DUAL OPERATOR ALGEBRAS GENERATED
BY A JORDAN OPERATOR

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Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. Suppose that $\mathcal{C}_1 = \mathcal{C}_1(\mathcal{H})$ is the von Neumann-Schatten ideal of trace class operators in $\mathcal{L}(\mathcal{H})$ under the trace norm. Then it is well known that $\mathcal{C}_1^* \cong \mathcal{L}(\mathcal{H})$ under the pairing

$$< T, [L] > = \text{trace}(TL), \quad T \in \mathcal{L}(\mathcal{H}), \quad L \in \mathcal{C}_1.$$  

(1)

Note that the weak* topology that accrues to $\mathcal{L}(\mathcal{H})$ by virtue of the above duality coincides with the ultraweak operator topology on $\mathcal{L}(\mathcal{H})$ (cf. [6]). A dual algebra is a subalgebra of $\mathcal{L}(\mathcal{H})$ that contains the identity operator $I_\mathcal{H}$ and is closed in the ultraweak operator topology on $\mathcal{L}(\mathcal{H})$. For $T \in \mathcal{L}(\mathcal{H})$, let $\mathcal{A}_T$ denote the dual algebra generated by $T$. The theory of dual algebras is applied to the study of invariant subspaces, reflexivity and dilation theory. This theory is deeply related to properties $(\mathcal{A}_{m,n})$ which are the study of the problem of solving systems of the predual of a dual algebra (cf. [1], [3] and [4]). In this paper, we discuss property $(\mathcal{A}_{m,n})$ of the dual algebra singly generated by a Jordan block part of a Jordan operator.

The notation and terminology employed here agree with those in [2], [4] and [5]. We recall the essentials nonetheless for the convenience of the reader. Suppose that $\mathcal{A}$ is a dual algebra in $\mathcal{L}(\mathcal{H})$. Let $^\perp \mathcal{A}$ denote the
preannihilator of $\mathcal{A}$ in $\mathcal{C}_1$. Let $\mathcal{Q}_A$ denote the quotient space $\mathcal{C}_1 / \mathcal{A}$. One knows that $\mathcal{A}$ is the dual space of $\mathcal{Q}_A$ and that the duality is given by

$$< T, [L] > = \text{trace}(TL), \quad T \in \mathcal{A}, \quad [L] \in \mathcal{Q}_A. \quad (2)$$

For vectors $x$ and $y$ in $\mathcal{H}$, we write, as usual, $x \otimes y$ for the rank one operator in $\mathcal{C}_1$ defined by $(x \otimes y)(u) = (u, y)x, u \in \mathcal{H}$. Throughout this paper, we write $\mathbb{N}$ for the set of natural numbers. For a Hilbert space $\mathcal{K}$ and any operators $T_i \in \mathcal{L}(\mathcal{K}), \ i = 1, 2$, we write $T_1 \cong T_2$ if $T_1$ is unitarily equivalent to $T_2$. For $T \in \mathcal{L}(\mathcal{K})$ we write the $n$-th ampliation of $T$ by

$$T^{(n)} = T \oplus \cdots \oplus T.$$ 

Let $\mathcal{A}$ be an algebra in $\mathcal{L}(\mathcal{K})$. Then we write

$$\mathcal{A}^{(n)} = \{ T^{(n)} | T \in \mathcal{A} \}.$$ 

Suppose $m$ and $n$ are cardinal numbers such that $1 \leq m, n \leq \aleph_0$. A dual algebra $\mathcal{A}$ will be said to have property $(\mathcal{A}_{m,n})$ if every $m \times n$ system of simultaneous equations of the form

$$[x_i \otimes y_j] = [L_{ij}], \quad 0 \leq i < m, \quad 0 \leq j < n, \quad (3)$$

where $\{ [L_{ij}] \}_{0 \leq i < m, 0 \leq j < n}$ is an arbitrary $m \times n$ array from $\mathcal{Q}_A$, has a solution $\{ x_i \}_{0 \leq i < m}, \{ y_j \}_{0 \leq j < n}$ consisting of a pair of sequences of vectors from $\mathcal{H}$. Furthermore, if $m, n \in \mathbb{N}$ and $r$ is a fixed real number satisfying $r \geq 1$, then a dual algebra $\mathcal{A}$ has property $(\mathcal{A}_{m,n})(r)$ if for every $s > r$ and every $m \times n$ array from $\mathcal{Q}_A$, there exist sequences $\{ x_i \}_{0 \leq i < m}, \{ y_j \}_{0 \leq j < n}$ that satisfy (3) and also satisfy the following conditions:

$$\| x_i \| \leq \left( s \sum_{0 \leq j < n} \| [L_{ij}] \| \right)^{1/2}, \quad 0 \leq i < m \quad (4)$$

and

$$\| y_j \| \leq \left( s \sum_{0 \leq i < m} \| [L_{ij}] \| \right)^{1/2}, \quad 0 \leq j < n. \quad (5)$$

Finally, a dual algebra $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ has property $(\mathcal{A}_{m,\aleph_0}(r))$ (for some real number $r \geq 1$) if for every $s > r$ and every array $\{ [L_{ij}] \}_{0 \leq i < m} \in \mathcal{Q}_A$
with summable rows, there exist sequences \( \{x_i\}_{0 \leq i < m} \) and \( \{y_j\}_{0 \leq j < \infty} \) of vectors from \( \mathcal{H} \) that satisfy (3), (4) and (5) with the replacement of \( n \) by \( \aleph_0 \). Properties \( (A_{\aleph_0,n}(r)) \) and \( (A_{\aleph_0,\aleph_0}(r)) \) are defined similarly. For brevity, we shall denote \( (A_{n,n,n}) \) by \( (A_n) \). We shall denote by \( D \) the open unit disc in the complex plane \( \mathbb{C} \) and we write \( T \) for the boundary of \( D \). For \( 1 \leq p < \infty \), we denote by \( L^p = L^p(T) \) the Banach space of complex valued, Lebesgue measurable functions \( f \) on \( T \) for which \( |f|^p \) is Lebesgue integrable, and by \( L^\infty = L^\infty(T) \) the Banach algebra of all complex valued Lebesgue measurable, essentially bounded functions on \( T \). For \( 1 \leq p \leq \infty \) we denote by \( H^p = H^p(T) \) the subspace of \( L^p \) consisting of those functions whose negative Fourier coefficients vanish. Let us recall that a completely nonunitary contraction \( T \in \mathcal{L}(\mathcal{H}) \) is to be of class \( C_0 \) if there exists a non-zero function \( u \in H^\infty(T) \) such that (under the functional calculus) \( u(T) = 0 \) (cf. [2]). Let \( S \) be the unilateral shift of multiplicity one. Then the function \( S(\theta) \) defined by \( S(\theta) = (S^*|(H^2 \ominus \theta H^2))^* \), for an inner function \( \theta \), is called a Jordan block and that any operator of the form \( S(\theta_1) \oplus S(\theta_2) \oplus \cdots \oplus S(\theta_k) \oplus S(\theta) \), where \( \theta_1, \theta_2, \cdots, \theta_k \) are nonconstant (scalar valued) inner functions and \( 0 \leq k < \infty \), \( 0 \leq \ell \leq \infty \), is called a Jordan operator (cf. [13]).

We start the work from the following theorem which comes from [9, Corollary 4.8].

**Theorem 1.** If \( T = S^{(n)} \oplus S(\theta_1) \oplus \cdots \oplus S(\theta_k) \) is a Jordan operator, \( 1 \leq n < \infty \), and \( k \in \mathbb{N} \), then \( \mathcal{A}_T \) has property \( (A_{n,\aleph_0})(1) \) but not property \( (A_{n+1,1})(1) \).

The Jordan block part, \( S(\theta_1) \oplus \cdots \oplus S(\theta_k) \), of \( T \) in the above theorem doesn't give any role for the property \( (A_{n,\aleph_0})(1) \). But by considering only the Jordan block part without shift part, we obtain the following theorem.

**Theorem 2.** If \( T = S(\theta)^{(n)} \) for \( n \in \mathbb{N} \), then \( \mathcal{A}_T \) has property \( (A_n)(1) \).

We need several lemmas to prove the above main theorem. The following lemma is elementary, but we sketch the proof here for the convenience of readers.

**Lemma 3.** Suppose that \( \mathcal{A} \subset \mathcal{L}(\mathcal{H}) \) is a dual algebra. Then the following are equivalent:

(a) For a weak*-continuous linear functional \( \varphi \) on \( \mathcal{A} \) and every \( \epsilon > 0 \), there exist vectors \( x, y \in \mathcal{H} \) such that \( \varphi(A) = (Ax, y) \) for all \( A \in \mathcal{A} \) and
\[
\|x\|\|y\| < (1 + \epsilon)\|\varphi\|;
\]
(b) For a weak*-continuous linear functional \(\varphi\) on \(A\) and every \(s > 1\), there exist vectors \(x, y \in \mathcal{H}\) such that \(\varphi(A) = (Ax, y)\) for all \(A \in A\) and
\[
\|x\|\|y\| < s\|\varphi\|;
\]
(c) For every \(s > 1\) and \([L] \in \mathcal{Q}_A\), there exist vectors \(x, y \in \mathcal{H}\) such that \([L] = [x \otimes y]\) and
\[
\|x\|\|y\| < s\|[L]\|;
\]
(d) For every \(s > 1\) and \([L] \in \mathcal{Q}_A\), there exist vectors \(x, y \in \mathcal{H}\) such that \([L] = [x \otimes y]\), \(\|x\|^2 < s\|[L]\|\) and \(\|y\|^2 < s\|[L]\|\);
(e) For every \(s > 1\) and \([L] \in \mathcal{Q}_A\), there exist vectors \(x, y \in \mathcal{H}\) with
\[
\|x\| = \|y\|\) such that \([L] = [x \otimes y]\) and \(\|x\|^2 < s\|[L]\|\).

Proof. (a) \(\Rightarrow\) (b): obvious.
(b) \(\Rightarrow\) (c): Let \([L] \in \mathcal{Q}_A\) and let \(\varphi : A \rightarrow \mathbb{C}\) be a weak*-continuous linear functional on \(A\) defined by \(\varphi(A) = \langle A, [L] \rangle\). Then there exist vectors \(x, y \in \mathcal{H}\) such that \(\varphi(A) = (Ax, y)\). Since \(\langle A, [L] \rangle = \langle A, [x \otimes y] \rangle\) for all \(A \in A\), \([L] = [x \otimes y]\). Furthermore, since

\[
\|x\| = \|y\|\)

we have this implication.
(c) \(\Rightarrow\) (e): Consider \(x' = \frac{1}{\lambda}x, y' = \lambda y\), where \(\lambda = \sqrt{\|x\|/\|y\|}\).
(e) \(\Rightarrow\) (d) \(\Rightarrow\) (c): obvious.
(e) \(\Rightarrow\)(a): Let \(\varphi\) be a weak*-continuous linear functional on \(A\). Then there exist square summable sequences \(\{x_i\}_{i=1}^\infty\) and \(\{y_i\}_{i=1}^\infty\) in \(\mathcal{H}\) such that

\[
\varphi(A) = \sum_{i=1}^\infty (Ax_i, y_i).
\]

By assumption of (e) there exist \(x\) and \(y\) in \(\mathcal{H}\) such that

\[
\sum_{i=1}^\infty [x_i \otimes y_j] = [x \otimes y]
\]

and

\[
\|x\|^2 = \|y\|^2 < (1 + \epsilon)\|[x \otimes y]\|
\]

which implies that

\[
\varphi(A) = \langle A, \sum_{i=1}^\infty [x_i \otimes y_i] \rangle = \langle A, [x \otimes y] \rangle = (Ax, y).
\]

Moreover, by (6) and (9) we have \(\|x\|^2 < (1 + \epsilon)\|\varphi\|\). Hence the proof is complete.
The following comes from Hadwin-Nordgren [7].

**Lemma 4.** Let $\mathcal{M}$ be a weak* -closed subspace of $\mathcal{L}(\mathcal{H})$ and $\varphi$ be a weak* -continuous functional on $\mathcal{M}$ with $\|\varphi\| \leq 1$. Then for every $\epsilon > 0$, there is an extension $\tilde{\varphi}$ of $\varphi$ to $\mathcal{L}(\mathcal{H})$ such that

$$\tilde{\varphi}(T) = \sum_{n=1}^{\infty} (Tx_n, y_n)$$

and

$$\left( \sum_{n=1}^{\infty} \|x_n\|^2 \right)^{\frac{1}{2}} \left( \sum_{n=1}^{\infty} \|y_n\|^2 \right)^{\frac{1}{2}} < 1 + \epsilon,$$

where $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are square summable sequences in $\mathcal{H}$.

Without loss of generality, we can assume that $\|\varphi\| = 1$ in Lemma 3 (a) (because, consider $\varphi' = \frac{\varphi}{\|\varphi\|}$). Hence if we follow the proof of the implication (a) $\Rightarrow$ (d) in Lemma 3, we can restate Lemma 4 as following:

**Lemma 5.** Let $\varphi$ be a weak* -continuous linear functional on a dual algebra $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$. Then for every $s > 1$, there is an extension $\tilde{\varphi}$ of $\varphi$ to $\mathcal{L}(\mathcal{H})$ with

$$\tilde{\varphi}(T) = \sum_{n=1}^{\infty} (Tx_n, y_n)$$

such that

$$\sum_{n=1}^{\infty} \|x_n\|^2 < s\|\varphi\|$$

and

$$\sum_{n=1}^{\infty} \|y_n\|^2 < s\|\varphi\|,$$

where $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are square summable sequences in $\mathcal{H}$.

Recall that if $\theta$ is an inner function, then it follows from [2, Proposition III 1.21] (or [11, Theorem 2]), the dual algebra $\mathcal{A}_{S(\theta)}$ has property $(A_1)(1)$. The following proposition improves [2, Proposition III 1.21 (iv)] (or [11, Theorem 2]).

**Proposition 6.** For an inner function $\theta$ and any $n \in \mathbb{N}$, the dual algebra $\mathcal{A}_{S(\theta)}$ has property $(A_{1,n})(1)$ and property $(A_{n,1})(1)$.
Proof. Note that

\[ S \cong \begin{pmatrix} * & * \\ 0 & S(\theta) \end{pmatrix} \]

relative to a decomposition \( \mathcal{K} = \mathcal{H}_0 \oplus \mathcal{H} \), where \( S \) is a unilateral shift of multiplicity one. Suppose that \( \varphi_i \) is a weak\(^*\)-continuous linear functional on \( \mathcal{A}_{S(\theta)} \) and \( s > 1, i = 1, 2, \ldots, n \). By Lemma 5, there are sequences \( \{x_k^{(i)}\}_{k=1}^\infty \) and \( \{y_k^{(i)}\}_{k=1}^\infty \) in \( \mathcal{H} \) satisfying

\[ \varphi_i(A) = \sum_{k=1}^\infty (Ax_k^{(i)}, y_k^{(i)}) \]

for all \( A \) in \( \mathcal{A}_T \) such that

\[ \sum_{k=1}^\infty \|x_k^{(i)}\|^2 < s \|\varphi_i\| \]

and

\[ \sum_{k=1}^\infty \|y_k^{(i)}\|^2 < s \|\varphi_i\|. \]

We denote by

\[ \tilde{\mathcal{K}} = \underbrace{\mathcal{K}_1^{(1)} \oplus \cdots \oplus \mathcal{K}_1^{(n)}}_{(n)} \oplus \underbrace{\mathcal{K}_2^{(1)} \oplus \cdots \oplus \mathcal{K}_2^{(n)}}_{(n)} \oplus \cdots, \]

\[ \tilde{x}^{(i)} = \underbrace{0, \ldots, 0, x_1^{(i)}, \ldots, 0}_{{\text{(n)}}} \underbrace{, 0, \ldots, 0, x_2^{(i)}, \ldots, 0}_{{\text{(n)}}} \cdots \]

and

\[ \tilde{y} = \underbrace{y_1^{(1)}, \ldots, y_1^{(n)}}_{(n)} \underbrace{, y_2^{(1)}, \ldots, y_2^{(n)}}_{(n)} \cdots, \]

where \( \mathcal{K}_k^{(i)} = \mathcal{K}, 1 \leq i \leq n, k \in \mathbb{N} \). Let \( \mathcal{M} = \bigvee_{k=1}^\infty \tilde{S}^k \tilde{y} \), where

\[ \tilde{S} = \underbrace{S_1^{(1)} \oplus \cdots \oplus S_1^{(n)}}_{(n)} \oplus \underbrace{S_2^{(1)} \oplus \cdots \oplus S_2^{(n)}}_{(n)} \oplus \cdots, \]

where \( S_k^{(i)} = S, 1 \leq i \leq n, k \in \mathbb{N} \). Since \( \tilde{S}|\mathcal{M} \) is a cyclic completely non-unitary isometry, it is unitarily equivalent to \( S \). Then there is an isometry \( W \) from \( \mathcal{K} \) into \( \tilde{\mathcal{K}} \) such that \( WK = \mathcal{M} \) and

\[ WS = \tilde{S}W. \]
Let $T^{(i)}_k = P_{k,i} \tilde{W}$, where $P_{k,i}$ is the projection from $\tilde{\mathcal{K}}$ onto $\mathcal{K}^{(i)}_k$. Then clearly $T^{(i)}_k \in \mathcal{L}(\mathcal{K})$ and for every $x \in \mathcal{K}$ we have

$$Wx = \underbrace{T^{(1)}_1 x \oplus \cdots \oplus T^{(n)}_1 x}_{(n)} \oplus \underbrace{T^{(1)}_2 x \oplus \cdots \oplus T^{(n)}_2 x}_{(n)} \oplus \cdots.$$  

It follows by (23) and (24) that

$$T^{(i)}_k S = ST^{(i)}_k$$

for any $k, i$. Let $y_0 = W^* \tilde{y}$. Then $T^{(i)}_k y_0 = y^{(i)}_k$ for any $k \in \mathbb{N}$. Furthermore, by (19) we have

$$\|y_0\|^2 = \|\tilde{y}\|^2 = \sum_{i=1}^{n} \sum_{k=1}^{\infty} \|y^{(i)}_k\|^2 < \sum_{i=1}^{n} \|\varphi_i\|.$$  

Since

$$\langle W^* \tilde{x}^{(i)}, z \rangle = \langle \tilde{x}^{(i)}, Wz \rangle = \sum_{k=1}^{\infty} \langle x^{(i)}_k, T^{(i)}_k z \rangle = \sum_{k=1}^{\infty} \langle T^{(i)}_k x^{(i)}_k, z \rangle$$

for every $z \in \mathcal{K}$, we can assert that the series $\sum_{k=1}^{\infty} T^{(i)}_k x^{(i)}_k$ converges weakly to some $x^{(i)}_0 = W^* \tilde{x}^{(i)} \in \mathcal{K}$, $i = 1, \ldots, n$. By (18) we have

$$\|x^{(i)}_0\|^2 = \|\tilde{x}^{(i)}\|^2 = \sum_{k=1}^{\infty} \|x^{(i)}_k\|^2 < s \sum_{i=1}^{n} \|\varphi_i\|.$$  

Moreover, $\mathcal{H}$ is a hyperinvariant subspace for $S^*$, so that $x^{(i)}_0 \in \mathcal{H}$ by (25). Now for every $A \in \mathcal{A}_{S(\theta)}$ we have

$$\varphi_i(A) = \sum_{k=1}^{\infty} (Ax^{(i)}_k, y^{(i)}_k) = \sum_{k=1}^{\infty} (Ax^{(i)}_k, T^{(i)}_k y_0)$$

$$= \sum_{k=1}^{\infty} (T^{(i)}_k x^{(i)}_k, y_0) = \sum_{k=1}^{\infty} (AT^{(i)}_k x^{(i)}_k, y_0) \text{ by (22)}$$

$$= (A \sum_{k=1}^{\infty} T^{(i)}_k x^{(i)}_k, y_0) = (Ax^{(i)}_0, y_0)$$

$$= (Ax^{(i)}_0, P_{\mathcal{H}} y_0), \text{ since } x^{(i)}_0 \in \mathcal{H}$$

$i = 1, 2, \ldots, n$. Hence the dual algebra $\mathcal{A}_{S(\theta)}$ has property $(\mathcal{A}_{1,n})(1)$. Furthermore, since $S(\theta)^* = S(\tilde{\theta})$ (see [2, Corollary III 1.7]), where $\tilde{\theta}(e^{it}) = $
and since $\mathcal{A}_{S(\theta)}$ has property $(A_{1,n})(1)$, the algebra $\mathcal{A}_{S(\theta)}$ has property $(A_{n,1})(1)$. Hence the proof is complete.

Proof of Theorem 2. If $\mathcal{A}$ is a dual algebra with property $(A_{1,n})$ for some positive integer $n$, then it follows from [10, Proposition 2.21] that the dual algebra $\mathcal{A}^{(n)}$ has property $(A_{n})$. Since

$$\mathcal{A}_{T(n)} = (\mathcal{A}_{T})^{(n)}$$

for any positive integer $n$, Proposition 6 implies the theorem.

References


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