f|_{(X, A)}) = L(f) - L(\tilde{f}) \) of \( f : (X, A) \to (X, A) \) and the trace subgroup of cyclic homotopies

\[ J(f, x_0) = \{ \xi \in \pi_1(X, f(x_0)) \mid \text{there exists a homotopy } H : f \simeq f : X \times I \to X \ni (H(x_0, t)) = \xi \}. \]

In [5, p.33, Theorem 4.2] where \( f_{\pi}(\pi_1(X)) \subset J(f) \), it follows from \( L(f) \neq 0 \) that \( N(f) = R(f) \). We prove the main theorems.

**Theorem 3.1.** Let \( f : (X, A) \to (X, A) \) be a selfmap of a pair of compact polyhedra with \( \hat{A} = \bigcup_{k=1}^{n} A_k \). If \( f_{\pi}(\pi_1(X)) \subset J(f) \), \( f_{k, \pi}(\pi_1(A_k)) \subset J(f_k) \), \( \iota_{k, \pi} \) is surjective and \( f_k^{w_k}|_{Ker \iota_{k, \pi}} \) is nilpotent for all \( k \), then

\[
n(f; X - A) = \begin{cases} \# \text{Coker } (1 - f_{u_0}^w), & \text{if } L(f|_{(X, A)}) \neq 0 \\ 0, & \text{otherwise.} \end{cases}
\]
Proof. If $L(f_k) = 0$ for all $k$, then this theorem is clear. We can assume that $L(f_k) \neq 0$ for all $k, 1 \leq k \leq m$ for some $m \leq n$. By [5, p.33, Theorem 4.2], when $f_k, n(\pi_1(A_k)) \subset J(f_k)$ for all $k, 1 \leq k \leq m$, the correspondence $\rho_k$ is bijective. Let $F$ be a fixed point class of $f : X \to X$. Then

$$\text{ind}(A, \bar{f}, F \cap A) = \text{ind}(A, \bar{f}, \cup_{k=1}^{n}(F \cap A_k))$$

$$= \sum_{k=1}^{n} \text{ind}(A_k, f_k, F \cap A_k)$$

$$= \sum_{k=1}^{m} \text{ind}(A_k, f_k, F_k) \quad \text{(by Lemma 3.3, 3.6)}$$

for some fixed $F_k \in \Phi(f_k)/\sim$.

Case 1) Suppose $L(f) = 0$. Then all the fixed point classes of $f : X \to X$ are inessential. If $L(f|_{(X,A)}) \neq 0$, then there exists a component $A_k$ such that

$$\text{ind}(A, \bar{f}, F \cap A) = \sum_{k=1}^{m} \text{ind}(A_k, f_k, F_k) = L(\bar{f})/N(f_k) \neq 0.$$

Hence all the fixed point classes of $f$ do not assume their index in $A$. If $L(f|_{(X,A)}) = 0$, then $L(\bar{f}) = 0$, and so

$$\text{ind}(A, \bar{f}, F \cap A) = \sum_{k=1}^{m} \text{ind}(A_k, f_k, F_k) = 0 = \text{ind}(X, f, F)$$

because $F$ is inessential. Thus all the fixed point classes of $f$ assume their index in $A$.

Case 2) Suppose $L(f) \neq 0$. By using [5, p.33, Theorem 4.2] again, $N(f) > 0$. Thus we have

$$\text{ind}(A, \bar{f}, F \cap A) = \frac{L(\bar{f})}{N(f)}$$

and

$$\text{ind}(X, f, F) = \frac{L(f)}{N(f)}.$$

This completes the theorem.

If $n = 1$, i.e. $A$ is connected, we can take $w_0 = w_1$ and $x_0 = a_1$. Then $\nu_1[(\alpha)] = [(i_1 \circ \alpha)]$ and $\nu_1 = i_* : \text{Coker} (1 - f_1^{w_0}) \to \text{Coker} (1 - f_1^{w_0})$. We shall get
COROLLARY 3.1. Let \( f : (X, A) \to (X, A) \) be a selfmap of a pair of compact polyhedra with \( \hat{A} \) connected. If \( f_\pi(\pi_1(\hat{X})) \subset J(f), f_{1,\pi}(\pi_1(\hat{A}_1)) \subset J(f_1), i_{1,\pi} \) is surjective and \( f_{1,\pi}^W|_{\text{Ker } i_{1,\pi}} \) is nilpotent, then

\[
\mathcal{N}(f; X - A) = \begin{cases} \# \text{Coker } (1 - f^W), & \text{if } L(f_1) \neq L(f) \\ 0, & \text{otherwise.} \end{cases}
\]

In [13], X.Zhao showed that if there is a component \( A_k \) of \( \hat{A} \) such that \( i_{k,\pi} \) is surjective, then \( \mathcal{N}(f; X - A) = 0 \). By Theorem 2.1 and Theorem 3.1, we have

THEOREM 3.2. Let \( f : (X, A) \to (X, A) \) be a selfmap of a pair of compact polyhedra with \( \hat{A} = \bigcup_{k=1}^n A_k \). Suppose \( f_\pi(\pi_1(\hat{X})) \subset J(f), f_{k,\pi}(\pi_1(A_k)) \subset J(f_k), i_{k,\pi} \) is surjective and \( f_{k,\pi}^W|_{\text{Ker } i_{k,\pi}} \) is nilpotent for all \( k \), then

\[
\tilde{\mathcal{N}}(f; X, A) = \begin{cases} \# \text{Coker } (1 - f^W), & \text{if } L(f|_{(X,A)}) \neq 0 \\ 0, & \text{otherwise.} \end{cases}
\]

COROLLARY 3.2. Let \( f : (X, A) \to (X, A) \) be a selfmap of a pair of compact polyhedra with \( \hat{A} \) connected. If \( f_\pi(\pi_1(\hat{X})) \subset J(f), f_{1,\pi}(\pi_1(\hat{A}_1)) \subset J(f_1), i_{1,\pi} \) is surjective and \( f_{1,\pi}^W|_{\text{Ker } i_{1,\pi}} \) is nilpotent, then

\[
\tilde{\mathcal{N}}(f; X, A) = \begin{cases} \# \text{Coker } (1 - f^W), & \text{if } L(f_1) \neq L(f) \\ 0, & \text{otherwise.} \end{cases}
\]

EXAMPLE 3.1. Let \( X = \{ x \in \mathbb{R}^2 | 1/2 \leq ||x|| \leq 1 \} \) be an annulus in \( \mathbb{R}^2 \) and let \( A = \bigcup_{k=1}^2 A_k \) be the boundary of \( X \) where \( A_k = \{ x \in X ||x|| = 1/k \} \). Define \( f : (X, A) \to (X, A) \) by \( f(re^{i\theta}) = re^{i3\theta} \) for \( 1/2 \leq r \leq 1 \). Take \( e^{i0} = 1 \) as base point of \( X \) and choose the path \( w_0 \) to be constant. Then, for all \( n \), \( 1 - f^n : \mathbb{Z} \to \mathbb{Z} \) is multiplication by \( 1 - 3^n \) and

\[ Coker (1 - f^n) = \mathbb{Z}_{3^n-1}. \]

Since \( A_k \)'s are H-spaces, \( L(f^n_k) = L(f^n) \) for each \( k \). Then we have

\[ \tilde{\mathcal{N}}(f^n; X, A) = n(f^n; X - A) = |3^n - 1| \]

for all \( n \). Also we have

\[ \mathcal{N}(f^n; X, A) = 2|3^n - 1| \]

for all \( n \).
EXAMPLE 3.2. Let $X$ be the solid torus in Euclidean 3-space $\mathbb{R}^3$ which is obtained by rotating the 2-disk in the $x_1x_3$-plane of radius 1 and centered at $(2,0,0)$ about the $x_3$-axis, and let $A$ be the 2-dimensional torus which bounds $X$. We consider $\mathbb{R}^3$ as $\mathbb{C} \times \mathbb{R}^1$, where $\mathbb{C}$ is the complex plane, and label the points of $X$ as $(r e^{i\theta}, t)$, where $r e^{i\theta} \in \mathbb{C}$ and $t \in \mathbb{R}^1$, with $1 \leq \theta < 2\pi$ and $-1 \leq t \leq 1$. Let $f : (X, A) \to (X, A)$ be the map given by

$$f(r e^{i\theta}, t) = (r e^{id\theta}, -|t|),$$

where $d \neq 1$ is an integer. As any circle of latitude is a deformation retract of $X$ we have $N(f) = |d - 1|$ [1, Ch. VIII, p.107; 5, p.21, Theorem 5.4 and p.33, Example 1], and it follows from [5, p.33, Example 2] that $N(f) = |d - 1|$. The fixed point set of $f$ lies in $t \leq 0$ and consists of $|d - 1|$ half-disks. Each half-disk forms an essential fixed point class of $f$ and contains one essential fixed point class of $\tilde{f}$ on its boundary because the arcs of the boundary $S^1$ of the rotated 2-disk from $(e^\frac{i\pi}{d-1}, 0)$ to $(3e^\frac{i\pi}{d-1}, 0)$ for $n = 0, 1, 2, \ldots, d - 2$, passing through the south pole show this. Hence

$$N(f; X, A) = |d - 1|.$$

$Ker_\pi \cong \mathbb{Z}$ is generated by the loop $\alpha$ obtained by travelling the boundary $S^1$ of the 2-disk once, starting $(e^{i0}, 0) = x_0$, in the counterclockwise direction. Now select the path $w_0$ to be the constant path at $x_0$. Then we have

$$\tilde{f}^{w_0}(\alpha) = \tilde{f}(\alpha) = 0.$$

It is easy to see that $\tilde{N}(f; X, A) = n(f; X - A) = 0$ by Theorem 3.1, Theorem 3.2 and thus, each essential fixed point class of $f$ assumes its index in $A$.

References

Some computations of relative Nielsen numbers


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