Kyungpook Mathematical Journal
Vol.34, No.1, 43-51, June 1994

DUAL OPERATOR ALGEBRAS GENERATED BY A JORDAN OPERATOR *

Kun Wook Choi, Young Soo Jo and Il Bong Jung

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. Suppose that $\mathcal{C}_1 = \mathcal{C}_1(\mathcal{H})$ is the von Neumann-Schatten ideal of trace class operators in $\mathcal{L}(\mathcal{H})$ under the trace norm. Then it is well known that $\mathcal{C}_1^* \cong \mathcal{L}(\mathcal{H})$ under the pairing

$$(1) \qquad < T, [L] > = trace(TL),\ T \in \mathcal{L}(\mathcal{H}), L \in \mathcal{C}_1.$$

Note that the weak* topology that accrues to $\mathcal{L}(\mathcal{H})$ by virtue of the above duality coincides with the ultraweak operator topology on $\mathcal{L}(\mathcal{H})$ (cf. [6]). A *dual algebra* is a subalgebra of $\mathcal{L}(\mathcal{H})$ that contains the identity operator $I_{\mathcal{H}}$ and is closed in the ultraweak operator topology on $\mathcal{L}(\mathcal{H})$. For $T \in \mathcal{L}(\mathcal{H})$, let $\mathcal{A}_T$ denote the dual algebra generated by T. The theory of dual algebras is applied to the study of invariant subspaces, reflexivity and dilation theory. This theory is deeply related to properties $(\mathbf{A}_{m,n})$ which are the study of the problem of solving systems of the predual of a dual algebra (cf. [1], [3] and [4]). In this paper, we discuss property $(\mathbf{A}_{m,n})$ of the dual algebra singly generated by a Jordan block part of a Jordan operator.

The notation and terminology employed here agree with those in [2], [4] and [5]. We recall the essentials nonetheless for the convenience of the reader. Suppose that $\mathcal{A}$ is a dual algebra in $\mathcal{L}(\mathcal{H})$. Let ${}^\perp\mathcal{A}$ denote the

Received September 10, 1993.

This work was partially supported by TGRC-KOSEF, 1993. The second author was partially supported by Research Institute programs of university, Korea Research Foundation, 1993. The third author was partially supported by the Basic Science Research Institute Program, Ministry of Education, 1993.

preannihilator of $\mathcal{A}$ in $\mathcal{C}_1$. Let $\mathcal{Q}_{\mathcal{A}}$ denote the quotient space $\mathcal{C}_1/{}^{\perp}\mathcal{A}$. One knows that $\mathcal{A}$ is the dual space of $\mathcal{Q}_{\mathcal{A}}$ and that the duality is given by

$$< T, [L] > = \; trace(TL),\ T \in \mathcal{A},\ [L] \in \mathcal{Q}_{\mathcal{A}}. \tag{2}$$

For vectors x and y in $\mathcal{H}$, we write, as usual, $x \otimes y$ for the rank one operator in $\mathcal{C}_1$ defined by $(x \otimes y)(u) = (u, y)x, u \in \mathcal{H}$. Throughout this paper, we write $\mathbf{N}$ for the set of natural numbers. For a Hilbert space $\mathcal{K}$ and any operators $T_i \in \mathcal{L}(\mathcal{K}),\ i = 1, 2$, we write $T_1 \cong T_2$ if T_1 is unitarily equivalent to T_2. For $T \in \mathcal{L}(\mathcal{K})$ we write the n-th ampliation of T by

$$T^{(n)} = \overbrace{T \oplus \cdots \oplus T}^{(n)}.$$

Let $\mathcal{A}$ be an algebra in $\mathcal{L}(\mathcal{K})$. Then we write

$$\mathcal{A}^{(n)} = \{T^{(n)} | T \in \mathcal{A}\}.$$

Suppose m and n are cardinal numbers such that $1 \le m, n \le \aleph_0$. A dual algebra $\mathcal{A}$ will be said to have property $(\mathbf{A}_{m,n})$ if every $m \times n$ system of simultaneous equations of the form

$$[x_i \otimes y_j] = [L_{ij}], 0 \le i < m,\ 0 \le j < n, \tag{3}$$

where $\{[L_{ij}]\}_{\substack{0 \le i < m \\ 0 \le j < n}}$ is an arbitrary $m \times n$ array from $\mathcal{Q}_{\mathcal{A}}$, has a solution $\{x_i\}_{0 \le i < m}, \{y_j\}_{0 \le j < n}$ consisting of a pair of sequences of vectors from $\mathcal{H}$. Furthermore, if $m,\ n \in \mathbf{N}$ and r is a fixed real number satisfying $r \ge 1$, then a dual algebra $\mathcal{A}$ has property $(\mathbf{A}_{m,n})(r)$ if for every $s > r$ and every $m \times n$ array from $\mathcal{Q}_{\mathcal{A}}$, there exist sequences $\{x_i\}_{0 \le i < m}, \{y_j\}_{0 \le j < n}$ that satisfy (3) and also satisfy the following conditions:

$$\|x_i\| \le \left(s \sum_{0 \le j < n} \|[L_{ij}]\| \right)^{1/2},\quad 0 \le i < m \tag{4}$$

and

$$\|y_j\| \le \left(s \sum_{0 \le < m} \|[L_{ij}]\| \right)^{1/2},\quad 0 \le j < n. \tag{5}$$

Finally, a dual algebra $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ has property $(\mathbf{A}_{m,\aleph_0}(r))$ (for some real number $r \ge 1$) if for every $s > r$ and every array $\{[L_{ij}]\}_{\substack{0 \le i < m \\ 0 \le j < \infty}}$ from $\mathcal{Q}_{\mathcal{A}}$

with summable rows, there exist sequences $\{x_i\}_{0\le i<m}$ and $\{y_j\}_{0\le j<\infty}$ of vectors from $\mathcal{H}$ that satisfy (3), (4) and (5) with the replacement of n by $\aleph_0$. Properties $(\mathbf{A}_{\aleph_0,n}(r))$ and $(\mathbf{A}_{\aleph_0,\aleph_0}(r))$ are defined similarly. For brevity, we shall denote $(\mathbf{A}_{n,n})$ by $(\mathbf{A}_n)$. We shall denote by $\mathbf{D}$ the open unit disc in the complex plane $\mathbf{C}$ and we write $\mathbf{T}$ for the boundary of $\mathbf{D}$. For $1 \le p < \infty$, we denote by $L^p = L^p(\mathbf{T})$ the Banach space of complex valued, Lebesgue measurable functions f on $\mathbf{T}$ for which $|f|^p$ is Lebesgue integrable, and by $L^\infty = L^\infty(\mathbf{T})$ the Banach algebra of all complex valued Lebesgue measurable, essentially bounded functions on $\mathbf{T}$. For $1 \le p \le \infty$ we denote by $H^p = H^p(\mathbf{T})$ the subspace of L^p consisting of those functions whose negative Fourier coefficients vanish. Let us recall that a completely nonunitary contraction $T \in \mathcal{L}(\mathcal{H})$ is to be of *class* C_0 if there exists a non-zero function $u \in H^\infty(\mathbf{T})$ such that (under the functional calculus) $u(T) = 0$ (cf. [2]). Let S be the unilateral shift of multiplicity one. Then the function $S(\theta)$ defined by $S(\theta) = (S^*|(H^2 \ominus \theta H^2))^*$, for an inner function θ, is called a *Jordan block* and that any operator of the form $S(\theta_1) \oplus S(\theta_2) \oplus \cdots \oplus S(\theta_k) \oplus S^{(l)}$, where $\theta_1, \theta_2, \cdots, \theta_k$ are nonconstant (scalar valued) inner functions and $0 \le k < \infty$, $0 \le l \le \infty$, is called a *Jordan operator* (cf. [13]).

We start the work from the following theorem which comes from [9, Corollary 4.8].

Theorem 1. *If $T = S^{(n)} \oplus S(\theta_1) \oplus \cdots \oplus S(\theta_k)$ is a Jordan operator, $1 \le n < \infty$, and $k \in \mathbf{N}$, then $\mathcal{A}_T$ has property $(\mathbf{A}_{n,\aleph_0})(1)$ but not property $(\mathbf{A}_{n+1,1})$.*

The Jordan block part, $S(\theta_1) \oplus \cdots \oplus S(\theta_k)$, of T in the above theorem doesn't give any role for the property $(\mathbf{A}_{n,\aleph_0})(1)$. But by considering only the Jordan block part without shift part, we obtain the following theorem.

Theorem 2. *If $T = S(\theta)^{(n)}$ for $n \in \mathbf{N}$, then $\mathcal{A}_T$ has property $(\mathbf{A}_n)(1)$.*

We need several lemmas to prove the above main theorem. The following lemma is elementary, but we sketch the proof here for the convenience of readers.

Lemma 3. *Suppose that $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ is a dual algebra. Then the following are equivalent:*

(a) *For a weak*-continuous linear functional φ on $\mathcal{A}$ and every $\epsilon > 0$, there exist vectors $x, y \in \mathcal{H}$ such that $\varphi(A) = (Ax, y)$ for all $A \in \mathcal{A}$ and*

$\|x\|\|y\| < (1+\epsilon)\|\varphi\|$;

(b) *For a weak*-continuous linear functional* φ *on* $\mathcal{A}$ *and every* $s > 1$, *there exist vectors* $x, y \in \mathcal{H}$ *such that* $\varphi(A) = (Ax, y)$ *for all* $A \in \mathcal{A}$ *and* $\|x\|\|y\| < s\|\varphi\|$;

(c) *For every* $s > 1$ *and* $[L] \in \mathcal{Q}_{\mathcal{A}}$, *there exist vectors* $x, y \in \mathcal{H}$ *such that* $[L] = [x \otimes y]$ *and* $\|x\|\|y\| < s\|[L]\|$;

(d) *For every* $s > 1$ *and* $[L] \in \mathcal{Q}_{\mathcal{A}}$, *there exist vectors* $x, y \in \mathcal{H}$ *such that* $[L] = [x \otimes y]$, $\|x\|^2 < s\|[L]\|$ *and* $\|y\|^2 < s\|[L]\|$;

(e) *For every* $s > 1$ *and* $[L] \in \mathcal{Q}_{\mathcal{A}}$, *there exist vectors* $x, y \in \mathcal{H}$ *with* $\|x\| = \|y\|$ *such that* $[L] = [x\otimes]$ *and* $\|x\|^2 < s\|[L]\|$.

Proof. (a) $\Rightarrow$ (b): obvious.
(b) $\Rightarrow$ (c): Let $[L] \in \mathcal{Q}_{\mathcal{A}}$ and let $\varphi : \mathcal{A} \longrightarrow \mathbf{C}$ be a weak*-continuous linear functional on $\mathcal{A}$ defined by $\varphi(A) =< A, [L] >$. Then there exist vectors $x, y \in \mathcal{H}$ such that $\varphi(A) = (Ax, y)$. Since $< A, [L] >=< A, [x \otimes y] >$ for all $A \in \mathcal{A}$, $[L] = [x \otimes y]$. Furthermore, since

$$\|\varphi\| = \sup\{|\varphi(A)| : A \in \mathcal{A}, \|A\| \le 1\} = \|[L]\|, \tag{6}$$

we have this implication.
(c) $\Rightarrow$ (e) : Consider $x' = \frac{1}{\lambda}x, y' = \lambda y$, where $\lambda = \sqrt{\|x\|/\|y\|}$.
(e) $\Rightarrow$ (d) $\Rightarrow$ (c): obvious.
(e) $\Rightarrow$(a): Let φ be a weak*-continuous linear functional on $\mathcal{A}$. Then there exist square summable sequences $\{x_i\}_{i=1}^{\infty}$ and $\{y_i\}_{i=1}^{\infty}$ in $\mathcal{H}$ such that

$$\varphi(A) = \sum_{i=1}^{\infty}(Ax_i, y_i). \tag{7}$$

By assumption of (e) there exist x and y in $\mathcal{H}$ such that

$$\sum_{i=1}^{\infty}[x_i \otimes y_j] = [x \otimes y] \tag{8}$$

and

$$\|x\|^2 = \|y\|^2 < (1+\epsilon)\|[x \otimes y]\| \tag{9}$$

which implies that

$$\varphi(A) =< A, \sum_{i=1}^{\infty}[x_i \otimes y_i] >=< A, [x \otimes y] >= (Ax, y). \tag{10}$$

Moreover, by (6) and (9) we have $\|x\|^2 < (1+\epsilon)\|\varphi\|$. Hence the proof is complete.

The following comes from Hadwin-Nordgren [7].

Lemma 4. *Let $\mathcal{M}$ be a weak*-closed subspace of $\mathcal{L}(\mathcal{H})$ and φ be a weak*-continuous functional on $\mathcal{M}$ with $\|\varphi\| \leq 1$. Then for every $\epsilon > 0$, there is an extension $\tilde{\varphi}$ of φ to $\mathcal{L}(\mathcal{H})$ such that*

$$\tilde{\varphi}(T) = \sum_{n=1}^{\infty}(Tx_n, y_n) \tag{11}$$

and

$$\left(\sum_{n=1}^{\infty}\|x_n\|^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty}\|y_n\|^2\right)^{\frac{1}{2}} < 1+\epsilon, \tag{12}$$

where $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are square summable sequences in $\mathcal{H}$.

Without loss of generality, we can assume that $\|\varphi\| = 1$ in Lemma 3 (a) (because, consider $\varphi' = \frac{\varphi}{\|\varphi\|}$). Hence if we follow the proof of the implication (a) $\Rightarrow$ (d) in Lemma 3, we can restate Lemma 4 as following:

Lemma 5. *Let φ be a weak*-continuous linear functional on a dual algebra $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$. Then for every $s > 1$, there is an extension $\tilde{\varphi}$ of φ to $\mathcal{L}(\mathcal{H})$ with*

$$\tilde{\varphi}(T) = \sum_{n=1}^{\infty}(Tx_n, y_n) \tag{13}$$

such that

$$\sum_{n=1}^{\infty}\|x_n\|^2 < s\|\varphi\| \tag{14}$$

and

$$\sum_{n=1}^{\infty}\|y_n\|^2 < s\|\varphi\|, \tag{15}$$

where $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are square summable sequences in $\mathcal{H}$.

Recall that if θ is an inner function, then it follows from [2, Proposition III 1.21] (or [11, Theorem 2]), the dual algebra $\mathcal{A}_{S(\theta)}$ has property $(\mathbf{A}_1)(1)$. The following proposition improves [2, Proposition III 1.21 (iv)] (or [11, Theorem 2]).

Proposition 6. *For an inner function θ and any $n \in \mathbf{N}$, the dual algebra $\mathcal{A}_{S(\theta)}$ has property $(\mathbf{A}_{1,n})(1)$ and property $(\mathbf{A}_{n,1})(1)$.*

Proof. Note that

$$S \cong \begin{pmatrix} * & * \\ 0 & S(\theta) \end{pmatrix} \tag{16}$$

relative to a decomposition $\mathcal{K} = \mathcal{H}_0 \oplus \mathcal{H}$, where S is a unilateral shift of multiplicity one. Suppose that φ_i is a weak*-continuous linear functional on $\mathcal{A}_{S(\theta)}$ and $s > 1, i = 1, 2, \cdots, n$. By Lemma 5, there are sequences $\{x_k^{(i)}\}_{k=1}^{\infty}$ and $\{y_k^{(i)}\}_{k=1}^{\infty}$ in $\mathcal{H}$ satisfying

$$\varphi_i(A) = \sum_{k=1}^{\infty}(Ax_k^{(i)}, y_k^{(i)}) \tag{17}$$

for all A in $\mathcal{A}_T$ such that

$$\sum_{k=1}^{\infty} \|x_k^{(i)}\|^2 < s\|\varphi_i\| \tag{18}$$

and

$$\sum_{k=1}^{\infty} \|y_k^{(i)}\|^2 < s\|\varphi_i\|. \tag{19}$$

We denote by

$$\widetilde{\mathcal{K}} = \underbrace{\mathcal{K}_1^{(1)} \oplus \cdots \oplus \mathcal{K}_1^{(n)}}_{(n)} \oplus \underbrace{\mathcal{K}_2^{(1)} \oplus \cdots \oplus \mathcal{K}_2^{(n)}}_{(n)} \oplus \cdots,$$

$$\widetilde{x}^{(i)} = (\underbrace{\overbrace{0, \cdots, 0}^{(i-1)}, x_1^{(i)}, \cdots, 0}_{(n)}, \underbrace{\overbrace{0, \cdots, 0}^{(i-1)}, x_2^{(i)}, \cdots, 0}_{(n)}, \cdots) \tag{20}$$

and

$$\widetilde{y} = (\underbrace{y_1^{(1)}, \cdots, y_1^{(n)}}_{(n)}, \underbrace{y_2^{(1)}, \cdots, y_2^{(n)}}_{(n)}, \cdots), \tag{21}$$

where $\mathcal{K}_k^{(i)} = \mathcal{K},\ 1 \le i \le n,\ k \in \mathbf{N}$. Let $\mathcal{M} = \bigvee_{k=1}^{\infty} \widetilde{S}^k \widetilde{y}$, where

$$\widetilde{S} = \underbrace{S_1^{(1)} \oplus \cdots \oplus S_1^{(n)}}_{(n)} \oplus \underbrace{S_2^{(1)} \oplus \cdots \oplus S_2^{(n)}}_{(n)} \oplus \cdots, \tag{22}$$

where $S_k^{(i)} = S,\ 1 \le i \le n,\ k \in \mathbf{N}$. Since $\widetilde{S}|\mathcal{M}$ is a cyclic completely non-unitary isometry, it is unitarily equivalent to S. Then there is an isometry W from $\mathcal{K}$ into $\widetilde{\mathcal{K}}$ such that $W\mathcal{K} = \mathcal{M}$ and

$$WS = \widetilde{S}W. \tag{23}$$

Let $T_k^{(i)} = P_{k,i}W$, where $P_{k,i}$ is the projection from $\widetilde{\mathcal{K}}$ onto $\mathcal{K}_k^{(i)}$. Then clearly $T_k^{(i)} \in \mathcal{L}(\mathcal{K})$ and for every $x \in \mathcal{K}$ we have

$$(24) \qquad Wx = \underbrace{T_1^{(1)}x \oplus \cdots \oplus T_1^{(n)}x}_{(n)} \oplus \underbrace{T_2^{(1)}x \oplus \cdots \oplus T_2^{(n)}x}_{(n)} \oplus \cdots.$$

It follows by (23) and (24) that

$$(25) \qquad T_k^{(i)}S = ST_k^{(i)}$$

for any k, i. Let $y_0 = W^*\widetilde{y}$. Then $T_k^{(i)}\, y_0 = y_k^{(i)}$ for any $k \in \mathbf{N}$. Furthermore, by (19) we have

$$(26) \qquad \|y_0\|^2 = \|\widetilde{y}\|^2 = \sum_{i=1}^{n}\sum_{k=1}^{\infty} \|y_k^{(i)}\|^2 < s\sum_{i=1}^{n}\|\varphi_i\|.$$

Since

$$(27) \qquad (W^*\widetilde{x}^{(i)}, z) = (\widetilde{x}^{(i)}, Wz) = \sum_{k=1}^{\infty}(x_k^{(i)}, T_k^{(i)}z) = \sum_{k=1}^{\infty}(T_k^{(i)*}x_k^{(i)}, z)$$

for every $z \in \mathcal{K}$, we can assert that the series $\sum_{k=1}^{\infty} T_k^{(i)*}x_k^{(i)}$ converges weakly to some $x_0^{(i)} (= W^*\widetilde{x}^{(i)}) \in \mathcal{K}$, $i = 1, \cdots, n$. By (18) we have

$$(28) \qquad \|x_0^{(i)}\|^2 = \|\widetilde{x}^{(i)}\|^2 = \sum_{k=1}^{\infty}\|x_k^{(i)}\|^2 < s\|\varphi_i\|.$$

Moreover, $\mathcal{H}$ is a hyperinvariant subspace for S^*, so that $x_0^{(i)} \in \mathcal{H}$ by (25). Now for every $A \in \mathcal{A}_{S(\theta)}$ we have

$$(29) \qquad \begin{aligned} \varphi_i(A) &= \sum_{k=1}^{\infty}(Ax_k^{(i)}, y_k^{(i)}) = \sum_{k=1}^{\infty}(Ax_k^{(i)}, T_k^{(i)}y_0) \\ &= \sum_{k=1}^{\infty}(T_k^{(i)*}Ax_k^{(i)}, y_0) = \sum_{k=1}^{\infty}(AT_k^{(i)*}x_k^{(i)}, y_0) \text{ by (22)} \\ &= (A\sum_{k=1}^{\infty}T_k^{(i)*}x_k^{(i)}, y_0) = (Ax_0^{(i)}, y_0) \\ &= (Ax_0^{(i)}, P_{\mathcal{H}}y_0), \qquad \text{since } x_0^{(i)} \in \mathcal{H} \end{aligned}$$

$i = 1, 2, \cdots, n$. Hence the dual algebra $\mathcal{A}_{S(\theta)}$ has property $(\mathbf{A}_{1,n})(1)$. Furthermore, since $S(\theta)^* = S(\widetilde{\theta})$ (see [2, Corollary III 1.7]), where $\widetilde{\theta}(e^{it}) =$

$\overline{\theta(e^{-it})}$, and since $\mathcal{A}_{S(\widetilde{\theta})}$ has property $(\mathbf{A}_{1,n})(1)$, the algebra $\mathcal{A}_{S(\theta)}$ has property $(\mathbf{A}_{n,1})(1)$. Hence the proof is complete.

Proof of Theorem 2. If $\mathcal{A}$ is a dual algebra with property $(\mathbf{A}_{1,n})$ for some positive integer n, then it follows from [10, Proposition 2.21] that the dual algebra $\mathcal{A}^{(n)}$ has property $(\mathbf{A}_n)$. Since

$$\mathcal{A}_{T^{(n)}} = (\mathcal{A}_T)^{(n)}$$

for any positive integer n, Proposition 6 implies the theorem.

References

[1] C. Apostol, H. Bercovici, C. Foiaş and C. Pearcy, *Invariant subspaces, dilation theory, and the structure of the predual of a dual algebra. I*, J. Funct Anal. 63(1985), 369–404.

[2] H. Bercovici, *Operator theory and arithemetic in H^∞*, Math. Surveys and Monographs, No 26, A.M.S. Providence, R.I., 1988.

[3] H. Bercovici, C. Foiaş and C. Pearcy, *Dilation theory and systems of simultaneous equations in the predual of an operator algebra. I*, Michigan Math. J. 30(1983), 335–354.

[4] ________, *Dual algebra with applications to invariant subspaces and dilation theory*, CBMS Conf. Ser. in Math. No. 56, Amer. Math. Soc., Providence, R.I., 1985.

[5] A. Brown and C. Pearcy, *Introduction to operator theory, I, Elements of functional analysis*, Springer-Verlag, New York, 1977.

[6] J. Dixmier, *Von Neumann algebras*, North-Holland Publishing Company, Amst. New York, Oxford, 1969.

[7] D. Hadwin and E. Nordgren, *Subalgebras of reflexive algebras*, J. Operator Theory, 7(1982), 3–23

[8] K. Hoffman, *Banach spaces of analytic functions*, Prentic-Hall, Englewood Cliffs, NJ, 1965.

[9] I. Jung, *Dual Operator Algebras and the Classes $\mathbf{A}_{m,n}$, I*, J. Operator Theory, 27(1992), 309-323.

[10] M. Marsalli, *Ph.D. Thesis*, University of Michigan, 1985.

[11] S. Rosenoer, *Note on operators of class $C_0(1)$*, Acta sci Math, 46(1983), 287-293.

[12] Sz.-Nagy and C. Foiaş, *Harmonic analysis of operators on the Hilbert space*, North Holland Akademiai Kiado, Amsterdam/Budapest, 1970.

[13] Sz.-Nagy, *Jordan model for contractions of $C_{\cdot 0}$*, Acta Sci. Math.(Szeged) 36(1974), 305–322.

DEPARTMENT OF MATHEMATICS, KYUNGPOOK NATIONAL UNIVERSITY, TAEGU 702-701, KOREA.

DEPARTMENT OF MATHEMATICS, KEIMYUNG UNIVERSITY, TAEGU 704-200, KOREA.

DEPARTMENT OF MATHEMATICS, KYUNGPOOK NATIONAL UNIVERSITY, TAEGU 702-701, KOREA.