1. Introduction

Let $M^n$ be a Riemannian manifold and $\gamma$ a geodesic joining two points of $M^n$. Recall that Myers[13] actually showed that if along $\gamma$ the Ricci curvature, $\text{Ric}$, satisfies

$$\text{Ric}(T, T) \geq a > 0$$

and the length of $\gamma$ exceeds $\pi \sqrt{n-1}/\sqrt{a}$ where $T$ is the unit tangent to $\gamma$, then $\gamma$ is not minimal.

Moreover, there have been several applications of Myers method to general relativity. T. Frankel[7] has used Myers theorem to obtain a bound on the size of a fluid mass in stationary space-time universe. In [8], G. Galloway made use of Frankel’s method to obtain a closure theorem which has as its conclusion the “finiteness” of the “spatial part” of a space-time obeying certain cosmological assumptions for cosmological models more general than the classical Friedmann models. To prove the closure theorem he generalized the Myers theorem on a Riemannian manifold. S. Markvorsen[12] obtained another extension of the Myers theorem.

On the other hand, J. K. Beem and P. E. Ehrlich[1,2] proved that if $(M, g)$ is a globally hyperbolic space-time with all Ricci curvature positive and bounded away from zero, then $(M, g)$ has finite timelike diameter.

In this paper, we used the generalized Myers theorem on Riemannian manifolds given by G. Galloway[8] to extend the Lorentzian version of Myers theorem given by J. K. Beem and P. E. Ehrlich. Moreover,
we compute the upper bound of Lorentzian arc lengths of all future-directed nonspacelike curves starting from a compact spacelike submanifold $K$ to any chronologically related point $q$ of $M$ for the suitable curvature tensor and second fundamental tensor conditions.

2. Preliminaries

Let $(M, g)$ be an arbitrary space-time. Given $p, q \in M$, $p \leq q$ means that $p = q$ or there is a piecewise smooth future directed nonspacelike curve from $p$ to $q$. Let $\Omega_{p,q}$ denote the path space of all piecewise smooth future directed nonspacelike curves $\gamma : [0, 1] \to M$ with $\gamma(0) = p$ and $\gamma(1) = q$. The Lorentzian arc length $L : \Omega_{p,q} \to \mathbb{R}$ is then defined as follows. Given a piecewise smooth curve $\gamma \in \Omega_{p,q}$, choose a partition $0 = t_0 < t_1 < t_2 < \ldots < t_n = 1$ such that $\gamma|_{(t_i, t_{i+1})}$ is smooth for each $i = 0, 1, 2, \ldots, n - 1$, and set

$$L(\gamma) = \sum_{i=0}^{n-1} \int_{t=t_i}^{t=t_{i+1}} \sqrt{-g(\gamma'(t), \gamma'(t))} dt.$$ 

Moreover, the Lorentzian distance $d : M \times M \to \mathbb{R} \cup \{\infty\}$ of $(M, g)$ is defined as follows. Given $p \in M$, if $q \in J^+(p) = \{q \in M | p \leq q\}$, set $d(p, q) = \sup \{L(\gamma) : \gamma \in \Omega_{p,q}\}$, and zero otherwise. Now, we define the timelike diameter, $\text{diam}(M, g)$, of the space-time by

$$\text{diam}(M, g) = \sup \{d(p, q) | p, q \in M\}.$$ 

A space-time $(M, g)$ is strongly causal if $(M, g)$ does not contain any point $p$ of $M$ such that there are future-directed nonspacelike curves leaving arbitrarily small neighborhood of $p$ and then returning. Moreover, a strong causal space-time $(M, g)$ is said to be globally hyperbolic if $J^+(p) \cap J^-(q)$ is compact for all $p, q \in M$ where $J^-(p) = \{q \in M | q \leq p\}$. It should be noted that global hyperbolicity does not imply any of geodesic completeness. This may be seen by fixing points $p$ and $q$ in the Minkowski space $L^2$ with $p << q$ ($p << q$ means that there is a future-directed piecewise smooth timelike curve from $p$ to $q$). Now set $M = I^+(p) \cap I^-(q)$ (here $I^+(p) = \{q \in L^2 | p << q\}$ and $I^-(q) = \{p \in L^2 | p << q\}$) equipped with the induced Lorentzian
A focal Myers-Galloway theorem on space-times

metric as an open subset of $L^2$. Clearly, $M$ is totally geodesic and globally hyperbolic. Therefore, a geodesic joining any pair of causally related points in $M$ is a geodesic segment in $L^2$ defined on a finite time interval.

However, the global hyperbolicity still guarantees the existence of a maximal geodesic $\gamma \in \Omega_{p,q}$, i.e., a future directed nonspacelike geodesic $\gamma$ from $p$ to $q$ with $L(\gamma) = d(p,q)$. c.f. [2, Theorem 5.1]. This fact makes Theorem 4.2 available for the globally hyperbolic space-times.

With respect to the conjugate points it is well known that a timelike geodesic is not maximal beyond the first conjugate point (c.f. [2, p.228]).

Let $\gamma : [0, b] \rightarrow (M, g)$ be a unit timelike geodesic segment. One considers an $\mathbb{R}$-vector space $V^\perp(\gamma)$ of continuous piecewise smooth vector fields $Y$ along $\gamma$ perpendicular to $\gamma'$ and let $V^\perp_0(\gamma) = \{Y \in V^\perp(\gamma) \mid Y(0) = Y(b) = 0\}$. Then, from the second variation formula of $\gamma$, the Lorentzian index form $I : V^\perp(\gamma) \times V^\perp(\gamma) \rightarrow \mathbb{R}$ is given by, for $X, Y \in V^\perp(\gamma)$

$$I(X, Y) = - \int_0^b \left[ g(X', Y') - g(R(X, \gamma'), \gamma Y) \right] dt$$

where $R$ is the curvature tensor with respect to the Levi-Civita connection $\nabla$ on $(M, g)$. Moreover, $t_1, t_2 \in [0, b]$ with $t_1 \neq t_2$ are conjugate if respect to the timelike geodesic $\gamma$ if there is a nontrivial Jacobi field $J$ (i.e., $J'' + R(J, \gamma')\gamma' = 0$) along $\gamma$ with $J(t_1) = J(t_2) = 0$. Then we have the following maximality property of Jacobi fields with respect to the index form, cf. [1,2].

**Proposition 2.1.** Let $\gamma : [0, b] \rightarrow (M, g)$ be a unit speed timelike geodesic with no conjugate points and let $J \in V^\perp(\gamma)$ be any Jacobi field. Then, for any $Y \in V^\perp(\gamma)$ with $Y \neq J$ and $Y(0) = J(0), Y(b) = J(b)$, we have $I(J, J) > I(Y, Y)$.

**Corollary 2.2.** Let $\gamma : [0, b] \rightarrow M$ have no conjugate points. Then the index form $I$ is negative definite on $V^\perp_0(\gamma) \times V^\perp_0(\gamma)$.

In [1,2], J. K. Beem and P. E. Ehrlich used Corollary 2.2 to prove the Lorentzian version of Myers theorem for complete Riemannian manifolds given in [3,9] as follows.
THEOREM 2.3. Let \((M, g)\) be a globally hyperbolic space-time of dimension \(n \geq 2\) satisfying
\[
Ric(\gamma', \gamma') \geq (n - 1)k > 0
\]
for any unit timelike geodesic \(\gamma\). Then
\[
diam(M, g) \leq \pi/\sqrt{k}.
\]
In fact, if \((n - 1)k = a\), we may check that this theorem reduces to Myers result on complete Riemannian manifolds.

THEOREM 2.4(MYERs-GALLOWAY). Let \(M^n\) be a complete Riemannian manifold. Suppose there exist constants \(a > 0\) and \(c \geq 0\) such that for every pair of points in \(M^n\) and unit minimal geodesic \(\gamma\) joining those points, the Ricci curvature satisfies
\[
Ric(\gamma', \gamma') \geq a + \frac{df}{ds}
\]
along \(\gamma\), where \(f\) is some function of arc lengths satisfying \(|f(s)| \leq c\) along \(\gamma\). Then \(M^n\) is compact and
\[
diam(M^n) \leq \frac{\pi}{a} \left(c + \sqrt{c^2 + a(n - 1)}\right).
\]
In above Myers-Galloway theorem on a Riemannian manifold we may find a differentiable function \(f\) of arc length \(s\) such that \(|f(s)| \leq c\) for some \(c > 0\). Such a function may be applied to prove a closure theorem of a more generalized model \((M^4, <, >)\) than the Friedmann model of general relativity. More in detail, let \(s\) be the arc length of a geodesic \(\gamma\) with unit tangent \(X\) in the "spatial part" \(V^3\) of \(M^4\) and let \(U\) be a smooth unit future-directed timelike vector field on \(M^4\) orthogonal to \(V^3\). Extend \(X\) along the flow lines through \(\gamma\) by making it invariant under the flow generated by \(U\). Then we may set \(f(s) = <X, \nabla_U U > (s)\) (c.f. [8]).

Now, we may prove the Lorentzian version of Myers-Galloway theorem as follows.
Proposition 2.5. Let \((M, g)\) be an arbitrary space-time of dimension \(n \geq 2\) and, let \(\gamma : [0, b] \rightarrow (M, g)\) be any unit timelike geodesic joining any pair of causally related points of \(M\) with length \(L\). Suppose that

\[\text{Ric}(\gamma', \gamma') \geq a + \frac{df}{ds}\]

where \(a > 0\), \(f\) is a differentiable function of arc length \(s\) with \(|f(s)| \leq c\) along \(\gamma\), and \(L > \frac{\pi}{a} \left( c + \sqrt{c^2 + a(n-1)} \right)\). Then \(\gamma\) can not be maximal.

The proof is similar to Theorem 2.3 (c.f. [2]). Note that if \(f = c = 0\) then Proposition 2.5 reduces to Theorem 2.3 (Myers theorem on space-times). Moreover, in this Proposition 2.5 the Ricci curvature does not require positiveness along \(\gamma\). Similarly, we have the Lorentzian analogue of Myers-Galloway diameter theorem for complete Riemannian manifolds.

Theorem 2.6. Let \((M, g)\) be a globally hyperbolic space-time of dimension \(n \geq 2\) and suppose there exist constants \(a > 0\) and \(c \geq 0\) such that for every pair of causally related points in \(M\) and any unit maximal timelike geodesic \(\gamma\) joining those points,

\[\text{Ric}(\gamma', \gamma') \geq a + \frac{df}{ds}\]

where \(f\) is some function of arc lengths satisfying \(|f(s)| \leq c\) along \(\gamma\). Then

\[\text{diam}(M, g) \leq \frac{\pi}{a} \left( c + \sqrt{c^2 + a(n-1)} \right)\].

3. Existence of Maximal Geodesics orthogonal to the Spacelike Submanifolds

Let \(K\) be a spacelike submanifold of dimension \(k \geq 0\) and let for \(q \in M\), \(K << q\) if there exists \(p \in K\) such that \(p << q\). \(K \leq q\) if there exists \(p \in K\) with \(p \leq q\). And let \(I^+(K) = \{q \in M | K << q\}\) chronological future of \(K\), \(I^-(K) = \{q \in M | q << K\}\) chronological past of \(K\), \(J^+(K) = \{q \in M | K \leq q\}\) causal future of \(K\), \(J^-(K) = \{q \in M | q \leq K\}\) causal past of \(K\). Clearly, \(I^+(K) = \bigcup_{p \in K} I^+(p)\).
Now, let $\Omega_{K,q}$ be the path space of all piecewise smooth future directed nonspacelike curves $\gamma : [0, b] \to (M, g)$ with $\gamma(0) \in K$ and $\gamma(b) = q$. The Lorentzian arc length $L : \Omega_{K,q} \to \mathbb{R}$ is defined as in Section 2.

Now we define the Lorentzian distance from $K$ to $q$ by

$$d(K, q) = \begin{cases} 0, & \text{if } q \notin J^+(K); \\ \sup \{ L(\gamma) | \gamma \in \Omega_{K,q} \}, & \text{if } q \in J^+(K). \end{cases}$$

Clearly, $d(K, q) > 0$ iff $q \in I^+(K)$. $q \in J^+(K) - I^+(K)$ implies that $d(K, q) = 0$. But the converse does not hold, since $d(K, q) = 0$ for $q \notin J^+(K)$.

Given a timelike curve $\gamma$ from $K$ to $q$, we have a variation $\alpha$ of $\gamma(t)$ and define the variation vector filed $V$ of $\alpha$ along $\gamma$ by

$$V(t) = \frac{\partial}{\partial s} \alpha(t, s)|_{s=0}, V(b) = 0, V(0) \in T_{\gamma(0)}K.$$  

Then we have some facts:

if $\gamma : [0, b] \to (M, g)$ is a unit speed timelike geodesic from $K$ to $q$, then $L'(0) = g(V(0), \gamma'(0))$. Thus, $\gamma$ is extremal iff $\gamma$ is orthogonal at $\gamma(0)$ to $K$.

Moreover, if $\gamma : [0, b] \to (M, g)$ is a unit timelike geodesic which is orthogonal at $\gamma(0)$ to the spacelike submanifold $K$ and if $V$ is a piecewise smooth vector field along $\gamma$ orthogonal to $\gamma'$, then we have

$$L''(0) = g(S_{\gamma'(0)}V(0), V(0)) + I(V, V)$$

where $I(V, V) = -\int_0^b [g(V', V') - g(R(V, \gamma')\gamma', V)] \, dt$ and, $S_{\gamma'(0)}$ is the second fundamental tensor given by $S_{\gamma'}x = -(\nabla_x \gamma'(0))^T$ for $x \in T_pK$ where $T$ means “tangential part”.

Hence we may define the Lorentzian submanifold index form

$$I_{(b, K)} : V(\perp, K) \times V(\perp, K) \to \mathbb{R}$$

on $V(\perp, K)$ the vector space of piecewise smooth vector fields $Y$ with $Y \perp \gamma', Y(0) \in T_pK$ as follows; for $X, Y \in V(\perp, K),$

$$I_{(b, K)}(X, Y) = g(S_{\gamma'(0)}X(0), Y(0)) + I(X, Y)$$

where $I$ is the index form on $V(\perp, \gamma)$. 
A focal Myers-Galloway theorem on space-times

Now a smooth vector field \( J \in V^\perp(\gamma, K) \) is called a \( K \)-Jacobi field along \( \gamma \) if \( J \) satisfies

1. \( J'(0) + S_{\gamma'(0)}J(0) \in (T_pK)^\perp \),
2. \( J'' + R(J, \gamma')\gamma' = 0 \).

Hence we may define a \( K \)-focal point \( \gamma(t_0), t_0 \in (0, b) \) if there is a nontrivial \( K \)-Jacobi field with \( J(t_0) = 0 \). Now, we may prove the maximality theorem of \( K \)-Jacobi fields among piecewise smooth vector fields in \( V^\perp(\gamma, K) \) (c.f. [6]).

**Theorem 3.1.** (Maximality of \( K \)-Jacobi fields) Let \( \gamma : [0, b] \to M \) be a timelike geodesic orthogonal at \( \gamma(0) \) to the spacelike submanifold \( K \) with no \( K \)-focal points and let \( X \in V^\perp(\gamma, K) \). If \( J \in V^\perp(\gamma, K) \) is a \( K \)-Jacobi field along \( \gamma \) with \( J(b) = X(b) \), then

\[
I_{(b,K)}(X,X) \leq I_{(b,K)}(J,J),
\]

and equality holds if and only if \( X = J \).

Let \( V^\perp_0(\gamma, K) \) be the subspace of \( V^\perp(\gamma, K) \) with \( Y(b) = 0 \).

**Corollary 3.2.** If such a \( \gamma \) in Theorem 3.1 has no \( K \)-focal points. Then the index form \( I_{(b,K)} \) is negative definite on \( V^\perp_0(\gamma, K) \times V^\perp_0(\gamma, K) \).

Recently in [4,5], P. E. Ehrlich and S. B. Kim used the maximality theorem of \( K \)-Jacobi fields to extend the Morse index theorem and the Rauch comparison theorem to the \( K \)-focal sense for nonspacelike geodesics.

Using the index form \( I_{(b,K)} \) it is well known that a timelike geodesic orthogonal to a spacelike hypersurfaces \( K \) fails to maximize arc length after the first \( K \)-focal point (c.f. [2,10]). Moreover, even if \( K \) is a spacelike submanifolds of codimension arbitrary, we can easily show the proof of the following proposition as in [2] by using the fact that given a \( K \)-Jacobi field \( J_1 \), there is a vector \( n \in (T_pK)^\perp \) such that

\[
J_1'(0) = -S_{\gamma'(0)}J_1(0) + n \quad \text{with} \quad g(n, J_1(0)) = 0.
\]

**Proposition 3.3.** Let \( \gamma : [0, b] \to M \) be a unit speed timelike geodesic segment orthogonal to a spacelike submanifold \( K \) at \( \gamma(0) = p \in K \). If there exists \( t_0 \in (0, b) \) such that \( \gamma(t_0) \) is a \( K \)-focal point along \( \gamma \), then there exists a variation vector field \( Z \in V^\perp(\gamma, K) \) such
that $I_{(k,K)}(Z,Z) > 0$, i.e., there exists a timelike curve from $K$ to $q$ longer than $\gamma$.

Let $K$ be a spacelike submanifold of a space-time $(M, g)$. If $(M, g)$ is a globally hyperbolic space-time, we know that there is a future directed maximal nonspacelike geodesic between any causally related two points. However, we cannot guarantee the existence of the future directed maximal nonspacelike geodesic from $K$ to a point in $M$ (even if $K$ is closed). It may be seen by fixing points $p = (0, -1)$ and $q = (2, 3)$ in Minkowski space $L^2$, and by setting $K = \{(x, y)| -1 < x < 1, y = 0\}$. Then $M = I^+(p) \cap I^-(q)$ is globally hyperbolic and $K$ is closed in $M$. Thus, we can not find any future directed maximal timelike geodesic from $K$ to the point $r = (2, 2)$. Moreover, we need to find such a geodesic $\gamma$ orthogonal at $\gamma(0)$ to $K$ as follows (c.f. [14]).

If $M$ is globally hyperbolic and if $J^-(q) \cap K$ is compact, then the function $x \to d(x, q)$ is continuous on the compact set $J^-(q) \cap K$. Hence, it has a maximum at $p \in J^-(q) \cap K$. Thus, $d(K, q) = d(p, q)$. Therefore, there is a geodesic $\gamma$ from $p$ to $q$ of length $d(K, q) = d(p, q)$. We may assume that $q \notin K$ and $p << q$. From the first variation formula, it is normal.

**Proposition 3.4.** Let $(M, g)$ be a globally hyperbolic space-time and let $K$ be a spacelike submanifold of $(M, g)$. Then for any $q \in I^+(K)$ with $J^-(q) \cap K$ compact, there is a future directed maximal timelike geodesic $\gamma$ perpendicular at $\gamma(0)$ to $K$ in $\Omega_{K,q}$.

**4. The Main Results**

Now, we generalize Proposition 2.5 to the $K$-focal sense.

**Theorem 4.1.** Let $(M, g)$ be a space-time of dimension $\geq 2$ and $\gamma$ any unit speed timelike geodesic with length $L$ in $\Omega_{K,q}$ perpendicular at $\gamma(0)$ to the spacelike submanifold $K$ of dimension $k \geq 0$ for any point $q \in M$. Suppose

$$g\left(R(u, \gamma'(t))\gamma'(t), u\right) \geq \frac{1}{n-1} \left(a + \frac{df}{dt}\right)$$

for all $u \in (\gamma'(t))^\perp$ with $g(u, u) = 1$ along $\gamma$, and suppose
A focal Myers-Galloway theorem on space-times 105

\[ g(S_{\gamma'(0)}w, w) \geq \frac{f(0)}{n-1} \]

for all \( w \in T_{\gamma(0)}K \) with \( g(w, w) = 1 \), where \( a > 0, c \geq 0 \) and \( f \) is a differentiable function with \( |f(t)| \leq c \).

Assume

\[ L(\gamma) > \]

\[ \frac{\pi}{a} \left( \left( 1 - \frac{k}{2(n-1)} \right) c + \sqrt{\left( 1 - \frac{k}{2(n-1)} \right)^2 c^2 + a \left( n - 1 - \frac{3k}{4} \right)} \right). \]

Then \( \gamma \) can not be maximal.

**Proof.** Suppose that \( \gamma : [0, L] \to M \) be a unit speed timelike geodesic with length \( L \) orthogonal at \( \gamma(0) \) to the spacelike submanifold \( K \). Set \( E_n(t) = \gamma'(t) \) and let \( \{E_1, E_2, \ldots, E_{n-1}\} \) be \( n-1 \) spacelike parallel fields such that \( \{E_1(0), E_2(0), \ldots, E_k(0)\} \) forms an orthonormal basis of \( T_{\gamma(0)}K \) and \( \{E_1(t), E_2(t), \ldots, E_n(t)\} \) the orthonormal basis of \( T_{\gamma(t)}M \). Set

\[ W_i = \begin{cases} 
\cos\left( \frac{\pi i}{2L} \right) E_i, & i = 1, 2, \ldots, k \\
\sin\left( \frac{\pi i}{2L} \right) E_i, & i = k + 1, \ldots, n - 1.
\end{cases} \]

Then

\[ W_i(0) = \begin{cases} 
E_i(0) \in T_{\gamma(0)}K, & i = 1, 2, \ldots, k \\
0 \in T_{\gamma(0)}K, & i = k + 1, \ldots, n - 1.
\end{cases} \]

Since \( W_i(L) = 0, \ i = 1, 2, \ldots, n - 1, \) we have \( W_i \in V_0^+(\gamma, K) \). Moreover, \( |f(t)| \leq c \) implies \( -c \leq \sin p(t)f(t) \leq c \) for any function \( p \).

Now, we compute the Lorentzian submanifold index form

\[ I_{(b, K)}(W_i, W_i) = g(S_{\gamma'(0)}W_i(0), W_i(0)) + \int_0^L [g(R(W_i, \gamma')\gamma', W_i) - g(W_i', W_i')] dt \]

as follows.
For i=1,2,......,k,

\[ I_{(b,K)}(W_i, W_i) \]

\[ = g(S_{\gamma(0)} E_i(0), E_i(0)) \]

\[ + \int_0^L \left[ \cos^2 \left( \frac{\pi t}{2L} \right) g(R(E_i, \gamma'), E_i) - \left( \frac{\pi}{2L} \right)^2 \sin^2 \left( \frac{\pi t}{2L} \right) g(E_i, E_i) \right] dt, \]

\[ \geq \frac{f(0)}{n-1} + \int_0^L \left[ \cos^2 \left( \frac{\pi t}{2L} \right) \frac{1}{n-1} (a + \frac{df}{dt}) - \left( \frac{\pi}{2L} \right)^2 \sin^2 \left( \frac{\pi t}{2L} \right) \right] dt \]

\[ = \frac{f(0)}{n-1} + \frac{a}{n-1} \int_0^L \cos^2 \left( \frac{\pi t}{2L} \right) dt + \frac{1}{n-1} \int_0^L \cos^2 \left( \frac{\pi t}{2L} \right) \frac{df}{dt} dt \]

\[ - \int_0^L \left( \frac{\pi}{2L} \right)^2 \sin^2 \left( \frac{\pi t}{2L} \right) dt \]

\[ = \frac{f(0)}{n-1} + \frac{a}{n-1} \frac{L}{2} \]

\[ + \frac{1}{n-1} \left[ \cos^2 \left( \frac{\pi t}{2L} \right) f(t) \bigg|_0^L + \int_0^L \left( \frac{\pi}{2L} \right) \sin \left( \frac{\pi t}{L} \right) f(t) dt \right] - \left( \frac{\pi}{2L} \right)^2 \frac{L}{2} \]

\[ \geq \frac{1}{n-1} \frac{a}{n-1} \frac{L}{2} \left( -Lc \right) - \left( \frac{\pi}{2L} \right)^2 \frac{L}{2} \]

For i=k+1,......,n-1,

\[ I_{(b,K)}(W_i, W_i) \]

\[ = \int_0^L \left[ \sin^2 \left( \frac{\pi t}{L} \right) g(R(E_i, \gamma'), E_i) - \left( \frac{\pi}{L} \right)^2 \cos^2 \left( \frac{\pi t}{L} \right) g(E_i, E_i) \right] dt \]

\[ \geq \int_0^L \left[ \sin^2 \left( \frac{\pi t}{L} \right) \frac{1}{n-1} (a + \frac{df}{dt}) - \left( \frac{\pi}{L} \right)^2 \cos^2 \left( \frac{\pi t}{L} \right) \right] dt \]

\[ = \frac{a}{n-1} \int_0^L \sin^2 \left( \frac{\pi t}{L} \right) dt + \frac{1}{n-1} \int_0^L \sin^2 \left( \frac{\pi t}{L} \right) \frac{df}{dt} dt \]

\[ - \int_0^L \left( \frac{\pi}{L} \right)^2 \cos^2 \left( \frac{\pi t}{L} \right) dt \]

\[ = \frac{a}{n-1} \frac{L}{2} + \frac{1}{n-1} \left[ \sin^2 \left( \frac{\pi t}{L} \right) f(t) \bigg|_0^L - \int_0^L \left( \frac{\pi}{L} \right) \sin \left( \frac{2\pi t}{L} \right) f(t) dt \right] - \left( \frac{\pi}{L} \right)^2 \frac{L}{2} \]

\[ \geq \frac{a}{n-1} \frac{L}{2} + \frac{1}{n-1} \left( - \left( \frac{\pi}{L} \right) Lc \right) - \left( \frac{\pi}{L} \right)^2 \frac{L}{2}. \]
Therefore, we have
\[
\sum_{i=1}^{n-1} I_{(b,K)}(W_i, W_i) \\
\geq \frac{aL}{2} - \frac{k\pi c}{2(n-1)} - \frac{(n-1)\pi^2}{2L} + \frac{3k\pi^2}{8L} \\
= \frac{2}{2L} \left[ aL^2 - 2 \left( 1 - \frac{k}{2(n-1)} \right) \pi cL - \left( n - 1 - \frac{3k}{4} \right) \pi^2 \right] \\
> 0.
\]

The last inequality is given by our hypothesis:

\[
L > \frac{\pi}{a} \left( \left( 1 - \frac{k}{2(n-1)} \right) c + \sqrt{\left( 1 - \frac{k}{2(n-1)} \right)^2 c^2 + a \left( n - 1 - \frac{3k}{4} \right)} \right).
\]

By Corollary 3.2, \( \gamma \) has a \( K \)-focal point. By Proposition 3.3, \( \gamma \) can not be maximal.

The following theorem is a Myers type diameter theorem. Set \( \text{diam}_K(M, g) = \sup \{d(K, q) | q \in I^+(K)\} \).

**Theorem 4.2.** Let \((M, g)\) be a globally hyperbolic space-time of dimension \( n \geq 2 \) and \( K \) the compact spacelike submanifold of dimension \( k \geq 0 \). Suppose there exist constants \( a > 0 \) and \( c \geq 0 \) such that for any point \( q \in M \), and any unit maximal timelike geodesic \( \gamma \) in \( \Omega_{K,q} \) with length \( L \) perpendicular at \( \gamma(0) \) to \( K \),

\[
g(\mathcal{R}(u, \gamma'(t))\gamma'(t), u) \geq \frac{1}{n-1}(a + \frac{df}{dt})
\]

for all \( u \in (\gamma'(t))^\perp \) with \( g(u, u) = 1 \) along \( \gamma \)

\[
g(S_{\gamma'(0)}w, w) \geq \frac{f(0)}{n-1}
\]

for all \( w \in T_{\gamma(0)}K \) with \( g(w, w) = 1 \), where \( f \) is some function with \( |f(t)| \leq c \) along \( \gamma \). Then
\[ \text{diam}_K(M, g) \leq \]
\[ \frac{\pi}{a} \left( \left( 1 - \frac{k}{2(n-1)} \right) c + \sqrt{\left( 1 - \frac{k}{2(n-1)} \right)^2 c^2 + a \left( n - 1 - \frac{3k}{4} \right)} \right). \]

Proof. Suppose

\[ \text{diam}_K(M, g) > \]
\[ \frac{\pi}{a} \left( \left( 1 - \frac{k}{2(n-1)} \right) c + \sqrt{\left( 1 - \frac{k}{2(n-1)} \right)^2 c^2 + a \left( n - 1 - \frac{3k}{4} \right)} \right). \]

Then there exist a point \( q \) of \( M \) with \( K \ll q \) such that

\[ d(K, q) > \]
\[ \frac{\pi}{a} \left( \left( 1 - \frac{k}{2(n-1)} \right) c + \sqrt{\left( 1 - \frac{k}{2(n-1)} \right)^2 c^2 + a \left( n - 1 - \frac{3k}{4} \right)} \right). \]

by definition of \( \text{diam}_K(M, g) \).

Since \( M \) is a globally hyperbolic space-time, \( d(K, q) > 0 \) iff \( q \in I^+(K) \), and since \( J^-(q) \cap K \) is compact, by Proposition 3.4, there is a timelike maximal geodesic \( \gamma \) perpendicular to \( K \) starting from \( K \) to \( q \) with

\[ L = d(K, q) > \]
\[ \frac{\pi}{a} \left( \left( 1 - \frac{k}{2(n-1)} \right) c + \sqrt{\left( 1 - \frac{k}{2(n-1)} \right)^2 c^2 + a \left( n - 1 - \frac{3k}{4} \right)} \right). \]

By Theorem 4.1, \( \gamma \) can not be maximal, in contradiction.
REMARK 4.3. (1) If $k = 0$, we obtain the same result of Theorem 2.6 and some-what a generalization of Theorem 4.1 in [].

(2) If $K$ is any compact spacelike hypersurface of $M$, we have

$$diam_K(M, g) \leq \frac{\pi}{2a} \left( c + \sqrt{c^2 + (n-1)a} \right),$$

which is exactly a half of the upper bound of $diam(M, g)$ given in Theorem 2.6 and some-what a generalization of Theorem 4.1 in [].

(3) If $K$ is a Cauchy hypersurface, i.e., a subset of $M$ which every inextendible timelike curve intersects exactly once, Theorem 4.2 still holds.

References

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