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Optimal Plan of Partially Accelerated Life Tests under Type I Censoring

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ABSTRACT

In this paper, we consider optimum plan to determine stress change times under the three-step stress PALTs, assuming that each test units follows an exponential distribution. The tampered random variable(TRV) model for the three-step stress PALTs setup are introduced, and maximum likelihood estimators(MLEs) of the failure rate and the acceleration factors are obtained. The change times to minimize the generalized asymptotic variance(GAVR) of MLEs of the failure rate and the acceleration factors are proposed for the three-step stress PALTs.

1. INTRODUCTION

In many reliability studies, it may require a long testing time because the lifetimes of test units under the usual conditions tend to be large for extremely reliable units. As a common approach to shorten the lifetimes of test units, the accelerated life tests(ALTs) and the partially accelerated life tests(PALTs) are widely used. In ALTs, samples of tests units are subjected

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to conditions of greater stress than the usual conditions, and in PALTs, samples of test units are subjected to both the usual and the accelerated conditions.

ALTs is more appropriate than PALTs, in which the relationships that relates test condition to the failure rate of test units must be known in order to estimate the failure rates of the test units at the usual condition by using data from ALTs. However, if such relationships are not known or can not be assumed, data from ALTs can not be extrapolated to the usual condition.

DeGroot and Goel(1979) considered the two-step stress PALTs and proposed TRV model, which the effect of changing the stress from s_1 to s_2 is to multiply the remaining lifetime of the unit at changing time τ by some unknown factor α. Let X_1 denote the lifetime under the use stress s_1 and let X denote the lifetime under the two-step stress PALTs. Then a lifetime of a test unit is represented as

$$X = \begin{cases} X_1 & \text{if} \quad X_1 \leq \tau \ , \\ \tau + \alpha^{-1}(X_1 - \tau) & \text{if} \quad X_1 > \tau \ . \end{cases}$$

In general, α is larger than 1 since the effect of changing to the higher stress is to shorten the lifetime of the test unit. Bai and Chung(1992) considered the optimum plan to search the stress change time τ that minimizes the generalized asymptotic variance of MLEs of the acceleration factor and the failure rate at the usual condition with Type I censored data under the two-step stress PALTs. In Section 2, the TRV model under the three-step stress PALTs, which is considered by Mohamed(1989) is introduced, and then MLEs of the failure rate and the acceleration factors are obtained. By the Fisher information matrix, the plan that determine the optimal stress change times minimizing the generalized asymptotic variance of MLEs of the failure rate and the acceleration factors are proposed. In practice to use this optimum plan, the unknown parameters in optimal change times must be approximated by experience, or preliminary tests. But wrong pre-estimated values may result in a plan far from optimum. This possibility can be checked if this plan is examined for different parameter values. Thus the effects of the pre-estimated values of parameters are investigated in Section

2. MODELS AND MAXIMUM LIKELIHOOD ESTIMATORS

For the three-step stress PALTs, all test units are simultaneously put on the use stress $s_1 = s_u$ until a preassigned time τ_1, but if all units do not fail before τ_1, the surviving units are subjected to the increased stress s_2 and observed until time τ_2. For the surviving units at time τ_2, the stress is also increased to a larger stress s_3 and held constant until the preassigned censoring time T. Then a lifetime of a test unit can be represented by

$$X = \begin{cases} X_1 \quad if X_1 \le \tau_1, \\ \tau_1 + \alpha_1^{-1}(X_1 - \tau_1) \quad if \tau_1 < X_1 \le \alpha_1^{-1}(\tau_2 - \tau_1) + \tau_1, \\ \tau_2 - \alpha_2^{-1}(\tau_2 - \tau_1) + \alpha_1^{-1}\alpha_2^{-1}(X_1 - \tau_1) if X_1 > \alpha_1^{-1}(\tau_2 - \tau_1) + \tau_1. \end{cases} \tag{2.1}$$

Let F_1 be the distribution function under the use stress s_1. Then the distribution function corresponding to equation (2.1) is given by

$$F(x) = \begin{cases} F_1(x) & \text{if } x \le \tau_1 , \\ F_1(\alpha_1(x - \tau_1) + \tau_1) & \text{if } \tau_1 < x \le \tau_2 , \\ F_1(\alpha_1\alpha_2(x - \tau_2) + \alpha_1(\tau_2 - \tau_1) + \tau_1) & \text{if } \tau_2 < x . \end{cases} \tag{2.2}$$

The likelihood function of observations $x_p, p = 1, 2, \cdots, n,$ under the three-step stress PALTs takes the form given by

$$\begin{aligned} L = & \prod_{p_1=1}^{n_1} \lambda \exp(-\lambda x_{p_1}) \\ & \times \prod_{p_2=1}^{n_2} \alpha_1 \lambda \exp(-\alpha_1\lambda(x_{p_2} - \tau_1) - \lambda\tau_1) \\ & \times \prod_{p_3=1}^{n_3} \alpha_1\alpha_2\lambda \exp(-\alpha_1\alpha_2\lambda(x_{p_3} - \tau_2) - \alpha_1\lambda(\tau_2 - \tau_1) - \lambda\tau_1) \\ & \times \exp(-n_c\alpha_1\alpha_2\lambda(T - \tau_2) - n_c\alpha_1\lambda(\tau_2 - \tau_1) - n_c\lambda\tau_1) , \end{aligned} \tag{2.3}$$

where n_i is the number of test units failed at the stress s_i, $i = 1, 2, 3$ and n_c is the number of test units censored at a fixed censoring time T. From equation (2.3), MLEs of the failure rate and the acceleration factors can be obtained which are given by

$$\begin{aligned} \widehat{\lambda} &= \frac{n_1}{\sum_{p_1=1}^{n_1} x_{p_1} + (n_2 + n_3 + n_c)\tau_1} , \\ \widehat{\alpha}_1 &= \frac{n_2[\sum_{p_2=1}^{n_1} x_{p_2} + (n_2 + n_3 + n_c)\tau_1]}{n_1[\sum_{p_3=1}^{n_2}(x_{p_3} - \tau_1) + (n_3 + n_c)(\tau_2 - \tau_1)]} \end{aligned} \tag{2.4}$$

and

$$\widehat{\alpha}_2 = \frac{n_3[\sum_{p_2=1}^{n_2}(x_{p_2} - \tau_1) + (n_3 + n_c)(\tau_2 - \tau_1)]}{n_2[\sum_{p_3=1}^{n_3}(x_{p_3} - \tau_2) + n_c(T - \tau_2)]} .$$

Now, the plan to determine optimal stress change times minimizing the generalized asymptotic variance(GAVR) of MLEs of the failure rate and the acceleration factors under the three-step stress PALTs is presented in the following section.

3. OPTIMAL STRESS CHANGE TIMES

In the step stress PALTs, it is known that the asymptotic variance of the MLE $\widehat{\lambda}$ of the failure rate λ at the usual condition does not seem to be a reasonable criterion for optimality because optimal stress change time τ_1^* to minimize the asymptotic variance of MLE of the failure rate is ∞, which means that the test must be carry out at the usual condition. Therefore the generalized asymptotic variance (GAVR) of MLEs of the failure rate and the acceleration factors is used as the criterion to determine optimal stress change times for the three-step stress PALTs.

The expectations for the number of test units failed at the stresses s_i, $i = 1, 2, 3$, can be obtained as follows :

$$\begin{aligned} E(n_1) &= n(1 - \exp(-\lambda\tau_1)) , \\ E(n_2) &= n\exp(-\lambda\tau_1)(1 - \exp(-\alpha_1\lambda(\tau_2 - \tau_1)) , \\ E(n_3) &= n\exp(-\lambda\tau_1 - \alpha_1\lambda(\tau_2 - \tau_1))(1 - \exp(-\alpha_1\alpha_2\lambda(T - \tau_2))) , \\ E(n_c) &= n\exp(-\lambda\tau_1 - \alpha_1\lambda(\tau_2 - \tau_1) - \alpha_1\alpha_2\lambda(T - \tau_2)) , \end{aligned}$$

and the expectations for the sum of lifetimes of test units failed at stresses s_i, $i = 2, 3$, are given by

$$\begin{aligned} E(\sum_{p_2=1}^{n_2} X_{p_2}) &= E(\sum_{p=1}^{n} I(\tau_1 < X_p \le \tau_2)X_p) \\ &= n\exp(-\lambda_1\tau_1)\left\{\tau_1 + \frac{1}{\alpha_1\lambda} - \exp(-\alpha_1\lambda(\tau_2 - \tau_1))\left(\tau_2 + \frac{1}{\alpha_1\lambda}\right)\right\} , \\ E(\sum_{p_3=1}^{n_3} X_{p_3}) &= E(\sum_{p=1}^{n} I(\tau_2 < X_p \le T)X_p) \\ &= n\exp(-\alpha_1\lambda(\tau_2 - \tau_1) - \lambda\tau_1) \\ &\quad \times \left\{\tau_2 + \frac{1}{\alpha_1\alpha_2\lambda} - \exp(-\alpha_1\alpha_2\lambda(T - \tau_2))\left(T + \frac{1}{\alpha_1\alpha_2\lambda}\right)\right\} . \end{aligned}$$

Then from the above facts, the Fisher information matrix I can be obtained by

$$H = \begin{pmatrix} \frac{n}{\lambda^2}(1-a_3) & \frac{n}{\alpha_1\lambda}(a_1-a_3) & \frac{n}{\alpha_2\lambda}(a_2-a_3) \\ & \frac{n}{\alpha_1^2}(a_1-a_3) & \frac{n}{\alpha_1\alpha_2}(a_2-a_3) \\ & & \frac{n}{\alpha_2^2}(a_2-a_3) \end{pmatrix},$$

where

$$\begin{aligned} a_1 &= \exp(-\lambda\tau_1)\ , \\ a_2 &= \exp(-\lambda\tau_1 - \alpha_1\lambda(\tau_2-\tau_1))\ , \\ a_3 &= \exp(-\lambda\tau_1 - \alpha_1\lambda(\tau_2-\tau_1) - \alpha_1\alpha_2\lambda(T-\tau_2))\ . \end{aligned} \tag{2.5}$$

From the information matrix, GAVR of MLEs of the failure rate and the acceleration factors is given by

$$GAVR = \frac{\alpha_1^2\alpha_2^2\lambda^2}{n^3(1-a_1)(a_1-a_2)(a_2-a_3)}\ . \tag{2.6}$$

Theorem Optimal change times τ_1^* and τ_2^* minimizing the generalized asymptotic variance(GAVR) are unique solutions of

$$\begin{aligned} &\frac{\lambda a_1}{1-a_1} - \frac{\alpha_1\lambda a_2}{a_1-a_2} + \alpha_1\lambda - 2\lambda = 0\ , \\ &\frac{\alpha_1\lambda a_2}{a_1-a_2} - \frac{\alpha_1\alpha_2\lambda a_3}{a_2-a_3} - \alpha_1\lambda = 0\ , \end{aligned} \tag{2.7}$$

where a_1, a_2 and a_3 are given in equation (2.5)

Proof. To show that GAVR has the unique minimum value at τ_1^* and τ_2^*, it is sufficient to prove that the second order partial derivatives matrix H is positive definite, which is given by

$$H = \begin{pmatrix} \frac{\partial^2 \text{GAVR}}{\partial\tau_1^2} & \frac{\partial^2 \text{GAVR}}{\partial\tau_1\partial\tau_2} \\ & \frac{\partial^2 \text{GAVR}}{\partial\tau_2^2} \end{pmatrix}.$$

Let $p_i = 1 - \exp(-\lambda_i T)$, $i = 1,2,3$, $\zeta_1 = \frac{\tau_1}{T}$ and $\zeta_2 = \frac{\tau_2}{T}$. Then equation (2.5) is represented as

$$\begin{aligned} a_1 &=(1-p_1)^{\zeta_1} , \\ a_2 &=(1-p_1)^{\zeta_1}(1-p_2)^{\zeta_2-\zeta_1} , \\ a_3 &=(1-p_1)^{\zeta_1}(1-p_2)^{\zeta_2-\zeta_1}(1-p_3)^{1-\zeta_2} , \end{aligned}$$

and hence equation (2.7) can be written as for censoring time T,

$$\frac{(1-p_1)^{\zeta_1}\log(1-p_1)}{1-(1-p_1)^{\zeta_1}} - \frac{(1-p_2)^{\zeta_2-\zeta_1}\log(1-p_2)}{1-(1-p_2)^{\zeta_2-\zeta_1}} + \log\frac{(1-p_2)}{(1-p_1)^2} = 0 ,$$

$$\frac{(1-p_2)^{\zeta_2-\zeta_1}\log(1-p_2)}{1-(1-p_2)^{\zeta_2-\zeta_1}} - \frac{(1-p_3)^{1-\zeta_2}\log(1-p_3)}{1-(1-p_3)^{1-\zeta_2}} - \log(1-p_2) = 0 . \tag{2.8}$$

Optimal change times are given by $\tau_1^* = \zeta_1^* T$ and $\tau_2^* = \zeta_2^* T$, where ζ_1^* and ζ_2^* are unique solutions of equation (2.8) for $0 < \zeta_1,\ \zeta_2 < 1$.

For the combinations of different values of p_1 and p_2 when p_3 is assumed to be fixed, optimal change times ζ_1^* and ζ_2^* can be obtained from equation (2.8) and some facts are observed from Table 3.1. As p_2 becomes to increase, ζ_1^* goes to be large and ζ_2^* to be small, and then the ratio of ζ_2^* to ζ_1^* tends to decrease. When p_2 has small value, ζ_2^* tends to be about two times as large as ζ_1^*. In order to use this plan for searching optimal stress change times to minimize GAVR, unknown parameters in ζ_i^*, $i = 1,2$, must be approximated by the past data set. However, the wrong pre-estimated values of parameters may not lead to optimal change times and result in the poor estimators of parameters at the usual condition. Thus the effects of the pre-estimated values of parameters are investigated. The true values of p_1 and p_2 are assumed to be 0.3 and 0.65, respectively and $p_3 = 0.9$ is fixed. The behaviors of GAVR relative to optimal GAVR due to wrong pre-estimated values of p_1 and p_2 are shown in Figure 3.1. If the pre-estimated values of p_1 and p_2 are not too far from true values, the behaviors of GAVR seem to be less sensitive. If the pre-estimated value of p_2 are near true value, the behaviors of GAVR are likely to be stable even though the pre-estimated value of p_1 is far from true value.

Table 3.1 Optimal Change Times

p_1	p_2: 0.4	0.45	0.5	0.55	0.6	0.65	0.7	0.75	0.8
.15	.37052	.37600	.38180	.38797	.39455	.40163	.40931	.41772	.42712
	.69811	.69433	.69034	.68612	.68162	.67682	.67163	.66596	.65967
.20	.35832	.36367	.36933	.37536	.38179	.38871	.39622	.40445	.41363
	.69239	.68829	.68396	.67936	.67445	.66919	.66350	.65727	.65034
.25	.34572	.35090	.35640	.36225	.36850	.37523	.38252	.39052	.39944
	.68649	.68204	.67733	.67231	.66696	.66119	.65494	.64809	.64045
.30	.33271	.33770	.34301	.34865	.35468	.36118	.36822	.37594	.38454
	.68039	.67557	.67045	.66499	.65913	.65282	.64596	.63842	.62999
.35	.31930	.32408	.32916	.33456	.34035	.34657	.35332	.36070	.36893
	.67411	.66889	.66332	.65737	.65098	.64408	.63655	.62825	.61895

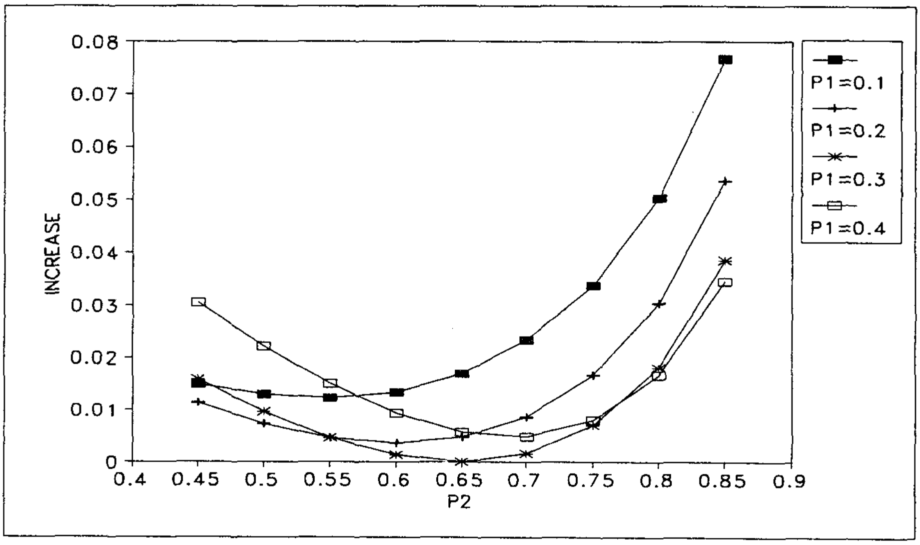

Figure 3.1 The Behavior of GAVR

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