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Approximate MLE for the Scale Parameter of the Weibull Distribution with Type-II Censoring

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ABSTRACT

It is known that the maximum likelihood method does not provide explicit estimator for the scale parameter of the Weibull distribution based on Type-II censored samples. In this paper we provide an approximate maximum likelihood estimator (AMLE) of the scale parameter of the Weibull distribution with Type-II censoring. We obtain the asymptotic variance and simulate the values of the bias and the variance of this estimator based on 3000 Monte Carlo runs for $n = 10(10)30$ and $r, s = 0(1)4$. We also simulate the absolute biases of the MLE and the proposed AMLE for complete samples. It is found that the absolute bias of the AMLE is smaller than the absolute bias of the MLE.

1. INTRODUCTION

The Weibull distribution has in recent years assumed a position of

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prominence in the field of reliability and life testing where samples are often either truncated or censored. From a computational point of view, this distribution is particularly appealing, since its cumulative distribution function can be expressed explicitly as a simple function of the random variable. Various topics associated with this distribution have been considered by numerous authors. The results about the truncated and censored samples from the several distributions including the Weibull distribution may be found in Cohen (1991). But, for the Weibull distribution, the maximum likelihood method based on Type-II censored samples does not provide explicit estimators. Hence, it is desirable to develop approximation to this maximum likelihood method of estimation which would provide us with estimators that are explicit functions of order statistics. The approximate maximum likelihood estimation method was first developed by Balakrishnan (1989 a, b) for the purpose of providing the explicit estimators of the scale parameter in the Rayleigh distribution and the mean and standard deviation in the normal distribution with censoring. For the generalized logistic distribution, the AMLEs of the location and scale parameters were obtained by Balakrishnan (1990). Balakrishnan and Wong (1991) obtained the AMLEs of the location and scale parameters in the half-logistic distribution with Type-II right censoring. Balakrishnan and Varadan (1991) obtained AMLEs of the location and scale parameters in the extreme value distribution with censoring. Some historical remarks and a good summary of the approximate maximum likelihood estimation may be found in Balakrishnan and Cohen (1991).

In this paper, we provide approximate maximum likelihood estimation method of deriving explicit estimator by approximating differentiate formula the log-likelihood equation for the Weibull distrbution with general Type-II censoring. We obtain the asymptotic variance and simulate the values of the biases of the MLE and AMLE and the variance of the AMLE (based on 3000 Monte Carlo runs).

2. APPROXIMATE MAXIMUM LIKELIHOOD ESTIMATOR AND ASYMPTOTIC PROPERTIES

Consider the Weibull distribution with probability density function (pdf)

$$f(x;\beta,\theta) = \frac{\beta}{\theta^{\beta}} x^{\beta-1} e^{-(x/\theta)^{\beta}}, \quad x > 0,\ \theta > 0,\ \beta > 0 \tag{2.1}$$

and cumulative distribution function (cdf)

$$F(x;\beta,\theta) = 1 - \exp\{-(x/\theta)^{\beta}\}, \tag{2.2}$$

where β is known.

The Weibull distribution is widely used in engineering practice due to its versatility. It provides a close approximation to the distribution of the item. Suppose some initial observations are censored and some final observations are also censored. Then let

$$X_{r+1:n} \leq X_{r+2:n} \leq \cdots \leq X_{n-s:n} \tag{2.3}$$

be the available Type-II censored sample from the Weibull distribution with pdf (2.1), where the first r and last s observations are censored. Based on the censored sample in (2.3), we shall derive the AMLE of θ in this section.

The likelihood function based on the censored sample in (2.3) is given by

$$L = \frac{n!}{r!s!}\theta^{-A}\{F(Z_{r+1:n})\}^{r}\{1 - F(Z_{n-s:n})\}^{s}\prod_{i=r+1}^{n-s} f(Z_{i:n}) \tag{2.4}$$

where $A = n-r-s$ is the size of the censored sample in (2.3), $Z_{i:n} = X_{i:n}/\theta$, and $f(z) = \beta z^{\beta-1}e^{-z^{\beta}}$ and $F(z) = 1 - \exp(-z^{\beta})$. From equation (2.4), we differentiate the logarithm of the likelihood function for θ as follows;

$$\begin{aligned}\frac{d\ln L}{d\theta} =& -\frac{1}{\theta}\Big[A + rZ_{r+1:n}\frac{f(Z_{r+1:n})}{F(Z_{r+1:n})} \\ & - sZ_{n-s:n}\frac{f(Z_{n-s:n})}{1 - F(Z_{n-s:n})} + \sum_{i=r+1}^{n-s}\frac{f'(Z_{i:n})}{f(Z_{i:n})}Z_{i:n}\Big] \\ =& 0.\end{aligned} \tag{2.5}$$

Equation (2.5) does not admit an explicit solution for θ, so we will expand the functions $f(Z_{r+1:n})/F(Z_{r+1:n})$, $f(Z_{n-s:n})/\{1 - F(Z_{n-s:n})\}$ and $f'(Z_{i:n})/f(Z_{i:n})$ appearing in (2.5) to Taylor series around the points $F^{-1}(p_{r+1}) = (-\ln\, q_{r+1})^{1/\beta}$, $F^{-1}(p_{n-s}) = (-\ln\, q_{n-s})^{1/\beta}$, and $F^{-1}(p_i) = (-\ln\, q_i)^{1/\beta}$, respectively, and then approximate them by

$$\frac{f(Z_{r+1:n})}{F(Z_{r+1:n})} \simeq \alpha - \delta Z_{r+1:n}, \tag{2.6}$$

$$\frac{f(Z_{n-s:n})}{\{1-F(Z_{n-s:n})\}} \simeq \kappa + \eta Z_{n-s:n}, \tag{2.7}$$

and

$$\frac{f'(Z_{i:n})}{f(Z_{i:n})} \simeq \nu_i + \gamma_i Z_{i:n}, \tag{2.8}$$

where $p_i = i/(n+1),\ q_i = 1-p_i,$

$$\begin{aligned}\alpha =&\beta\big((-\ln q_{r+1})^{1/\beta}\big)^{\beta-1} q_{r+1}/p_{r+1}\\ &- (-\ln q_{r+1})^{1/\beta}\Big[\Big(\beta(\beta-1)\big((-\ln q_{r+1})^{1/\beta}\big)^{\beta-2} q_{r+1}\\ &- \beta^2\big((-\ln q_{r+1})^{1/\beta}\big)^{2\beta-2} q_{r+1}\Big)/p_{r+1}\\ &- \beta^2\big((-\ln q_{r+1})^{1/\beta}\big)^{2\beta-2} q_{r+1}^2/p_{r+1}^2\Big],\end{aligned} \tag{2.9}$$

$$\begin{aligned}\delta =&\beta^2\big((-\ln q_{r+1})^{1/\beta}\big)^{2\beta-2} q_{r+1}^2/p_{r+1}^2 - \Big(\beta(\beta-1)\big((-\ln q_{r+1})^{1/\beta}\big)^{\beta-2}\\ &- \beta^2\big((-\ln q_{r+1})^{1/\beta}\big)^{2\beta-2}\Big) q_{r+1}/p_{r+1},\end{aligned} \tag{2.10}$$

$$\kappa = \beta\big((-\ln q_{n-s})^{1/\beta}\big)^{\beta-1} - (-\ln q_{n-s})^{1/\beta}\beta(\beta-1)\big((-\ln q_{n-s})^{1/\beta}\big)^{\beta-2}, \tag{2.11}$$

$$\eta = \beta(\beta-1)\big((-\ln q_{n-s})^{1/\beta}\big)^{\beta-2}, \tag{2.12}$$

$$\begin{aligned}\nu_i =&\Big((\beta-1)(-\ln q_i)^{-1/\beta} - \beta\big((-\ln q_i)^{1/\beta}\big)^{\beta-1}\Big)\\ &- (-\ln q_i)^{1/\beta}\Big[(\beta-1)(\beta-2)(-\ln q_i)^{-2/\beta}\\ &- 3\beta(\beta-1)\big((-\ln q_i)^{1/\beta}\big)^{\beta-2} + \beta^2\big((-\ln q_i)^{1/\beta}\big)^{2\beta-2}\\ &- \Big((\beta-1)(-\ln q_i)^{-1/\beta} - \beta\big((-\ln q_i)^{1/\beta}\big)^{\beta-1}\Big)^2\Big],\end{aligned} \tag{2.13}$$

and

$$\begin{aligned}\gamma_i =& (\beta-1)(\beta-2)(-\ln q_i)^{-2/\beta} - 3\beta(\beta-1)\big((-\ln q_i)^{1/\beta}\big)^{\beta-2}\\ &+ \beta^2\big((-\ln q_i)^{1/\beta}\big)^{2\beta-2}\\ &- \Big((\beta-1)(-\ln q_i)^{-1/\beta} - \beta\big((-\ln q_i)^{1/\beta}\big)^{\beta-1}\Big)^2.\end{aligned} \tag{2.14}$$

The special case with $\beta = 2$ is known as the Rayleigh distribution. From (2.6), (2.7) and (2.8), we may approximate the equation (2.5) by

$$\frac{d\ln L}{d\theta} \simeq \frac{d\ln L^*}{d\theta} = -\frac{1}{\theta}\Big[A + rZ_{r+1:n}(\alpha - \delta Z_{r+1:n}) - sZ_{n-s:n}(\kappa + \eta Z_{n-s:n}) + \sum_{i=r+1}^{n-s}(\nu_i + \gamma_i Z_{i:n})Z_{i:n}\Big] = 0 \tag{2.15}$$

Upon solving equation (2.15) for θ, we derive the AMLE of θ as follows;

$$\hat{\theta} = \frac{\{-B + (B^2 + 4AC)^{1/2}\}}{2A}, \tag{2.16}$$

where

$$B = r\alpha X_{r+1:n} - s\kappa X_{n-s:n} + \sum_{i+r+1}^{n-s} \nu_i X_{i:n},$$

and

$$C = r\delta X^2_{r+1:n} + s\eta X^2_{n-s:n} - \sum_{i+r+1}^{n-s} \gamma_i X_{i:n}{}^2.$$

Since the AMLE is the solution of the approximate maximum likelihood equation (2.15), it immediately follows that $\hat{\theta}$ is asymptotically normally distributed with mean θ and variance $1/E(-d^2\ln L^*/d\theta^2)$ (See Kendall and Stuart(1973)). From (2.15), we can derive

$$E\Big(-\frac{d^2\ln L^*}{d\theta^2}\Big) = D/\theta^2, \tag{2.17}$$

where

$$D = 3\Big(r\delta E(Z^2_{r+1:n}) + s\eta E(Z^2_{n-s:n}) - \sum_{i=r+1}^{n-s} \gamma_i E(Z^2_{i:n})\Big) - 2\Big(r\alpha E(Z_{r+1:n}) - s\kappa E(Z_{n-s:n}) + \sum_{i=r+1}^{n-s} \nu_i E(Z_{i:n})\Big) - A.$$

Therefore, the variance of AMLE $\hat{\theta}$ may be approximated from the preceding expression (2.17) by using the following formulas;

$$E(Z_{i:n}) = \frac{n!}{(i-1)!(n-i)!}\Gamma(1+\frac{1}{\beta})\sum_{r=0}^{i-1}(-1)^r\binom{i-1}{r}\Big/(n-i-r+1)^{1+1/\beta}$$

and

$$E(Z_{i:n}^2) = \frac{n!}{(i-1)!(n-i)!}\Gamma(1+\frac{2}{\beta})\sum_{r=0}^{i-1}(-1)^r\binom{i-1}{r}\Big/(n-i-r+1)^{1+2/\beta}$$

(See Lieblein(1955)).

In Table 1, we simulate the absolute biases of the MLE and the AMLE $\hat{\theta}$ (based on 3000 Monte Carlo runs) for complete samples. From Table 1, the absolute bias of the AMLE is smaller than the absolute bias of the MLE. Hence, the AMLE is better than the MLE for complete samples in the sence of bias. We also simulate the values of the bias and the variance of the AMLE $\hat{\theta}$ (based on 3000 Monte Carlo runs) for n = 10, 20, 30 and $r, s = 0(1)4$ when $\beta = 3$ and $\theta = 0.5$, and then use them to compute the values of $E(\hat{\theta}-\theta)/\theta$, $\mathrm{Var}(\hat{\theta})/\theta^2$ in Table 2. The values of asymptotic variance of the AMLE that are given in Table 2 are computed from (2.17) in the asymptotic normality. From Table 2, we observe that asymptotic variances of the AMLE is close to the variance of the AMLE. We also observe that the bias of the AMLE decreases, for fixed r and s, as n increases. This is to be expected, as the estimator $\hat{\theta}$ in (2.16) is approximate solution to the likelihood equation (2.5).

Table 1. The absolute biases of the MLE and the AMLE of the scale parameter in the Weibull distribution for complete sample ($n = 10$)

	$\beta = 1$		$\beta = 2$		$\beta = 3$	
θ	MLE	AMLE	MLE	AMLE	MLE	AMLE
0.5	0.00044	0.00044	0.00589	0.00067	0.00522	0.00417
1.0	0.00089	0.00089	0.01178	0.00134	0.01044	0.00833
2.0	0.00178	0.00178	0.02357	0.00269	0.02089	0.01666

Table 2. The relative biases, the relative variances and the asymptotic variances of the AMLE $\hat{\theta}$ of the Weibull scale parameter from the Type-II censored samples.

r	s	n	$E(\hat{\theta}-\theta)/\theta$	$\text{VAR}(\hat{\theta})/\theta^2$	$\text{AVAR}(\hat{\theta})/\theta^2$
0	0	10	-0.00987	0.01134	0.01148
		20	-0.00386	0.00539	0.00692
		30	-0.00244	0.00357	
0	1	10	-0.01105	0.01262	0.01269
		20	-0.00408	0.00566	0.00727
		30	-0.00224	0.00376	
0	2	10	-0.01003	0.01421	0.01452
		20	-0.00441	0.00606	0.00833
		30	-0.00218	0.00389	
0	3	10	-0.01115	0.01640	0.01628
		20	-0.00400	0.00638	0.00846
		30	-0.00207	0.00406	
0	4	20	-0.00437	0.00673	0.00729
		30	-0.00175	0.00424	
1	0	10	0.00757	0.01183	0.01087
		20	0.00041	0.00545	0.00681
		30	-0.00059	0.00359	
1	1	10	0.00822	0.01323	0.01197
		20	0.00041	0.00571	0.00715
		30	-0.00034	0.00377	
1	2	10	0.01152	0.01494	0.01334
		20	0.00032	0.00612	0.00816
		30	-0.00021	0.00390	
1	3	10	0.01335	0.01736	0.01510
		20	0.00100	0.00644	0.00829
		30	-0.00003	0.00408	
1	4	20	0.00094	0.00681	0.00716
		30	0.00037	0.00426	
2	0	10	0.04337	0.01288	0.00983
		20	0.00954	0.00557	0.00658
		30	0.00342	0.00363	
2	1	10	0.04782	0.01451	0.01071
		20	0.00999	0.00585	0.00699
		30	0.00381	0.00381	
2	2	10	0.05586	0.01653	0.01180
		20	0.01041	0.00627	0.00784
		30	0.00408	0.00395	

$\text{AVAR}(\hat{\theta}) = 1/E(-d^2\ln L^*/d\theta^2)$

Table 2. (Continued)

r	s	n	$E(\hat{\theta}-\theta)/\theta$	$\mathrm{VAR}(\hat{\theta})/\theta^2$	$\mathrm{AVAR}(\hat{\theta})/\theta^2$
2	3	10	0.06385	0.01942	0.01316
		20	0.01167	0.00661	0.00796
		30	0.00441	0.00413	
2	4	20	0.01226	0.00700	0.00691
		30	0.00498	0.00431	
3	0	10	0.09799	0.01474	0.00860
		20	0.02365	0.00576	0.00627
		30	0.00969	0.00367	
3	1	10	0.10825	0.01675	0.00927
		20	0.02480	0.00606	0.00655
		30	0.01029	0.00386	
3	2	10	0.12362	0.01926	0.01017
		20	0.02602	0.00651	0.00740
		30	0.01078	0.00400	
3	3	10	0.14128	0.02279	0.01104
		20	0.02816	0.00688	0.00750
		30	0.01136	0.00418	
3	4	20	0.02976	0.00730	0.00657
		30	0.01218	0.00437	
4	0	20	0.04247	0.00605	0.00589
		30	0.01810	0.00375	
4	1	20	0.04456	0.00637	0.00614
		30	0.01897	0.00395	
4	2	20	0.04684	0.00685	0.00688
		30	0.01977	0.00409	
4	3	20	0.05017	0.00726	0.00697
		30	0.02067	0.00428	
4	4	20	0.05312	0.00773	0.00616
		30	0.02184	0.00447	

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