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ON BAER-*-SEMIGROUP, NEARLY RANGE CLOSED, STRONGLY BOUNDED AND ATOMISTIC BAER-*-SEMIGROUP*

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In this paper various forms of new Baer-*-Semigroups are introduced. New concepts of nearly range closed, strongly bounded and atomistic Baer-*-Semigroups are given. It is shown that a *-regular multiplicative Baer-*-Semigroup is nearly range closed and a *-regular strongly bounded Baer-*-Semigroup is also nearly range closed. Further it is proved that a *-regular (SSC)-Baer-*-Semigroup $\mathcal{L}(X,Y)$ of bounded linear operators from X to Y is an atomistic Baer-*-Semigroup.

1. Introduction

The purpose of this paper is to introduce various forms of new Baer-*-Semigroups called nearly range closed semigroup, strongly bounded Semigroups and an atomistic Baer-*-Semigroup. These semigroups are not merely of mathematical interest but these are also useful in quantum mechanics [5]. It is generally agreed that quantum propositions form Baer-*-Semigroup. I shall give the concept of an atomistic Baer-*-Semigroup which is based on an atomistic lattice [3]. It is observed that a *-regular multiplicative Baer-*-Semigroup is nearly range closed and if a *-regular strongly bounded abelian Baer-*-Semigroup is modular, then is is nearly range closed.

In this paper I suppose X and Y are complex Banach spaces, write $\mathcal{L}(X,Y)$ for the set of bounded linear operators from X to Y and abbreviated $\mathcal{L}(X,X)$ to $\mathcal{L}(X)$ [6].

We recall that $T \in \mathcal{L}(X,Y)$ is said to be bounded below if there is $k > 0$ for which

$$\|x\| \leq k\|Tx\|$$

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for all $x \in X$ and is said to be regular if and only if $T(X)$ is closed and both $T^{-1}(0)$ and $T(X)$ are complemented and that $(0,1)$: T regular and one-one $\Rightarrow T$ bounded below $\Rightarrow T(X)$ closed [6]. (cf. [2,3]. Recall also, that $T \in \mathcal{L}(X,Y)$ is said to be Fredholm if $T^{-1}(0)$ and $Y/T(X)$ are finite dimensional. If $T \in \mathcal{L}(X,Y)$ is Fredholm then the index of T is defined by

$$\text{index } (T) = \dim T^{-1}(0) - \dim Y/T(X) \text{ [6]} .$$

Further we show that a *-regular (SSC)-Baer-*-Semigroup of bounded linear operators from X to Y is an atomistic Baer-*-Semigroup.

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2. We generally follow in the sequel, Foulis [1], [2],

Maeda [3], Schelp [5] and Woo Young Lee [6], for the definitions and notations. However, for the sake of completeness we give the following definitions.

A *-semigroup is a semigroup S with an involuterial anti-automorphism $x \to x^*$ such that (i) $(xy)^* = y^*x^*$ and (ii) $x^{**} = x$ for all $x, y \in S$. A projective in such an S is an element e in S with $e = e^2 = e^*$. The partially ordered set of all projections in S is denoted by $P(S)$, the partial ordering being defined by $e \leq f$ if and only if $e = ef$ $(e, f \in P(S))$. A Baer-*-Semigroup is a *-Semigroup S with a two sided zero 0 with the property: For each element $a \in S$, there exists a projection $a' \in P(S)$ such that $\{x \in S; ax = 0\} = a'S$. We define $P'(S) = P(S)$ by the condition $P'(S) = \{a' : a \in S\}$.

A projection $e \in P'(S)$ is closed if and only if $e = a'$ for some $a \in S$. An element $a \in S$ is said to be right-*-regular in S if $aS = (a^*)''$, a is left-*-regular in S, then a is said to be *-regular in S.

A slight different but equivalent definition of *-regular in S as defined that if a is an element of a Baer-*-Semigroup S, then a is *-regular in S if there exists a unique element a^+ in S such that $a = aa^+a$, $a^+ = a^+aa^+$, $aa^+ = (a^*)''$ and $a^+a = a''$ [1].

Let b be an element belonging to Baer-*-Semigroup S. If for every $e, f \in P(S)$, we have $(eb)'' \wedge (fb)'' = ((e \wedge f)b)''$, then we say that b is multiplicative [2]. Baer-*-Semigroup having multiplicative elements is called multiplicative Baer-*-Semigroup [2]. We give the following definitions which will play main role in proving the theorems.

Definition 1. An element a in a Baer-*-Semigroup S is nearly range closed if the following condition holds: $1 \neq g \in P'(S)$ with $a'' \nleq g \nleq 1$, $g \leq a''$ and $(ga^*)'' = (a^*)''$ necessarily implies $[(g'a^*)'a]'' = a''$.

A Baer-*-Semigroup in which all elements are nearly range closed is called nearly range closed Baer-*-Semigroup.

Definition 2. A projection g in a Baer-*-Semigroup is called strongly bounded if $1 \neq g \in P'(S)$ with $a'' \nleq g \nleq 1$, $g \leq a''$ and $(ga^*)'' = (a^*)''$ for every $a \in S$. A Baer-*-Semigroup in which all projection are strongly bounded is called strongly bounded Baer-*-Semigroup.

By using above definitions and different techniques we derive the following results:

Theorem 1. *A *-regular multiplicate Baer-*-Semigroup is nearly range closed Baer-*-Semigroup.*

Proof. Let $1 \neq g \in P'(S)$ with $a'' \nleq g \nleq 1$, $g \leq a''$ and $(ga^*)'' = (a^*)''$. Then from Theorem 3[4],

$$(g'a^*)' \wedge (a^*)'' = (ga^+)'' \text{ for } 1 \neq g \in P(S). \qquad (i)$$

So by theorem 2[4],

$$(ga^*)'' = (a^*)'' \qquad (ii)$$

as $1 \neq g \leq a''$ and $(ga^*)'' = (a^*)''$. From (i) and (ii), we get $(g'a^*) \wedge (a^*)'' = (a^*)''$, which gives $([g'a^*)' \wedge (a^*)'']a)'' = ((a^*)a)''$. By multiplicativity, we obtain

$$((g'a^*)'a)'' \wedge ((a^*)''a)'' = ((a^*)''a)''.$$

Not by corollary of Theorem 11[1].

$$((g'a^*)'a)'' \wedge a'' = a''.$$

Therefore $a'' \leq ((g'a^*)'a)''$. But $((g'a^*)'a)'' \leq a''$ by Theorem 1[1]. Finally, we get

$$a'' = [(g'a^*)'a)'a]''.$$

Hence the result.

Theorem 2. *If a is *-regular strongly bounded abelian Baer-*-Semigroup S is modular, then S is nearly range closed Baer-*-Semigroup.*

Proof. As S is strongly bounded, so $1 \neq g \leq a''$, $(ga^*)'' = (a^*)''$ and $a'' \nleq g \nleq 1$ for all projections $g \in S$. By Theorem 12[1],

$$(ga^+)'' = [(g' \wedge a'')a^*]' \wedge (a^*)''. \qquad (i)$$

Let $h = [(g' \wedge a'')a^*]$. Hence $h' = [(g' \wedge a'')a^*]''$ which gives $h' \leq (a^*)''$ by Theorem 1[1]. Now we get $hC(a^*)''$. By definition of bounded Baer-*-Semigroup $1 \neq h \leq a'', (ha^*) = (a^*)''$, $a'' \not< h \not< 1$. Therefore by Theorem 7[4].

$([(h \wedge (ga^*)''a)]^*)'' = [(a^+h)]''$ and by Theorem 2[4], $(ha^+)'' = (a^*)''$. Hence $(a^*)'' = ((h \wedge ((ga^*)''a)^*)''$

$$\begin{aligned} a'' &= ((a^*)''a)'' = ((h \wedge ((ga^*)''a)^*)''a'' = (a^*(h \wedge (ga^*)'') \\ &= ((a^*)''(h \wedge (ga^*)'')''a)'' = ((a^*a)''(h \wedge (ga^*)'')'')'' \\ &= ((a^*)''(h \wedge (ga^*)'')''a)'' = ((a^*a)''(h \wedge (ga^*)'')'')'' \\ &= ((h \wedge (ga^*)''a)''a)'' \text{ by Theorem (i) [1] .} \end{aligned}$$

Further by Theorem 3[4], $(ga^+)'' = (g'a^*)' \wedge (a^*)''$, so we get

$$(g'a^*)' \wedge (a^*)'' = [(g' \wedge a'')a^*]' \wedge (a^*)'' \text{ from (i) .}$$

Therefore by modularity;

$$(g'a^*)'' \vee (a^*)'] \vee (a^*)'' = \{[(g' \wedge a'')a^*]'' \vee (a^*)'\} \wedge (a^*)''.$$

and $(g'a^*)' = [(g' \wedge a'')a^*]''$. Now $a'' = ((g'a^*)' \wedge ((ga^*)''a)''a)''$ which gives

$$\begin{aligned} a'' &= [(ga^*)((ga^*)''a)''a]'' = [(g'a^*)'a]''((ga^*)''a)'' \\ &= [(g'a^*)'a]((a^*)''a)'' = [(g'a^*)'a]''(a^*a)'' \\ &= [(a''((g'a^*)')''a]'' = [(g'a^*)'a]'' \text{ by Theorem 1[1].} \end{aligned}$$

Hence $a'' = [(g'a^*)'a]''$.

Here we are defining an atomistic Baer-*-Semigroup and (SSC)-Baer-*-Semigroup.

Definition 3. A Baer-*-Semigroup is an atomistic Baer-*-Semigroup if the following condition holds: For every non-zero element $a \in S$,

$$a = \vee(x/x \leq a) \text{ for } 1 \neq x \in S.$$

Definition 4. A Baer-*-Semigroup is an (SSC)-Baer-*-Semigroup if the following condition hold: For any $a, b \in S$, if b¿a, then there exists a projection $e \neq 1 > e \geq a$ and $e \wedge b = 0$.

We give the following theorem on the basis of above definitions:

Theorem 3. *A *-regular (SSC)-Baer-*-Semigroup $\mathcal{L}(X,Y)$ of bounded linear operators on complex Banach space X to Y is an atomistic Baer-*-Semigroup.*

Proof. Suppose $T \in \mathcal{L}(X,Y)$ is regular and $T^{-1}(0) \cap T(X)$ is finite dimensional by (0.1) [6]. We have T regular $\Leftrightarrow$ T bounded below $\Leftrightarrow$ $T(X)$ closed. Since $T \in \mathcal{L}(X,Y)$ is regular, so it is bounded below. For every non-zero element $a \in S$, we have a lower bound $1 \neq x \in S$ such that x is contained in a. $S \subseteq \mathcal{L}(X,Y)$. From this, we have

$$\vee(x/x \leq a) \leq a.$$

If we had $\vee(x/x \leq a) < a$, then there would exist a projection c such that $1 > c \geq a$ and

$$c \wedge \{\vee(x/x \leq a)\} = 0 \text{ by def 4.}$$

It gives $(a, \vee(x/x \leq a))M$ by (1.2) [3], $(c,a)M$ by (1.2) [3], and $c \wedge \{a \vee (\vee(x/x \leq a)\} \leq a$. Hence by 1.6[3], we obtain

$$(c \vee a) \wedge \{\vee(x/x \leq a)\} = a \wedge \{\vee(x/x \leq a)\},$$

which gives $\{\vee(x/x \leq a)\} = 0$. It is a contradiction. Therefore $\vee(x/x \leq a) = a$.

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